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GENERALIZED COMMUTATORS OF MULTILINEAR SINGULAR INTEGRALS

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Abstract. The boundedness of generalized commutators of multilinear singular integrals in Lebesgue spaces is obtained.

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1. Introduction

In recently years, multilinear singular integrals of Calderón-Zygmund type have attracted more attention. Many results obtained parallel the linear theory of classical Calderón-Zygmund operators. For details, one can see [2], [3], [4], [7] and [9]. Simultaneously, commutators of singular integral operators continue to attract many authors' attention. See [1], [5], [6], [8], [10], [11] and the references therein. In fact, the multilinear commutator as a generalization of commutator was introduced by Perez and Trujillo-Gonzalez in [10] recently. In [11] Perez and Trujillo-Gonzalez obtained sharp weighted estimates for vector valued singular integral operators and commutators. Motivated by [9], we will consider the analog of [10] for the multilinear singular integrals.

Let T be a multilinear operator which was initially defined on the m-fold product of Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and take their values into the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. We will assume that the distributional kernel on $(\mathbb{R}^n)^{m+1}$ of the operators coincides away from the diagonal $y_0 =$

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$y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$ with a function K so that

$$T\vec{f}(x) = T(f_1, \dots, f_m)(x) = \\ = \int\limits_{(\mathbb{R}^n)^m} K(x, y_1, \cdots, y_m) f_1(y_1) \dots f_m(y_m) dy_1 \dots dy_m, \qquad (1.1)$$

whenever $f_1, \cdots, f_m$ are $L^\infty_C(\mathbb{R}^n)$ and $x \notin \bigcap_{j=1}^m supp\ f_j$, where as usual, $L^\infty_C(\mathbb{R}^n)$ denotes all $L^\infty(\mathbb{R}^n)$ functions with compact support. Moreover, the kernel function K satisfies the standard estimates

$$\big|K(y_0, y_1, \cdots, y_m)\big| \le A\bigg(\sum_{k,l=0}^{m} |y_k - y_l|\bigg)^{-mn}, \qquad (1.2)$$

and, for some $\epsilon > 0$,

$$\big|K(y_0, y_1, \cdots, y_j, \cdots, y_m) - K(y_0, y_1, \dots, y_j', \dots, y_m)\big| \le \\ \le \frac{A|y_j - y_j'|^\epsilon}{\Big(\sum\limits_{k,l=0}^{m} |y_k - y_l|\Big)^{mn+\epsilon}}, \qquad (1.3)$$

provided that $0 \le j \le m$ and $|y_j - y_j'| \le \frac{1}{2} \max_{0 \le k \le m} |y_j - y_k|$. Such kernels are called m-linear Calderón-zygmund kernels and the collection of such functions is denoted by $m - CZK(A, \epsilon)$ in [2].

For these operators, it is well known that a boundedness estimate $T : L^{q_1} \times \cdots \times L^{q_m} \to L^q$, for some $1 < q_1, \cdots, q_m < \infty$ and

$$\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}, \qquad (1.4)$$

implies the boundedness of the operator for all possible exponents in such range of values and a end-point estimate

$$T : \ L^{q_1} \times \cdots \times L^{q_m} \to L^{q,\infty}$$

for $1 \le q_1, \cdots, q_m < \infty$ satisfying (1.4), where L^p and $L^{p,\infty}$ denote Lebesgue spaces and weak Lebesgue spaces for $0 < p < \infty$ respectively. In particular, it holds that

$$T : \ L^1 \times \cdots \times L^1 \to L^{1/m,\infty},$$

which extends the classical results in the linear case $T : \ L^1 \to L^{1,\infty}$.

Let T be as in (1.2) with a $m - CZK(A, \epsilon)$ kernel. If T is bounded from $L^{q_1} \times \cdots \times L^{q_m}$ to $L^q(\mu)$ with $1 < q_1, \dots, q_m < \infty$ and $\frac{1}{q_1} + \cdots + \frac{1}{q_m} = \frac{1}{q}$, then we say T is a m-CZO.

