Partiality and non-determinism

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Motivation: π-calculus


Robin Milner

$$\bar{a} \mid a \quad \longrightarrow \quad \mathbf{0}$$

Motivation: π-calculus

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$$\bar{a} \mid b \quad \not\longrightarrow$$

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$$\bar{a} \mid b \quad \not\longrightarrow$$

substitute a for b

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Motivation: π-calculus

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$$\bar{a} \mid b$$

substitute a for b

Two problems.

1. Classify substitution-frames as coalgebras.
2. Find an expressive modal logic for substitution frames.

$$\bar{a} \mid a \longrightarrow \mathbf{0}$$

Next-state function

$$\bar{a} \mid a \longmapsto \{\mathbf{0}\}$$

Next-state function

$$\bar{a} \mid b \longmapsto \emptyset$$

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substitute a for b $\qquad\qquad \nparallel$

$$\bar{a} \mid a \longmapsto \{\mathbf{0}\}$$

Next-state function

$$\bar{a} \mid b \longmapsto \{\text{when } a = b \text{ then } \mathbf{0}\}$$

substitute a for b

$$\{\text{when } a = a \text{ then } \mathbf{0}\}$$

$$=$$

$$\bar{a} \mid a \longmapsto \{\mathbf{0}\}$$

1. Partial map classifiers

2. Relation classifiers

3. Modal logic

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2. Relation classifiers
3. Modal logic

- A partial function $X \rightharpoonup Y$ is
  - a subset $D \subseteq X$
  - a total function $D \to Y$

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X & & Y
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\end{tikzcd}
$$

Admissible monos $\mathcal{M}$
- contain all isomorphisms;
- closed under composition;
- and stable under pullback.

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Partial maps into Y
correspond to
(Total) maps into $\mathrm{p}\,Y$

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NEXT we will characterize $\mathrm{p}\,Y$ as a free algebra.

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NEXT we will characterize $\mathrm{p}\,Y$ as a free algebra.

c.f. M. Jibladze. A presentation of the initial lift algebra.
A. Kock. Algebras for the partial map classifier monad.

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Enriched algebraic theories

Max Kelly & John Power

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Operations:

Classically, a signature consists of

- A collection of operations
- For each operation, an arity.

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A signature enriched in a cartesian closed category $\mathcal{C}$ consists of

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An algebra consists of

- An object A of $\mathcal{C}$ *(carrier)*
- For each op^n o, a morphism $coar(o) \times A^{ar(o)} \to A$

Enriched algebraic theory for partial maps

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  an operation when_m
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$x+(x+(x+(x+y)))$

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  - lhs, rhs $: Y \to \mathrm{T}(X)$ $\quad (x+y),(y+x) : 1 \to \mathrm{T}(\{x,y\})$

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- Build equations from generic effects using metalanguage

$$\vdash \mathsf{let}\ \mathrm{x} \Leftarrow \mathsf{choose}()\ \mathsf{in}\ \mathsf{not}(\mathrm{x}) \approx \mathsf{choose}() \quad : \{\mathsf{tt}, \mathsf{ff}\}$$

Enriched algebraic theories

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$$\frac{\Gamma \vdash \mathrm{t} : Y \quad \Gamma, \mathrm{y} : Y \vdash \mathrm{u} : Z}{\Gamma \vdash \mathsf{let}\ \mathrm{y} : Y \Leftarrow \mathrm{t}\ \mathsf{in}\ \mathrm{u} : Z}$$

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$x+y$

- Equatic

$(x+(x+y)))$

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- The generic effect is
  $$\mathsf{assert}_m \colon X \to T(D)$$

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THEOREM. If there is a partial map classifier then $\mathrm{p}X$ is the free algebra over X.

