DYNAMIC MEREOTOPOLOGY: A POINT-FREE THEORY OF CHANGING REGIONS. I. Stable and unstable mereotopological relations

Dimiter Vakarelov

Faculty of Mathematics and Computer Science,
Department of Mathematical logic,
Sofia University

tu-logo

ur-logo

International Workshop on Topological Methods in Logic II,
June 8-10, 2010, Tbilisi, Georgia

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead’s view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Outline

1. Historical introduction
   - Whitehead's view on point-free theory of space and time
2. Contact algebras
   - Topological representations of contact algebras
   - Abstract points in contact algebras
3. Dynamic contact algebras (DCA)
   - Examples of DCA by products of contact algebras
4. Representation theory of DCA
   - Abstract moments of time
   - reconstruction of coordinate contact algebras
   - Representation theorem for DCA
   - DCL - a quantifier-free logic related to DCA
5. Concluding remarks: other versions of DCA

tu-logo

ur-logo

Historical introduction

tu-logo

ur-logo

Outline

tu-logo

ur-logo

A quote from Whitehead

Whitehead is known not only as one of the authors of the famous book "Principia Mathematica". He also was an initiator of a **new approach to the theory of space and time** influenced by the Einstein's Theory of Relativity. Here is a quote from his first writings on this subject

A. N. Whitehead, **The Organization of Though**, London, William and Norgate, 1917, page 195

"...It follows from the relative theory that **a point should be definable in terms of the relations between material things**. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning.Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. **Similar explanations apply to time.**Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared.**This is an unwritten chapter of mathematics...**"

"...It follows from the relative theory that **a point should be definable in terms of the relations between material things**. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning.Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. **Similar explanations apply to time.** Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared.**This is an unwritten chapter of mathematics...**"

"...It follows from the relative theory that **a point should be definable in terms of the relations between material things**. So far as I am aware, this outcome of the theory has escaped the notice of mathematicians, who have invariably assumed the point as the ultimate starting ground of their reasoning.Many years ago I explained some types of ways in which we might achieve such a definition, and more recently have added some others. **Similar explanations apply to time.**Before the theories of space and time have been carried to a satisfactory conclusion on the relational basis, a long and careful scrutiny of the definitions of points of space and instants of time will have to be undertaken, and many ways of effecting these definitions will have to be tried and compared.**This is an unwritten chapter of mathematics...**"

Whitehead's theory of space

The theory of space

based on the Whiteheadean approach is known now as **Region Based Theory of Space (RBTS)**.

- Unlike the classical Euclidean approach, based on the primitive notions **point, line** and **plane**, which are abstract features having no existence in reality as separate things,
- RBTS is based on a more realistic primitive notions like **region** and some spatial relations between regions like **part-of** and **contact**.

Proces and Reality

In his book

A. N. Whitehead, **Process and Reality**, New York, MacMillan, 1929.

Whitehead presented a detailed programme how to build a mathematical theory of RBTS in which points (as well as lines and planes) can be defined by the primitives of the theory.

Contact algebras and RBTS

It is now commonly accepted that the simplest version of RBTS, capturing only some topological aspects of regions is the notion of **contact algebra**. Contact algebras are in fact Boolean algebras extended with one binary relation called contact, satisfying some obvious axioms. The elements of the Boolean algebra are called regions and the Boolean operations can be considered as operations of how to define new regions from given ones. The Boolean part of contact algebra can be considered as a formalization of the **mereological part of RBTS**, considered in mereology as the **theory of parts**. For instance Boolean ordering corresponds to **part-of** relation.

Mereotopology

The contact relation captures some properties of Whitehead's connection relation. Standard point models of contact algebra are of topological nature, namely the Boolean algebra of regular closed subsets of a given topological space and the main representation theorem for contact algebras in fact realizes the Whitehead's idea to define points by means of regions and some relations between regions. Topological models of contact algebras show that contact has some topological nature and that in certain sense

CONTACT ALGEBRA = MEREOLOGY + TOPOLOGY

This suggests also to use another terminology and equation:

CONTACT ALGEBRA = MEREOTOPOLOGY

Whitehead's theory of time

was developed mainly in

A. N. Whitehead,**Science and the Modern World.** New Work, MacMillan , 1925.

and was called in [**Process and Reality**] **Epochal Theory of Time (ETT).Whitehead claims that the theory of time can not be separated from the theory of space and have to be extracted from the existing things in reality and some of their spatio-temporal relations considered as primitives. For instance, like points, moments of time do not have separate existence in reality and consequently have to be defined on the base of the primitives of the theory.**

Unfortunately, unlike his program how to build mathematical theory of space given in [**Proces and Reality**], Whitehead did not describe analogous program for his theory of time.