Let $\vec{b} = (b_1, \dots, b_m)$ and b_j are BMO functions for $1 \le j \le m$, and $\vec{f} = (f_1, \dots, f_m)$ for suitable functions $f_1, \dots, f_m$. We will consider the following type operators

$$[\vec{b},T]\,\vec{f}(x)=\sum_{\sigma}(-1)^{|\sigma'|}b_\sigma(x)Tb_{\sigma'}\vec{f}(x),$$

where σ denotes a subset of $\{1,2,\dots,m\}$, σ' is the complement of σ in $\{1,2,\dots,m\}$, $|\sigma'|$ denotes the number of elements of σ', $b_\sigma(x)=\prod_{j\in\sigma}b_j(x)$, $b_{\sigma'}\vec{f}=(g_1,\dots,g_m)$, when $j\in\sigma'$, $g_j=b_jf_j$, otherwise $g_j=f_j$.

If $f_1,\cdots,f_m$ are $L^\infty_C(\mathbb{R}^n)$ and $x\notin\bigcap_{j=1}^m supp\ f_j$, then we have

$$[\vec{b},T]\,\vec{f}(x)=\int\limits_{\mathbb{R}^{nm}}\prod_{j=1}^m\big(b_j(x)-b(y_j)\big)f_j(y_j)K(x,y_1,\cdots,y_m)\,dy_1\cdots dy_m.$$

These operators can be thought as generalized commutators for multilinear operators. In this paper, for simplicity, we will restrict the case $m=2$. Actually, we will consider

$$\begin{aligned}[b_1,b_2,T](f_1,f_2)(x)&=b_1(x)b_2(x)T(f_1,f_2)(x)-b_1(x)T(f_1,b_2f_2)(x)-\\&-b_2(x)T(b_1f_1,f_2)(x)+T(b_1f_1,b_2f_2)(x).\end{aligned}$$

Our main result is the following.

Theorem 1.1. *Let $\frac{1}{q}=\frac{1}{q_1}+\frac{1}{q_2}$, and $1<q_i<\infty$, and $b_i\in BMO(\mathbb{R}^n)$, for $i=1,2$. If T is a 2-CZO, then $[b_1,b_2,T]$ can be extended to a bounded operator from $L^{q_1}\times L^{q_2}$ into L^q.*

One can easily see that our special case can be extended to general cases. The proof of Theorem 1.1 will be given in the following section. Finally, we point out that in the sequel C denotes constants, but it may be different at different places.

2. Proof of the result

The idea of the proof will use a variant of sharp maximal operator $M^\#$ of Fefferman and Stein [12]. This idea have already been exploited in [8], [9], [10] and [11]. First we recall some notation.

Let f be a function in $L^1_{loc}(\mathbb{R}^n)$, the sharp maximal function of f is defined by

$$M^\#f(x)=\sup_Q\frac{1}{|Q|}\int\limits_Q|f(y)-f_Q|\,dy,$$

where the supremum is taking over all the cubes Q containing the point x, and as usual, f_Q denotes the average of f over Q, and $|Q|$ denotes the Lebesgue measure of Q. If $M^\#f$ belongs to L^∞, then we say f is a BMO function and denote $\|M^\#f\|_{L^\infty}$ as BMO norm of f by $\|f\|_*$. For $\delta>0$ we denote by $M^\#_\delta$ the operator

$$M^\#_\delta f=\big(M^\#(|f|^\delta)\big)^{1/\delta}. \tag{2.1}$$

Also as usual, we denote the standard Hardy-Littlewood maximal function by Mf. And similarly as above, for $\delta > 0$ we will denote by M_δ the operator $M_\delta(f) = (M(|f|^\delta))^{1/\delta}$.

As the same as in [11], we will use the Fefferman-Stein's inequalities as the following.

Let $0 < p,\ \delta < \infty$ and let ω be any A_∞ weight. Then there exists a constant C such that the inequality

$$\int M_\delta(f)(x)^p \omega(x)\,dx \leq C \int M_\delta^{\#}(f)(x)^p \omega(x)\,dx \tag{2.2}$$

holds for any function f for which the left hand side is finite.