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THEOREM. If $\mathcal{G}$ is dense in $\mathcal{C}$, then it is sufficient to consider the operations arising from monos $D \rightarrowtail X \in \mathcal{G}$

$$\begin{tikzcd} {} \arrow[r, "f'"] \arrow[d, "m'"'] & X \arrow[d, tail, "m"] \\ Y' \arrow[r, "f"'] & Y \end{tikzcd}$$

Example in posets

two admissible monos

$$m\colon 0 \rightarrowtail 1$$

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THEOREM. For any poset X,
the free algebra for this theory is $X_\perp$

Substitution actions

- In the category of substitution actions,

$$\mathbb{A}$$

$$\mathbb{A} \times \mathbb{A}$$

$$\mathbb{A} \times \mathbb{A} \times \mathbb{A}$$

$$\dots$$

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$$\left.\begin{array}{l}\mathbb{A} \\ \mathbb{A}\times\mathbb{A} \\ \mathbb{A}\times\mathbb{A}\times\mathbb{A} \\ \ldots\end{array}\right\}\text{ these are dense}$$

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- Hence our monos are predicates over names.

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1. Partial map classifiers
2. Relation classifiers
3. Modal logic

- We want:

> THEOREM:
>
> Image-finite relations $(\to) \subseteq X \times X$
>
> *correspond to*
>
> Morphisms $X \to T(X)$.

where $T(X)$ is the free algebra for some theory.

- To obtain this theory we combine the theory of semilattices with the theory of partial maps.

Combined theory

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<u>*Operations:*</u>

$+ \colon A^2 \to A$

$\mathsf{when}_m \colon X \times A^D \longrightarrow A$

for $m \colon D \rightarrowtail X$

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Operations:

$+ : A^2 \to A$ $\qquad$ $\mathsf{when}_m : X \times A^D \to A$

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Equations:

$$x + x = x$$
$$x + y = y + x$$
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$$x : X \vdash \mathsf{let}\ b = \mathsf{choose}()\ \mathsf{in}\ \mathsf{let}\ d = \mathsf{assert}_m(x)\ \mathsf{in}\ (b, d)$$
$$\approx \mathsf{let}\ d = \mathsf{assert}_m(x)\ \mathsf{in}\ \mathsf{let}\ b = \mathsf{choose}()\ \mathsf{in}\ (b, d)$$

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$$\begin{aligned} x \colon X \vdash \ & \big(\mathsf{if}\ \mathsf{choose}()\ \mathsf{then}\ m(\mathsf{assert}_m x)\ \mathsf{else}\ n(\mathsf{assert}_n x)\big) \\ \approx \ & (m \vee n)(\mathsf{assert}_{m \vee n} x) \end{aligned}$$

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$$+ : A^2 \longrightarrow A \qquad\qquad \mathsf{when}_m : X \times A^D \longrightarrow A$$

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$: T(X)$

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THEOREM:

In a topos, free algebras exist and classify a class of relations.

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$x : X \vdash \text{let } b =$

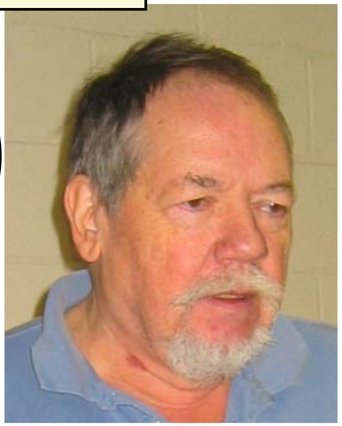
$\approx \quad \text{let } d =$

The free algebra X is $\tilde{K}(X)$.

c.f. P. Freyd, Numerology in topoi

$$x : X \vdash \left(\text{if choose() then } m(\text{assert}_m x) \text{ else } n(\text{assert}_n x)\right)$$

$$\approx \quad (m \vee n)(\text{assert}_{m \vee n} x)$$



Semi-lattices in substitution actions

$$+ \colon X \times X \to X \qquad \mathbf{0} \colon 1 \to X$$

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$$\begin{gathered} [a = b]\,[a = b]\,x = [a = b]\,x \\ [a = b]\,[c = d]\,x = [c = d]\,[a = b]\,x \\ [a = b]\,[b = c]\,x = [a = b]\,[a = c]\,x \\ [a = a]\,x = x \\ [a = b]\,x = [b = a]\,x \end{gathered}$$