Whitehead introduced and analyzed many notions related to ETT but mainly in an informal way, which fact makes extremely difficult to obtain clear mathematical theory corresponding to ETT.

So, any such attempt will be only an approximate partial formalization of Whitehead's ideas and the claim that it corresponds to what Whitehead had in mind always will be disputable.

THE AIM OF THE TALK

In this talk we will try to make the first step in obtaining an extension of region-based theory of space with the notion of time. The algebraic counterpart of this theory is the notion of DYNAMIC CONTACT ALGEBRA as a point-free version of an integrated theory incorporating both space and time.

The main reference later on about some technical results and definitions will be on the following paper:

[**Dimov, Vakarelov**]: Contact Algebras and Region-based Theory of Space: A Proximity Approach - I+II. Fundam. Inform. 74(2-3): 209-249, 251-282 (2006)

tu-logo

ur-logo

THE AIM OF THE TALK

In this talk we will try to make the first step in obtaining an extension of region-based theory of space with the notion of time. The algebraic counterpart of this theory is the notion of DYNAMIC CONTACT ALGEBRA as a point-free version of an integrated theory incorporating both space and time.

The main reference later on about some technical results and definitions will be on the following paper:

[**Dimov, Vakarelov**]: Contact Algebras and Region-based Theory of Space: A Proximity Approach - I+II. Fundam. Inform. 74(2-3): 209-249, 251-282 (2006)

Contact algebras

Definition

By a *Contact Algebra*(CA) we mean any system $\underline{B} = (B, C) = (B, 0, 1, ., +, *, C)$, where $(B, 0, 1, ., +, *)$ is a non-degenerate Boolean algebra with Boolean complement denoted by "$*$" and C – a binary relation in B, called **contact** and satisfying the following axioms:

$(C1)$ If xCy, then $x, y \neq 0$,

$(C2)$ $xC(y+z)$ iff xCy or xCz,

$(C3)$ If xCy, then yCx,

$(C4)$ If $x.y \neq 0$, then xCy.

The elements of B are called **regions**.

Zero region and mereological relations

Let us note that the Boolean part of the definition of contact algebra incorporates the mereological component of the notion. In mereology B is supposed to be Boolen algebra with excluded zero element 0, because 0 does not correspond to any existing in reality region. We, however, do not exclude it and **consider 0** as **non-existing region**. So axiom C1 then says that if aCb then a and b are existing regions.

On the base of Boolean algebra we can define the basic mereological relations between regions:

- **part-of:** $a \leq b$ – just the lattice ordering of B
- **overlap:** aOb iff $a.b \neq 0$.
- **underlap (dual overlap)** xUy iff $x + y \neq 1$

tu-logo

ur-logo

Zero region and mereological relations

Let us note that the Boolean part of the definition of contact algebra incorporates the mereological component of the notion. In mereology B is supposed to be Boolen algebra with excluded zero element 0, because 0 does not correspond to any existing in reality region. We, however, do not exclude it and **consider 0** as **non-existing region**. So axiom C1 then says that if aCb then a and b are existing regions.

On the base of Boolean algebra we can define the basic mereological relations between regions:

- **part-of:** $a \leq b$ – just the lattice ordering of B
- **overlap:** aOb iff $a.b \neq 0$.
- **underlap (dual overlap)** xUy iff $x + y \neq 1$

Topological example of CA

The CA of regular closed sets

Let X be an arbitrary topological space. A subset a of X is *regular closed* if $a = Cl(Int(a))$. The set of all regular closed subsets of X will be denoted by $RC(X)$. It is a well-known fact that regular closed sets with the operations

- $a + b = a \cup b$,
- $a.b = Cl(Int(a \cap b))$,
- $a^* = Cl(X \setminus a)$, and
- $0 = \varnothing$, $1 = X$

  form a Boolean algebra. If we define the contact by
- $a\, C_X\, b$ iff $a \cap b \neq \varnothing$, then $RC(X)$ becomes a contact algebra.