Also, we need the following lemmas, Lemma 2.1 is well known.

Lemma 2.1. (i) *Assume $b \in BMO$ and $M > 1$. Then for every ball $B(x;r)$, we have*

$$|b_B - b_{MB}| \leq C\,\|b\|_* \log M;$$

(ii) (*John-Nirenberg Lemma*) *Let $1 \leq p < \infty$ and $B \subset \mathbb{R}^n$, then $b \in BMO$ if and only if*

$$\frac{1}{\mu(B)} \int\limits_B |b - b_B|^p\, d(x) \leq \|b\|_*^p.$$

Lemma 2.2. *Suppose b_1 and b_2 belong to L^∞. Let $[b_1, b_2, T]$ be as in (1.6) and let $0 < \delta < \tau < 1/2$ and $1 < p_1 < q_1$, $1 < p_2 < q_2$. Then there exists a constant $C > 0$, such that*

$$\begin{aligned} M_\delta^{\#}[b_1, b_2, T](f_1, f_2)(x) &\leq C\big[\|b_1\|_*\|b_2\|_* M_\tau(T(f_1, f_2))(x) + \\ &+ \|b_1\|_* M_\tau([b_2, T](f_1, f_2))(x) + \|b_2\|_* M_\tau([b_1, T](f_1, f_2))(x) + \\ &+ \|b_1\|_*\|b_2\|_* M_{p_1} f_1(x) M_{p_2} f_2(x)\big], \end{aligned} \tag{2.3}$$

for any functions f_1, f_2 belong to $L_c^\infty(\mathbb{R}^n)$ and for every $x \in \mathbb{R}^n$, where $[b_1, T](f_1, f_2))(x) = b_1(x)T(f_1, f_2)(x) - T(b_1 f_1, f_2)(x)$ and $[b_2, T](f_1, f_2))(x)$ $= b_2(x)T(f_1, f_2)(x) - T(f_1, b_2 f_2)(x)$, where the constant C independent of f_1 and f_2.

Remark 1. For the m-linear case, the result of Lemma 2.2 is still true if $0 < \delta < \tau < 1/m$.

Proof. For fixed $x \in \mathbb{R}^n$ and a cube Q containing x. As it is well known, to obtain (2.3) it suffices to prove

$$\begin{aligned} &\left(\frac{1}{|Q|} \int\limits_Q \big||[b_1, b_2, T](f_1, f_2)(z)|^\delta - |c_Q|^\delta\big| dz\right)^{1/\delta} \leq \\ &\leq C\big[\|b_1\|_*\|b_2\|_* M_\tau(T(f_1, f_2))(x) + \|b_1\|_* M_\tau([b_2, T](f_1, f_2))(x) + \\ &+ \|b_2\|_* M_\tau([b_1, T](f_1, f_2))(x) + \|b_1\|_*\|b_2\|_* M_{p_1} f_1(x) M_{p_2} f_2(x)\big], \end{aligned} \tag{2.4}$$

for some constant c_Q to be determined. Using $||\alpha|^r - |\beta|^r| \leq |\alpha - \beta|^r,\ o < r < 1$, we only need to prove that

$$\left(\frac{1}{|Q|}\int_Q \left|[b_1,b_2,T](f_1,f_2)(z) - c_Q\right|^\delta dz\right)^{1/\delta} \leq$$
$$\leq C\big[\|b_1\|_*\|b_2\|_* M_\tau(T(f_1,f_2))(x) + \|b_1\|_* M_\tau([b_2,T](f_1,f_2))(x) +$$
$$+ \|b_2\|_* M_\tau([b_1,T](f_1,f_2))(x)\|b_1\|_*\|b_2\|_* M_{p_1} f_1(x) M_{p_2} f_2(x)\big]. \tag{2.5}$$