Semi-lattices in substitution actions

$$+ : X \times X \to X \qquad \mathbf{0} : 1 \to X$$

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$$\begin{aligned} [a = b]\,[a = b]\,x &= [a = b]\,x \\ [a = b]\,[c = d]\,x &= [c = d]\,[a = b]\,x \\ [a = b]\,[b = c]\,x &= [a = b]\,[a = c]\,x \\ [a = a]\,x &= x \\ [a = b]\,x &= [b = a]\,x \end{aligned}$$

$$\begin{aligned} [a = b]\,\mathbf{0} &= \mathbf{0} \\ [a = b]\,(x + y) &= ([a = b]\,x) + ([a = b]\,y) \\ [a = b]\,x + x &= x \\ [a = b]\,x &= [a = b]\,(\{{}^{a}/_{b}\}x) \end{aligned}$$

Semi-lattices in substitution actions

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$$\text{when} : \mathbb{A} \times \mathbb{A} \times X \to X$$

THEOREM:

Substitution-frames

correspond bijectively with

coalgebras for the free substitution semilattice

$$= b]\, x$$

$$= d]\,[a = b]\, x$$

$$= b]\,[a = c]\, x$$

$$= a]\, x$$

$$[a = b]\,(x + y) = ([a = b]\, x) + ([a = b]\, y)$$

$$[a = b]\, x + x = x$$

$$[a = b]\, x = [a = b]\,(\{^a/_b\}x)$$

Semi-lattices in substitution actions

$$+ : X \times X \to X \qquad \mathbf{0} : 1 \to X$$

$$\text{when} : \mathbb{A} \times \mathbb{A} \times X \to X$$

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THEOREM:

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coalgebras for the free substitution semilattice

e.g. $\bar{a} \mid b \longmapsto \{\text{when } a = b \text{ then } \mathbf{0}\}$

$\bar{a} \mid a \longmapsto \{\mathbf{0}\}$ = $\{\text{when } a = a \text{ then } \mathbf{0}\}$

Semi-lattices in substitution actions

$$+ : X \times X \to X \qquad \mathbf{0} : 1 \to X$$

$$\text{when} : \mathbb{A} \times \mathbb{A} \times X \to X$$

THEOREM:

Substitution-frames

correspond bijectively with

coalgebras for the free substitution semilattice

$$= b]\, x$$

$$= d]\, [a = b]\, x$$

$$x$$

$$[a = b]\, (x + y) = ([a = b]\, x) + ([a = b]\, y)$$

$$[a = b]\, x + x = x$$

$$[a = b]\, x = [a = b]\, (\{{}^{a}/_{b}\}x)$$

c.f. Davide Sangiorgi
A theory of bisimulation for the π-calculus

1. Partial map classifiers
2. Relation classifiers
3. Modal logic

- We want a modal logic for substitution frames such that

THEOREM.
If $p \Vdash \phi$ then $(f \cdot p \Vdash f \cdot \phi)$.

Two states satisfy the same formulae

iff

they are related by a substitution-closed bisimulation

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- *Aside: we do already have the following:*

THEOREM (Bartek Klin, 2008).
Every finitary functor on a strongly locally finitely presentable category admits an expressive logic.

A modal logic

$$\phi \ ::= \ \mathrm{T} \mid \mathrm{F} \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Diamond\phi \mid \boxed{(a = b)} \mid \phi \supset \phi \mid \Box\phi$$

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$p \Vdash \mathrm{T}$	always
$p \Vdash \mathrm{F}$	never
$p \Vdash \phi \wedge \psi$	iff$(\, p \Vdash \phi)$ and $(p \Vdash \psi)$
$p \Vdash \phi \vee \psi$	iff$(\, p \Vdash \phi)$ or $(p \Vdash \psi)$
$p \Vdash \Diamond\phi$	iff $\exists q.\ p \longrightarrow q$ and $(q \Vdash \phi)$

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