Outline

tu-logo

ur-logo

Topological representation of CA

In the representation theory of contact algebras more special topological spaces are used, namely **semi-regular topological spaces** X having the set $RC(X)$ as a topological base of closed sets of X.

Theorem

Topological representation theorem for contact algebras[Dimov,Vakarelov]. *For every contact algebra (B, C) there exists a semi-regular and compact T_0 space X and an embedding h into the contact algebra $RC(X)$.*

Topological representation of CA

In the representation theory of contact algebras more special topological spaces are used, namely **semi-regular topological spaces** X having the set $RC(X)$ as a topological base of closed sets of X.

Theorem

Topological representation theorem for contact algebras[Dimov,Vakarelov]. *For every contact algebra (B, C) there exists a semi-regular and compact T_0 space X and an embedding h into the contact algebra $RC(X)$.*

Products of topological spaces and contact algebras

Notations

Let $T \neq \varnothing$ be a set of indices, X_t, $t \in T$, be a family of non-empty topological spaces and let $X = \prod_{t \in T} X_t$ be the cartesian product of the sets X_t (we will write simply $\prod_t$ instead of $\prod_{t \in T}$). We consider X as a topological space (product of the spaces X_t, $t \in T$) taking as a closed base the family of subsets of X of the form $\prod_t a_t$, where a_t is a closed subset of X_t and $a_t = X_t$ except for some finite set of indices t. We denote closure and interior operations of X_t correspondingly by Cl_t, Int_t, and for X – by Cl and Int.

If $x \in X$ we denote by x_t the t-th coordinate of x and define the projection function $pr_t(x) = x_t$. For $a_t \subseteq X_t$ we define $pr_t^{-1}(a_t) = \{x \in X : x_t \in a_t\}$.

Products of topological spaces and contact algebras

Notations

Let $T \neq \varnothing$ be a set of indices, X_t, $t \in T$, be a family of non-empty topological spaces and let $X = \prod_{t \in T} X_t$ be the cartesian product of the sets X_t (we will write simply $\prod_t$ instead of $\prod_{t \in T}$).We consider X as a topological space (product of the spaces X_t, $t \in T$) taking as a closed base the family of subsets of X of the form $\prod_t a_t$, where a_t is a closed subset of X_t and $a_t = X_t$ except for some finite set of indices t. We denote closure and interior operations of X_t correspondingly by Cl_t, Int_t, and for X – by Cl and Int.

If $x \in X$ we denote by x_t the t-th coordinate of x and define the projection function $pr_t(x) = x_t$. For $a_t \subseteq X_t$ we define $pr_t^{-1}(a_t) = \{x \in X : x_t \in a_t\}$.

Products of topological spaces and contact algebras

Notations

Let $T \neq \varnothing$ be a set of indices, X_t, $t \in T$, be a family of non-empty topological spaces and let $X = \prod_{t \in T} X_t$ be the cartesian product of the sets X_t (we will write simply $\prod_t$ instead of $\prod_{t \in T}$).We consider X as a topological space (product of the spaces X_t, $t \in T$) taking as a closed base the family of subsets of X of the form $\prod_t a_t$, where a_t is a closed subset of X_t and $a_t = X_t$ except for some finite set of indices t. We denote closure and interior operations of X_t correspondingly by Cl_t, Int_t, and for X – by Cl and Int.
If $x \in X$ we denote by x_t the t-th coordinate of x and define the projection function $pr_t(x) = x_t$. For $a_t \subseteq X_t$ we define $pr_t^{-1}(a_t) = \{x \in X : x_t \in a_t\}$.

Lemma

Let X_t, $t \in T$ be a non-empty family of topological spaces and X be their product. Then:

1. *If all X_t are T_0 and compact then X is also T_0 and compact,*
2. *If all spaces X_t are semiregular, then X is also semiregular and moreover pr_t^{-1} is an embedding of $RC(X_t)$ into $RC(X)$.*

As a consequence we obtain

Theorem

Joint embedding theorem for contact algebras. *Let (B_t, C_t), $t \in T$ be a nonempty family of contact algebras. Then there exist a semiregular and compact T_0 space X and a family of embeddings g_t of (B_t, C_t) into $RC(X)$, $t \in T$.*

Lemma

Let X_t, $t \in T$ be a non-empty family of topological spaces and X be their product. Then:

1. *If all X_t are T_0 and compact then X is also T_0 and compact,*
2. *If all spaces X_t are semiregular, then X is also semiregular and moreover pr_t^{-1} is an embedding of $RC(X_t)$ into $RC(X)$.*

As a consequence we obtain

Theorem

Joint embedding theorem for contact algebras. *Let (B_t, C_t), $t \in T$ be a nonempty family of contact algebras. Then there exist a semiregular and compact T_0 space X and a family of embeddings g_t of (B_t, C_t) into $RC(X)$, $t \in T$.*

Outline

tu-logo

ur-logo

Clans

The abstract points which are used in the representation theory of contact algebra developed in [Dimov,Vakarelov] are called clans (the name “clan” is taken from the theory of proximity spaces)

Definition

Let (B, C) be a contact algebra. A subset $\Gamma \subseteq B$ is called a **clan** if the following conditions are satisfied:

- $1 \in \Gamma$ and $0 \notin \Gamma$,
- If $a \in \Gamma$ and $a \leq b$ then $b \in \Gamma$,
- If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$,
- If $a, b \in \Gamma$ then aCb.

Clans

The abstract points which are used in the representation theory of contact algebra developed in [Dimov,Vakarelov] are called clans (the name “clan” is taken from the theory of proximity spaces)

Definition

Let (B, C) be a contact algebra. A subset $\Gamma \subseteq B$ is called a **clan** if the following conditions are satisfied:

- $1 \in \Gamma$ and $0 \notin \Gamma$,
- If $a \in \Gamma$ and $a \leq b$ then $b \in \Gamma$,
- If $a + b \in \Gamma$ then $a \in \Gamma$ or $b \in \Gamma$,
- If $a, b \in \Gamma$ then aCb.

It is easy to see that every ultrafilter in $\underline{B} = (B, C)$ is a clan, but there are clans, which are not ultrafilters.

General construction of clans

- Define in the set of ultrafilters the relation $U_1 \rho U_2$ iff $U_1 \times U_2 \subseteq C$. Let U_i, $i \in I$ be a nonempty family of ultrafilters such that every two are in the relation ρ. Then $\Gamma = \bigcup_{i \in I} U_i$ is a clan and every clan can be represented in this form.
- The set of all clans of $\underline{B}$ is denoted by $CLANS(\underline{B})$.
- For $a \in B$ we denote by $h(a) = \{\Gamma \in CLANS(\underline{B}) : a \in \Gamma\}$.
- The set $CLANS(\underline{B})$ with the set $\{h(a) : a \in B\}$ as a closed base define a semiregular and compact T_0 space and h is an embedding of $\underline{B}$ into $RC(CLANS(\underline{B}))$ required in the Representation theorem for contact algebras.

It is easy to see that every ultrafilter in $\underline{B} = (B, C)$ is a clan, but there are clans, which are not ultrafilters.

General construction of clans

- Define in the set of ultrafilters the relation $U_1 \rho U_2$ iff $U_1 \times U_2 \subseteq C$. Let U_i, $i \in I$ be a nonempty family of ultrafilters such that every two are in the relation ρ. Then $\Gamma = \bigcup_{i \in I} U_i$ is a clan and every clan can be represented in this form.
- The set of all clans of $\underline{B}$ is denoted by $CLANS(\underline{B})$.
- For $a \in B$ we denote by $h(a) = \{\Gamma \in CLANS(\underline{B}) : a \in \Gamma\}$.
- The set $CLANS(\underline{B})$ with the set $\{h(a) : a \in B\}$ as a closed base define a semiregular and compact T_0 space and h is an embedding of $\underline{B}$ into $RC(CLANS(\underline{B}))$ required in the Representation theorem for contact algebras.

Later on we will need the following lemma from [Dimov,Vakarelov] in which $\underline{B}$ is a contact algebra and $UF(\underline{B})$ is the set of ultrafilters in $\underline{B}$.

Lemma

1. aCb *iff* $(\exists U_1, U_2 \in UF(\underline{B}))(U_1 \rho U_2$ *and* $a \in U_1$ *and* $b \in U_2)$,
2. aCb *iff* $(\exists \Gamma \in CLANS(\underline{B}))(a, b \in \Gamma)$ *iff* $h(a) \cap h(b) \neq \varnothing$.