Let Q^* be the cube with same center as Q and side-length $2\sqrt{n}$ times the side-length of Q. Split $f_j = f_j^0 + f_j^\infty$, for $j = 1, 2$, where $f_j^0 = f_j\chi_{2Q}$, $f_j^\infty = f_j - f_j^0$. Put $c_Q = T(f_1^\infty, f_2^\infty)(x)$. Set $\lambda_i = (b_i)_{2Q}$, $i = 1, \cdots, m$.

$$\left|[b_1,b_2,T](f_1,f_2)(z) - c_Q\right| = \Bigg|\int_{\mathbb{R}^{2n}} (b_1(z) - b_1(y_1))(b_2(z) -$$
$$- b_2(y_2))K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 - c_Q\Bigg| =$$
$$= \Bigg|\int_{\mathbb{R}^{2n}} (b_1(z) - \lambda_1 + \lambda_1 - b_1(y_1))(b_2(z) - \lambda_2 + \lambda_2 - b_2(y_2))\times$$
$$\times K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 - c_Q\Bigg| =$$
$$= \Bigg|(b_1(z) - \lambda_1)(b_2(z) - \lambda_2)T(f_1,f_2)(z) -$$
$$- (b_1(z) - \lambda_1)\int_{\mathbb{R}^{2n}} (b_2(y_2) - \lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 -$$
$$- (b_2(z) - \lambda_2)\int_{\mathbb{R}^{2n}} (b_1(y_1) - \lambda_1)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 +$$
$$+ \int_{\mathbb{R}^{2n}} (b_1(y_1) - \lambda_1)(b_2(y_2 - \lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 - c_Q\Bigg| =$$
$$= \Bigg|(b_1(z) - \lambda_1)(b_2(z) - \lambda_2)T(f_1,f_2)(z) - (b_1(z) - \lambda_1)\times$$
$$\times \int_{\mathbb{R}^{2n}} (b_2(y_2) - b_2(z) + b_2(z) - \lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 -$$
$$- (b_2(z) - \lambda_2)\times$$
$$\times \int_{\mathbb{R}^{2n}} (b_1(y_1) - b_1(z) + b_1(z) - \lambda_1)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2 +$$

$$+\int\limits_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)(b_2(y_2-\lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2-c_Q\Big|\le$$

$$\le\big|-(b_1(z)-\lambda_1)(b_2(z)-\lambda_2)T(f_1,f_2)(z)\big|+$$

$$+\Big|(b_1(z)-\lambda_1)\int\limits_{\mathbb{R}^{2n}}(b_2(z)-b_2(y_2))K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2\Big|+$$

$$+\Big|(b_2(z)-\lambda_2)\int\limits_{\mathbb{R}^{2n}}(b_1(z)-b_1(y_1))K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2\Big|+$$

$$+\Big|\int\limits_{\mathbb{R}^{2n}}(b_1(y_1)-\lambda_1)(b_2(y_2-\lambda_2)K(z,y_1,y_2)f_1(y_1)f_2(y_2)dy_1dy_2-c_Q\Big|=$$

$$=I(z)+II(z)+III(z)+IV(z),$$

$$\left(\frac{1}{|Q|}\int\limits_Q|[b_1,b_2,T](f_1,f_2)(z)-c_Q|^\delta dz\right)^{1/\delta}\le C\left(\frac{1}{|Q|}\int\limits_Q I(z)^\delta dz\right)^{1/\delta}+$$

$$+C\left(\frac{1}{|Q|}\int\limits_Q II(z)^\delta dz\right)^{1/\delta}+C\left(\frac{1}{|Q|}\int\limits_Q III(z)^\delta dz\right)^{1/\delta}+$$

$$+C\left(\frac{1}{|Q|}\int\limits_Q IV(z)^\delta dz\right)^{1/\delta}=C(I+II+III+IV). \tag{2.6}$$

Now we start to estimate first term I. By Hölder's inequality and Lemma 2.1, we obtain

$$\left(\frac{1}{|Q|}\int\limits_Q|I(z)|^\delta dz\right)^{1/\delta}\le\left(\frac{1}{|Q|}\int\limits_Q|b_1(z)-\lambda_1|^{\tau_1}dz\right)^{1/\tau_1}\times$$

$$\times\left(\frac{1}{|Q|}\int\limits_Q|b_2(z)-\lambda_2|^{\tau_2}dz\right)^{1/\tau_2}\left(\frac{1}{|Q|}\int\limits_Q|T(f_1,f_2)|^\tau dz\right)^{\frac{1}{\tau}}\le$$

$$\le C\|b_1\|_*\|b_2\|_*M_q(T(f_1,f_2))(x). \tag{2.7}$$

Where $\tau_1>1,\ \tau_2>1$, such that $1/\tau+1/\tau_1+1/\tau_2=1/\delta$.

