Stone duality above dimension zero

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  - *compact* if every family of open sets covering X contains a finite subset that covers X;
  - *Hausdorff* if every two distinct points of X are contained in disjoint open sets; and
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- Then $\mu(B)$ is a compact Hausdorff space, known as the *maximal spectrum* of B; the topology given by (*) is known as the *Zariski* or as the *hull-kernel* topology.
- It is zero-dimensional because $\{\mathbb{V}(\mathfrak{p})\}$, as $\mathfrak{p}$ ranges over the *principal* ideals of B, can be shown to be a basis of clopen sets for $\mu(B)$.

The adjoint functors: the $\mathrm{C}(X)$ functor

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$$(f \vee g)(x) = f(x) \vee g(x) \quad \text{for all } x \in X.$$

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- Then $\mathrm{C}(X)$ is a Boolean algebra, and $\mu(\mathrm{C}(X)) \cong X$.
- Conversely, if B is any Boolean algebra, $B \cong \mathrm{C}(\mu(B))$.

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- Most work originating in logic is motivated by the specular question: if you generalise Boolean algebras in a specified manner, what do you need to do on the topological side to regain a duality?
- Cf. e.g. Stone or Priestley duality for distributive lattices, Esakia duality for Heyting algebras, Jónnson-Gehrke-Priestley duality for certain expansions of distributive lattices (*double quasioperator algebras*), etc.

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We will presently see an instance of the latter statement for **KHaus**.

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This structure can be lifted to $\mathrm{C}(X)$ by defining operations pointwise:

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- fg is given by $(fg)(x) = f(x)g(x)$ for all $X \in X$.
- 1 is given by $1(x) = 1$ for all $x \in X$.
- Etc.

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- Points of X are in one-one correspondence with the *maximal ideals* of the ring $\mathrm{C}(X)$.
- A basis of *closed* sets for the topology of X is given by the *vanishing loci* (or *zero sets*) of ideals of functions:

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- So X can be recovered from the abstract ring $\mathrm{C}(X)$.

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- *Note*: By the term *Gelfand duality* one usually refers to the complex-valued case; indeed, the '$*$' in 'C^* algebras' refers to the the action of complex conjugation on $\mathbb{C}$. The real-valued case works beautifully too, although it is somewhat less well known among analysts.

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- λf is given by $(\lambda f)(x) = \lambda f(x)$ for all $X \in X$ and all $\lambda \in \mathbb{R}$.
- $||f||_\infty = \sup_{x \in X} ||f(x)||$ for all $x \in X$.
- Etc.

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- *Note*: Kakutani obtained a representation theorem for objects; morphisms were explicitly dualised by Banaschewski (1976).

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- $v \leqslant w \Rightarrow v + t \leqslant w + t$ for all $v, w, t \in V$. (This is known as *translation invariance*.)
- $\lambda v \geqslant 0$ whenever $\lambda \in \mathbb{R}$ satisfies $\lambda \geqslant 0$, and $v \geqslant 0$.

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So **UVectLat** is a subcategory of **VectLat**, but it is not full. (Also, not all vector lattices admit a unit, so the inclusion functor is not surjective on objects here.)

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$$\lambda(v \vee w) = \lambda(v) \vee \lambda(w)\,,$$

whenever $\lambda \geqslant 0$.
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Consider the *unit interval* of V given by

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(This is not totally ordered, of course, despite the name 'interval'.)

One can obtain an “addition” on $[0, u]$ by truncation:

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Notation: The category of Riesz MV-algebras is denoted **RieszMV**. Morphisms are homomorphisms.

Aside on MV-algebras

Introduced by C.C. Chang in 1959 as the equivalent algebraic semantics of Łukasiewicz infinite-valued propositional logic.

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Riesz MV-algebras also form a variety, as mentioned, albeit with continuum-many additional unary operations for scalar multiplication.
In this talk, Riesz MV-algebras are an important technical tool, but I am not concerned with the theory of MV-algebras *per se*.

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Why do we care about this? Where are compact Hausdorff spaces? Where is Kakutani duality gone?

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This gives a map $\|\cdot\|\colon V \to \mathbb{R}_{\geqslant 0}$. Is it actually a norm on the vector space V?
No: it may happen that $\|v\| = 0$ but $v \neq 0$ – i.e. there may exist non-zero vectors of zero length; this happens if v is an "infinitesimal vector", in an appropriate sense.

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Call a unital vector lattice (V, u) *norm-complete* if (i) its unit norm is a norm, and (ii) V is complete in that norm.
Write **UCoVectLat** for the category of unital, norm-complete vector lattices. Morphisms are just the unital vector lattice homomorphisms: we do not ask that they preserve $||\cdot||$. (It turns out that homomorphisms are automatically contractions w.r.t. $||\cdot||$.)

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By the equivalence of unital vector lattices and Riesz MV-algebras, there is a full subcategory of **RieszMV** corresponding to **UCoVectLat**; let us call it **CoRieszMV** for *norm-complete Riesz MV-algebras.*

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The category $\mathbf{KHaus}^{\mathrm{op}}$ *is equivalent both to* **UCoVectLat** *and to* **CoRieszMV**.

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Theorem (I. Leustean & V. Marra, 2009)

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Let me describe the functors that implement this duality; then I will explain why I call it 'almost equational'.