Note that the clan Γ from (2) is just $U_1 \cup U_2$ from (1).

tu-logo

ur-logo

A construction of contact algebra from a set of clans.

Definition

Let $\underline{B}$ be a contact algebra and $\alpha \subseteq CLANS(\underline{B})$, $\alpha \neq \varnothing$. We construct a contact algebra B_α corresponding to α as follows:

- Define $I(\alpha) = \{a \in B : \alpha \cap h(a) = \varnothing\}$. $I(\alpha)$ is a proper ideal in $\underline{B}$, i.e. $1 \notin I(\alpha)$.
- The congruence defined by $I(\alpha)$ is denoted by $\equiv_\alpha$. We have $a \equiv_\alpha b$ iff $a^*.b + a.b^* \in I(\alpha)$ iff $a^*.b \in I(\alpha)$ and $a.b^* \in I(\alpha)$. The congruence class determined by $a \in B$ is denoted by $|a|_\alpha$.
- Define $\underline{B}_\alpha$ to be the Boolean algebra $\underline{B}/\equiv_\alpha = B/I(\alpha)$. We define a contact relation C_α in $\underline{B}_\alpha$ as follows: $|a|_\alpha C_\alpha |b|_\alpha$ iff $\alpha \cap h(a) \cap h(b) \neq \varnothing$.

Dynamic contact algebras (DCA)

Definition

By a **dynamic contact algebra (DCA)** we mean any system $\underline{B} = (B, C^{\exists}, C^{\forall}) = (B, 0, 1, ., +, ^{*}, C^{\exists}, C^{\forall})$ where $(B, 0, 1, ., +, ^{*}, C^{\exists})$ is a contact algebra and $C^{\forall}$ is a binary relation in B satisfying the following axioms:

$(C^{\forall}1)$ $\quad 1C^{\forall}1$, $(C^{\forall}2)$ $\quad 0\overline{C}^{\forall}0$,

$(C^{\forall}3)$ $\quad$ If $pC^{\forall}q$ and $a\overline{C}^{\forall}b$ then $(p.a^{*})C^{\exists}(q.a^{*})$ or $(p.b^{*})C^{\exists}(q.b^{*})$.

The elements of B are called **dynamic regions** and the relations $C^{\forall}$ and $C^{\exists}$ are called correspondingly **stable contact** and **unstable contact**. These names are motivated by the standard examples of DCA which will be given below.
Since DCA-s are algebraic systems, we adopt for them the standard definitions of subalgebra, homomorphism, isomorphism, isomorphic embedding, etc.

Outline

tu-logo

ur-logo

Definition

Products of contact algebras. Let $T \neq \varnothing$ be a set and let for each $t \in T$, $(\underline{B}_t, C_t)$ be a contact algebra and let $B(T) = \prod_{t \in T} B_t$ denote the product of Boolean parts of contact algebras. For $x, y \in B$ we define the stable and unstable contact relations as follows:

$$xC^{\forall}y \text{ iff } (\forall t \in T)(x_t C_t y_t),$$
$$xC^{\exists}y \text{ iff } (\exists t \in T)(x_t C_t y_t),$$

where x_t and y_t denote the $t - th$ coordinate of x and y.

Lemma

The algebra $(B(T), C^{\exists}, C^{\forall})$ is a DCA, called **product DCA** *(PDCA). The components (B_t, C_t) are called coordinate algebras of the product and T is the set of* **time moments**.

Definition

Let $(\underline{B}_t, C_t)$, $t \in T$ be a non-empty family of contact algebras and $(B(T), C^{\exists}, C^{\forall})$ be the PDCA defined by this family.

- If for any $t \in T$ the algebra $(\underline{B}_t, C_t)$ is a subalgebra of $RC(X_t)$ for some topological space X_t, then PDCA is called **topological DCA** and the spaces X_t – its coordinate spaces.
- If all contact algebras from the product are subalgebras of $RC(X)$ of a single topological space X, then PDCA is called **coherent DCA**.
- Subalgebras of product (topological, coherent) DCA-s are called **standard PDCA-s, (standard topological DCA-s, standard coherent DCA-s)**.