Then, we turn to estimate term II. We have

$$\left(\frac{1}{|Q|}\int\limits_Q|II(z)|^\delta dz\right)^{1/\delta}\le\left(\frac{1}{|Q|}\int\limits_Q|b_1(z)-\lambda_1|^s dz\right)^{1/s}\times$$

$$\times\left(\frac{1}{|Q|}\int\limits_Q|[b_2,T](f_1,f_2)|^\tau dz\right)^{1/\tau}\le C\|b_1\|_*M_\tau([b_2,T](f_1,f_2))(x), \tag{2.8}$$

where $s>1$ such that $1/\tau+1/s=1/\delta$. Similarly, we have

$$\left(\frac{1}{|Q|}\int\limits_Q |III(z)|^\delta dz\right)^{1/\delta} \le \left(\frac{1}{|Q|}\int\limits_Q |b_2(z)-\lambda_2|^s dz\right)^{1/s}\times$$

$$\times\left(\frac{1}{|Q|}\int\limits_Q |[b_1,T](f_1,f_2)|^\tau dz\right)^{1/\tau} \le C\|b_2\|_* M_\tau([b_1,T](f_1,f_2))(x). \quad (2.9)$$

Finally, we estimate IV. To do this, we write

$$|IV(z)| \le \left|\int\limits_{\mathbb{R}^{2n}} (b_1(y_1)-\lambda_1)(b_2(y_2)-\lambda_2)K(z,y_1,y_2)f_1^0(y_1)f_2^0(y_2)dy_1dy_2\right| +$$

$$+\left|\int\limits_{\mathbb{R}^{2n}} (b_1(y_1)-\lambda_1)(b_2(y_2)-\lambda_2)K(z,y_1,y_2)f_1^0(y_1)f_2^\infty(y_2)dy_1dy_2\right| +$$

$$+\left|\int\limits_{\mathbb{R}^{2n}} (b_1(y_1)-\lambda_1)(b_2(y_2)-\lambda_2)K(z,y_1,y_2)f_1^\infty(y_1)f_2^0(y_2)dy_1dy_2\right| +$$

$$+\left|\int\limits_{\mathbb{R}^{2n}} (b_1(y_1)-\lambda_1)(b_2(y_2)-\lambda_2)K(z,y_1,y_2)f_1^\infty(y_1)f_2^\infty(y_2)dy_1dy_2 - c_Q\right| =$$

$$= IV_1(z)+IV_2(z)+IV_3(z)+IV_4(z).$$

By using Kolmogorov's inequality to the term $T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)(z)$, the boundedness of T, Lemma 2.1, and Hölder's inequality, we have

$$\left(\frac{1}{|Q|}\int\limits_Q IV_1(z)^\delta dz\right)^{1/\delta} \le C\left\|T((b_1-\lambda_1)f_1^0,(b_2-\lambda_2)f_2^0)(z)\right\|_{L^{1/2,\infty}\left(Q,\frac{|dx|}{|Q|}\right)} \le$$

$$\le C\frac{1}{|Q|}\int\limits_Q |b_1(z)-\lambda_1||f_1(z)|dz\ \frac{1}{|Q|}\int\limits_Q |(b_2(z)-\lambda_2)f_2(z)|dz \le$$

$$\le C\left(\frac{1}{|Q^*|}\int\limits_{Q^*} |b_1(z)-\lambda_1|^{p_1'}dz\right)^{1/p_1'}\left(\frac{1}{|Q^*|}\int\limits_{Q^*} |f_1(z)|^{p_1}dz\right)^{1/p_1}\times$$