The adjoint functors: the $\mathrm{C}(X)$ functor

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- *This makes $\mathrm{C}(X)$ into an object of* **UCoVectLat**.

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- *This makes* $\mathrm{C}\colon \mathbf{KHaus}^{\mathrm{op}} \to \mathbf{UCoVectLat}$ *into a functor.*

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- Topologise $\mu(V)$ using again the Zariski/hull-kernel topology: closed sets are precisely those of the form

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  as I ranges over all ideals of V.
- But (*) is not a zero-dimensional topology: $\{\mathbb{V}(\mathfrak{p})\}$, as $\mathfrak{p}$ ranges over the principal ideals of V, is indeed a basis of closed sets for $\mu(V)$ — but these need not be open.

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- In fact, more is true: **CoRieszMV** *is a reflective subcategory of the variety* **RieszMV** *of Riesz MV-algebras.*
- This means that the inclusion functor $\mathbf{CoRieszMV} \hookrightarrow \mathbf{RieszMV}$ has a left adjoint $\mathscr{C}$.

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- Conclusion: Although $\mathbf{UCoVectLat} \cong \mathbf{KHaus}^{\mathrm{op}}$ is not a variety, it is *almost* that – it is the *reflection* of a variety.
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- Composing the appropriate functors, we see that *there exist free objects in the category* **UCoVectLat** *of norm-complete unital vector lattices.*
- In fact, even more interestingly, there exist *finitely presentable* objects – but I will not discuss them.

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This is an 'almost equational' version of Kakutani duality in that $\mathbf{KHaus}^{\mathrm{op}}$ *is proved to be the reflection of a variety – namely,* $\mathbf{RieszMV}$ *– to within equivalences.*

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So we get a functor $\mathscr{A} \colon \mathbf{KHaus} \to \mathbf{CoAtBA}$.

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Any continuous function $f\colon X \to Y$ (for Y compact Hausdorff) induces a complete homomorphism of c.a. Boolean algebras $\mathscr{A}(f)\colon \mathscr{A}(X) \to \mathscr{A}(Y)$ as follows:

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$$A' = \{ g \in \mathrm{C}(Y) \mid g \circ f \in A \}.$$

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Theorem (V. Marra, 2010)

A compact Hausdorff space X is a κ-ball if and only if it is freely co-generated by a complete atomic Boolean algebra with κ atoms.

Two right adjoints (sketch)

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- Now $\mathbf{DL}^{\mathrm{op}}$ is of course one of the categories whose equivalent representations have been most intensively studied.
- There are at least three ways to represent $\mathbf{DL}^{\mathrm{op}}$: via coherent spaces (Stone), via ordered spaces (Priestley), and – possibly making for an optimal representation – via bitopological spaces (G. Bezhanishvili, N. Bezhanishvili, D. Gabelaia, and A. Kurz).

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- First, recall that a finite Priestley space is essentially the same thing as a finite poset.
- So $\mathscr{O}$ builds a compact Hausdorff space out of a poset.

The nerve (or order complex) of a poset

If S is a finite poset, let $\Delta(S)$ be the collection of chains of S (=totally ordered subsets). Then $\Delta(S)$ is an (abstract) simplicial complex on the vertex set S, called the *nerve of* S.

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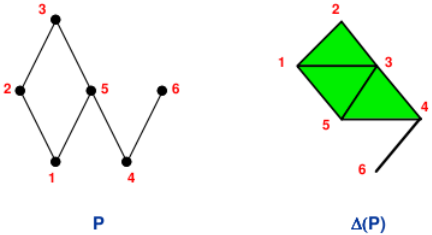

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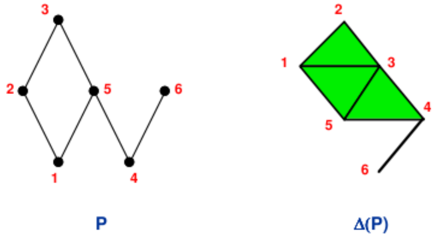

(To say that $\Delta(S)$ is a *simplicial complex* with vertex set S means that it is a family of subsets of S closed under taking subsets, and containing all singleton subsets of S.)

(Compact) polyhedra

A *polyhedron* is a subspace of $\mathbb{R}^n$ that can be triangulated, i.e. written as the underlying space of a (geometric) simplicial complex. (Thus, following the traditional terminology of PL-topology, a polyhedron is not convex.)

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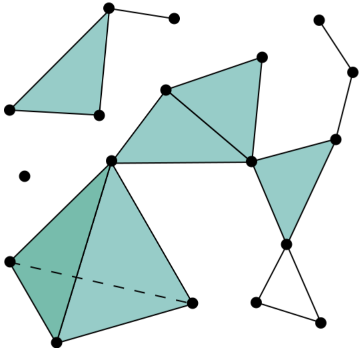

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It would be interesting to generalise the nerve construction $\Delta(S)$ to arbitrary Priestley spaces S so as to get an explicit description of the functor $\mathscr{O}$ in all cases.

Epilogue: The main obstruction

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However, several problems remain in this theory. They are, to say the least, rather stubborn.
In closing, I want to hint to at least one of them – one that I regard as a central obstacle to further developments.

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Understanding the properties of the embedding (†), I suggest, is the key to unlocking the deeper secrets of **KHaus**.

Thanks

Thank you for your attention.