Dynamic mereotopology

Since standard models of DCA are of topological nature we may introduce the integral name DYNAMIC MEREOTOPOLOGY. Analogous to the equation

- CONTACT ALGEBRAS = MEREOTOPOLOGY
  we may state its dynamic version
- DYNAMIC CONTACT ALGEBRAS = DYNAMIC MEREOTOPOLOGY

Outline

tu-logo

ur-logo

Introducing abstract moments of time

Let $\underline{B} = (B, C^{\exists}, C^{\forall})$ be a fixed DCA and let $CLANS(\underline{B})$ be the set of clans of the contact algebra $(B, C^{\exists})$.

Definition

Time moments. A subset α of $CLANS(\underline{B})$ is called a **time moment** of $\underline{B}$ if it satisfies the following condition:

(tm) $$(\forall p, q \in B)(pC^{\forall}q \Longrightarrow (\exists \Gamma \in \alpha)(p \in \Gamma \text{ and } q \in \Gamma)).$$

or equivalently

$$(\forall p, q \in B)(pC^{\forall}q \Longrightarrow \alpha \cap h(p) \cap h(q) \neq \varnothing).$$

We denote by $T(\underline{B})$ the set of all time moments of $\underline{B}$.

Time characterization of $C^{\exists}$ and $C^{\forall}$

Lemma

The following is true for the set of time moments $T(\underline{B})$:

1. $T(\underline{B}) \neq \varnothing$,
2. *Each time moment is a non-empty set of clans.*

Lemma

The following holds for all $a, b \in B$:

1. $aC^{\exists}b$ iff $(\exists\alpha \in T(\underline{B}))(\alpha \cap h(a) \cap h(b) \neq \varnothing)$.
2. $aC^{\forall}b$ iff $(\forall\alpha \in T(\underline{B}))(\alpha \cap h(a) \cap h(b) \neq \varnothing)$.

Outline

tu-logo

ur-logo

Definition

Coordinate contact algebras. Let $\alpha \in T(\underline{B})$. Since α is a nonempty set of clans the ideal $I(\alpha) = \{a \in B : \alpha \cap h(a) = \varnothing$ is proper, so the Boolean algebra $B/I(\alpha)$ is non-degenerate.

- To each time moment $\alpha \in T(\underline{B})$ we associate the contact algebra $(\underline{B}_\alpha, C_\alpha)$ determined by the ideal $I(\alpha)$ and let $(B(T(\underline{B})), C^{\exists}, C^{\forall})$ be the product DCA determined by the non-empty family $(\underline{B}_\alpha, C_\alpha)$, $\alpha \in T(\underline{B})$.

Embedding mapping g:

- We define a mapping g from the DCA $(B, C^{\exists}, C^{\forall})$ into $B(T(\underline{B}))$ as follows: for each $a \in B$ we define $g(a)$ as an element in $B(T(\underline{B}))$ such that for each $\alpha \in T(\underline{B})$ we have $g(a)_\alpha = |a|_\alpha$, so, $|a|_\alpha$ are the coordinates of $g(a)$.

Outline

tu-logo

ur-logo

Lemma

The mapping g is an isomorphic embedding of $(B, C^{\exists}, C^{\forall})$ *into the PDCA* $(B(T(\underline{B})), C^{\exists}, C^{\forall})$.

Theorem

Representation theorem for DCA-s. *For each DCA* $(\underline{B}, C^{\exists}, C^{\forall})$ *the following conditions hold:*

1. $(\underline{B}, C^{\exists}, C^{\forall})$ *is isomorphic to a standard PDCA.*
2. $(\underline{B}, C^{\exists}, C^{\forall})$ *is isomorphic to a standard topological DCA, with semiregular and compact* T_0 *coordinate spaces.*
3. $(\underline{B}, C^{\exists}, C^{\forall})$ *is isomorphic to a standard coherent DCA such that all coordinate topological algebras are subalgebras of* $RC(X)$ *of one semiregular and compact* T_0 *space* X.