$$\times\left(\frac{1}{|Q^*|}\int\limits_{Q^*} |b_2(z)-\lambda_2|^{p_2'}dz\right)^{1/p_2'}\left(\frac{1}{|Q^*|}\int\limits_{Q^*} |f_2(z)|^{p_2}dz\right)^{1/p_2} \le$$

$$\le C\|b_1\|_*\|b_2\|_* M_{p_1}f_1(x)M_{p_2}f_2(x). \quad (2.10)$$

For term IV_2, using (1.2) and Lemma 2.1, we have

$$\left(\frac{1}{|Q|}\int\limits_Q IV_2(z)^\delta dz\right)^{1/\delta} \le$$

$$\leq C\frac{1}{|Q|}\int\limits_{Q}\left|\int\limits_{Q^*}\int\limits_{\mathbb{R}^n}\frac{A|(b_1(y_1)-\lambda_1)f_1^0(y_1)||(b_2(y_2)-\lambda_2)f_2^\infty(y_2)|}{(|x-y_1|+|x-y_2|)^{2n}}dy_1dy_2\right|dz\leq$$

$$\leq C\frac{1}{|Q|}\int\limits_{Q}\int\limits_{Q^*}|(b_1(y_1)-\lambda_1)f_1^0(y_1)|dy_1\int\limits_{\mathbb{R}^n\setminus Q^*}\frac{|(b_2(y_2)-\lambda_2)f_2^\infty(y_2)|}{|x-y_2|^{2n}}dy_2dz\leq$$

$$\leq C\left(\frac{1}{|Q^*|}\int\limits_{Q^*}|b_1(y_1)-\lambda_1|^{p_1'}dy_1\right)^{1/p_1'}\left(\frac{1}{|Q^*|}\int\limits_{Q^*}|f_1(y_1)|^{p_1}dy_1\right)^{1/p_1}\times$$

$$\times\left(|Q^*|\int\limits_{\mathbb{R}^n\setminus Q^*}\frac{|(b_2(y_2)-\lambda_2)f_2(y_2)|}{|x-y_2|^{2n}}dy_2\right)\leq$$

$$\leq C\|b_1\|_*M_{p_1}f_1(x)\sum_{k=1}^{\infty}2^{-k}\frac{1}{|2^kQ^*|}\int\limits_{2^kQ^*}|b_2(y_2)-\lambda_2|\ |f_2(y_2)|dy_2\leq$$

$$\leq C\|b_1\|_*M_{p_1}f_1(x)\sum_{k=1}^{\infty}\left(\frac{1}{|2^kQ^*|}\int\limits_{2^kQ^*}|b_2(y_2)-\lambda_2|^{p_2'}dy_1\right)^{1/p_2'}\times$$

$$\times\left(\frac{1}{|2^kQ^*|}\int\limits_{2^kQ^*}|f_2(y_2)|^{p_2}\,dy\right)^{1/p_2}\leq$$

$$\leq C\|b_1\|_*M_{p_1}f_1(x)\sum_{k=1}^{\infty}2^{-k}k\|b\|_*M_{p_2}f_2(x)\leq$$

$$\leq C\|b_1\|_*\|b_2\|_*M_{p_1}f_1(x)M_{p_2}f_2(x). \tag{2.11}$$

Similarly, for term IV_3, we have

$$\left(\frac{1}{|Q|}\int\limits_{Q}IV_3(z)^\delta dz\right)^{1/\delta}\leq C\|b_1\|_*\|b_2\|_*M_{p_1}f_1(x)M_{p_2}f_2(x). \tag{2.12}$$