Outline

tu-logo

ur-logo

Dynamic Contact Logic (DCL)

We define Dynamic Contact Logic (DCL) as the quantifier-free version of the first-order theory of DCA. Since all axioms of DCA are universal formulas, DCL can be axiomatized by the axioms of DCA on the base of the Hylbert-style axiomatization of classical propositional logic plus the rule of Modus Ponens.

tu-logo

ur-logo

Theorem

Completeness Theorem for DCL. *The following conditions are equivalent for any formula A of DCL:*

1. *A is a theorem of DCL,*
2. *A is true in the class of all DCA-s,*
3. *A is true in the class of all standard topological DCA-s over semiregular and compact T_0 spaces,*
4. *A is true in the class of all standard coherent DCA-s over semiregular and compact T_0 spaces,*
5. *A is true in the class of all finite DCA-s with cardinality $\leq 2^{2^n}$, where n is the number of Boolean variables occurring in A.*

Concluding remarks: other versions of DCA

Dynamic representation of contact algebras

If we remove from the definition of dynamic contact algebra the relation $C^{\forall}$ we obtain a contact algebra. Then one may wonder if for contact algebra we can simulate a similar representation theory as for dynamic contact algebras. It can be shown that this is possible and in this way we obtained a new representation theorem for contact algebras. The new models of contact algebras are defined as follows.

Let $\underline{B}_t$, $t \in T$ be a nonempty family of non-degenerate Boolean algebras and let $B(T)$ be their cartesian product. Define a contact relation in $B(T)$ as follows: for $a, b \in B(T)$ put

- aCb iff $(\exists t \in T)(a_t \neq 0_t$ and $b_t \neq 0_t)$.

Contact as the Whiteheadean contemporaneity relation

Intuitively the above relation means that a and b exist simultaneously at some time t, i.e. a and b are **contemporaries** - one of the basic relations in Whiteheadean theory of space and time.

Every contact algebra can be represented in this way.

So this representation gives not the expected spatial meaning of the contact relation, but another, temporal meaning. So it is natural to call this relation (in this special representation) **time contact** and to denote it by C^t.

Dynamic bi-contact algebras with time and spatial contacts

If we want to have in the dynamic representation all coordinate algebras to be also contact algebras in which the corresponding contact to be considered in its spatial meaning (denoted by C^s), we have to consider bi-contact algebras with two contact relations - C^t and C^s interpreted in products of (ordinary) contact algebras as follows:

- **time contact** $aC^t b$ iff $(\exists t \in T)(a_t \neq 0_t$ and $b_t \neq 0_t)$
- **spatial contact** $aC^s b$ iff $(\exists t \in T)(a_t C_t b_t)$iff $aC^\exists b$.

In this model these two relations satisfy the following axiom:

- $aC^s b$ implies $aC^t b$,

which has very intuitive meaning: meeting somewhere in the space must be at some moment of time.

tu-logo

ur-logo

Contemporaneity as a property of a group of regions

In the above model one can generalize the temporal contact by defining the **contemporaneity relation** of a group as a property of the group as follows.
Let A be a group of regions in $B(T)$, then

- the members of A are contemporaries iff there exists a moment of time $t \in T$ such that for all $a \in A$, $a_t \neq 0_t$.

In Whitehead's terminology such groups are called **societies**. So the members of every society are contemporaries and exist in a certain epoch t.
All these considerations are strongly related to the Whiteheadean **epochal theory of time** and will be a subject of a subsequent paper.

tu-logo

ur-logo

Open problems

In this paper we introduced a point-free system - dynamic contact algebra, in which one can extract in a canonical way a spatio-temporal system containing time moments and space points. Dynamic contact algebras contain two spatio-temporal relations between dynamic regions - stable and unstable contact. **However, the obtained set of moments of time is too weak - it does not have any specific internal structure, for instance it does not have the ordering relation between time moments**. The reason is that the two spatio-temporal relations $C^{\exists}$ and $C^{\forall}$ do not need for their meaning such a structure. So, we formulate as an open problem to find suitable spatio-temporal relations between dynamic regions whose meaning depend on the order of time and to give for them point-free abstract characterization including the time order.

BIBLIOGRAPHY

beamericonarticle G. Dimov, D. Vakarelov: Contact Algebras and Region-based Theory of Space: A Proximity Approach - I+II. Fundam. Inform. 74(2-3): 209-249, 251-282 (2006)

beamericonarticle D. Vakarelov, *Region-Based Theory of Space: Algebras of Regions, Representation Theory and Logics.* In: Dov Gabbay et al. (Eds.) Mathematical Problems from Applied Logics. New Logics for the XXIst Century. II. Springer, 2007, 267-348.

beamericonarticle A. N. Whitehead, *The Organization of Though*, London, William and Norgate, 1917

beamericonarticle A. N. Whitehead, *Process and Reality*, New York, MacMillan, 1929.

tu-logo

ur-logo

E N D

THANK YOU

tu-logo

ur-logo