Finally, for term IV_4, using (1.3) and Lemma 2.1, we obtain

$$\left(\frac{1}{|Q|}\int\limits_{Q}IV_4(z)^\delta dz\right)^{1/\delta}\leq\frac{1}{|Q|}\int\limits_{Q}\big|T((b_1(y_1)-\lambda_1)f_1^\infty(y_1),(b_2(y_2)-\lambda_2)\times$$

$$\times f_2^\infty(y_2))(z)-T((b_1(y_1)-\lambda_1)f_1^\infty(y_1),(b_2(y_2)-\lambda_2)f_2^\infty(y_2))(x)\big|dz\leq$$

$$\leq\frac{1}{|Q|}\int\limits_{Q}\int\limits_{\mathbb{R}^{2n}\setminus(Q^*)^2}|K(z,y_1,y_2)-K(x,y_1,y_2)|\times$$

$$\times\left|\prod_{i=1}^{2}(b_i(y_i)-\lambda_i)f_i^\infty(y_i)dy_1dy_2\right|dz\leq$$

$$\leq C\frac{1}{|Q|}\int\limits_{Q}\int\limits_{\mathbb{R}^{2n}\setminus(Q^*)^2}\frac{A|x-z|^{\epsilon}}{(|x-y_1|+|x-y_2|)^{2n+\epsilon}}\times$$

$$\times\left|\prod_{i=1}^{2}(b_i(y_i)-\lambda_i)f_i^{\infty}(y_i)dy_1dy_2\right|dz\leq$$

$$\leq C\frac{1}{|Q|}\int\limits_{Q}\int\limits_{\mathbb{R}^{n}\setminus Q^*}\frac{A|x-z|^{\epsilon_1}}{|x-y_1|^{n+\epsilon_1}}|b_1(y_1)-\lambda_1||f_1^{\infty}(y_1)|dy_1\times$$

$$\times\int\limits_{\mathbb{R}^{n}\setminus Q^*}\frac{A|x-z|^{\epsilon_2}}{|x-y_2|^{n+\epsilon_2}}|b_2(y_2)-\lambda_2||f_2^{\infty}(y_2)|dy_2dz\leq$$

$$\leq C\frac{1}{|Q|}\int\limits_{Q}\sum_{k=1}^{\infty}\int\limits_{2^{k+1}Q^*\setminus 2^kQ^*}\frac{(2R)^{\epsilon_1}}{(2^kR)^{n+\epsilon_1}}|b_1(y_1)-\lambda_1|f_1^{\infty}(y_1)|dy_1\times$$

$$\times\sum_{k=1}^{\infty}\int\limits_{2^{k+1}Q^*\setminus 2^kQ^*}\frac{(2R)^{\epsilon_2}}{(2^kR)^{n+\epsilon_2}}|b_2(y_2)-\lambda_2|f_2^{\infty}(y_2)|dy_2dz\leq$$

$$\leq C\left(\sum_{k=1}^{\infty}\frac{(2R)^{\epsilon_1}}{(2^kR)^{n+\epsilon_1}}\left(\int\limits_{2^{k+1}Q^*}|b_1(y_1)-\lambda_1)|^{p_1'}dy_1\right)^{1/p_1'}\times\right.$$

$$\left.\times\left(\int\limits_{2^{k+1}Q^*}f_1^{\infty}(y_1)|^{p_1}dy_1\right)^{1/p_1}\right)\times$$

$$\times\left(\sum_{k=1}^{\infty}\frac{(2R)^{\epsilon_2}}{(2^kR)^{n+\epsilon_2}}\left(\int\limits_{2^{k+1}Q^*}|b_2(y_2)-\lambda_2)|^{p_2'}dy_1\right)^{1/p_2'}\times\right.$$

$$\left.\times\left(\int\limits_{2^{k+1}Q^*}f_2^{\infty}(y_2)|^{p_2}dy_2\right)^{1/p_2}\right)\leq$$

$$\leq C\left(\sum_{k=1}^{\infty}\frac{(2R)^{\epsilon_1}}{(2^kR)^{n+\epsilon_1}}\left(\int\limits_{2^{k+1}Q^*}|b_1(y_1)-b_{1,k}+b_{1,k}-\lambda_1)|^{p_1'}dy_1\right)^{1/p_1'}\times\right.$$

$$\left.\times\left(\int\limits_{2^{k+1}Q^*}|f_1^{\infty}(y_1)|^{p_1}dy_1\right)^{1/p_1}\right)\times$$

$$\times\left(\sum_{k=1}^{\infty}\frac{(2R)^{\epsilon_2}}{(2^kR)^{n+\epsilon_2}}\left(\int\limits_{2^{k+1}Q^*}|b_2(y_2)-b_{2,k}+b_{2,k}-\lambda_2)|^{p_2'}dy_2\right)^{1/p_2'}\times\right.$$

$$\times\left(\int_{2^{k+1}Q*}|f_2^\infty(y_2)|^{p_2}dy_2\right)^{1/p_2}\Bigg)\le$$

$$\le C\sum_{k=1}^{\infty}2^{-k\epsilon_1}(k+1)\|b_1\|_*M_{p_1}f_1(x)\sum_{k=1}^{\infty}2^{-k\epsilon_2}(k+1)\|b_2\|_*M_{p_2}f_2(x)\le$$
$$\le C\|b_1\|_*\|b_2\|_*M_{p_1}f_1(x)M_{p_2}f_2(x), \tag{2.13}$$

where $\epsilon_j>0$, $b_{j,k}=\frac{1}{2^{k+1}Q*}\int_{2^{k+1}Q*}b_j(x)dx$, for $j=1,\ 2$, and $\epsilon_1+\epsilon_2=\epsilon$.

So, adding (2.7) to (2.13) into (2.6), we obtain (2.5). This completes the proof of Lemma 2.2. □

The proof of Theorem 1.1. First, let us suppose that b_1 and b_2 are L^∞ functions and $f_1,\ f_2$ belong to $L_c^\infty(\mathbb{R}^n)$. As a similar argument as in [10, page 686], we obtain that $\int_{\mathbb{R}^n}|[b_1,b_2,T]f(x)|^q dx$ is finite. Thus $\int_{\mathbb{R}^n}|M_\delta[b_1,b_2,T]f(x)|^q dx$ is also finite. Since $\tau<1/2<q$, $1<p_1<q_1$, and $1<p_2<q_2$, by Lemma 2.2 and (2.2), we have

$$\big\|[b_1,b_2,T]f\big\|_{L^q}\le\big\|M_\delta[b_1,b_2,T]f\big\|_{L^q}\le C\big\|([b_1,b_2,T]f)_\delta^{\#}\big\|_{L^q}\le$$
$$\le C\big[\|b_1\|_*\|b_2\|_*\|M_\tau(T(f_1,f_2))\|_{L^q}+\|b_1\|_*\|M_\tau([b_2,T](f_1,f_2))\|_{L^q}+$$
$$+\|b_2\|_*\|M_\tau([b_1,T](f_1,f_2))\|_{L^q}+\|b_1\|_*\|b_2\|_*\|M_{p_1}f_1M_{p_2}f_2\|_{L^q}\big]\le$$
$$\le C\|b_1\|_*\|b_2\|_*\|f_1\|_{L^{q_1}}\|f_2\|_{L^{q_2}},$$

where we used Proposition 3.1 in [9], which implies that there exists a constant C independent of f_1 and f_2 such that $([b_2,T](f_1,f_2))\|_{L^q}\le C\|b_2\|_*\|f_1\|_{L^{q_1}}\|f_2\|_{L^{q_2}}$, and $([b_2,T](f_1,f_2))\|_{L^q}\le C\|b_2\|_*\|f_1\|_{L^{q_1}}\|f_2\|_{L^{q_2}}$.

Again as the argument as in [10, page 687], let b_1 and b_2 are BMO functions, there exists a constant C independent of f_1 and f_2 such that

$$\big\|[b_1,b_2,T]f\big\|_{L^q}\le C\|b_1\|_*\|b_2\|_*\|f_1\|_{L^{q_1}}\|f_2\|_{L^{q_2}}$$

holds for all $f_1,\ f_2\in L_C^\infty$.

Since L_C^∞ is dense in L^p for $1<p<\infty$, thus we can extend $[b_1,b_2,T]$ to the whole space $L^{q_1}\times L^{q_2}$. This finishes the proof of Theorem 1.1.

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