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TWO-WEIGHTED INEQUALITIES FOR A DISCRETE HILBERT TRANSFORM

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Abstract. Two-weight inequalities for a discrete Hilbert transform are obtained. The necessary and sufficient conditions for pairs of sequences for the validity of Hardy's inequalities are found in a discrete case.

The sufficient conditions for pairs of sequences guaranteeing the validity of two-weighted inequalities for a discrete Hilbert transform, are found. The sufficient condition for the corresponding inequalities are also found. Finally, an example of pairs of weights is given.

Two-weight inequalities for the classical Hilbert transform in the case of monotone weights have been established in [1]. Analogous problem for singular integrals defined on Euclidean spaces has been considered in [2]. In their paper [3] D. Edmunds and V. Kokilashvili found optimal conditions for pairs of weights ensuring the validity of two-weighted inequalities for Calderon-Zygmund singular integrals (for the other generalizations see [4–5]).

Define the Hilbert transform for a sequence $\{\beta_n\}$ as follows:

$$(T\beta)_n = \sum_{k\neq n} \frac{\beta_k}{n-k}.$$

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$(T\beta)_n$ will be taken as ∞ if the defining sums do not converge. In the sequel, an entire part of the rational number $\frac{k}{n}$ will be denoted by $[\frac{k}{n}]$. Moreover, we will assume $0 \cdot \infty = \frac{0}{0} = 0$.

Definition 1. Let $1 < p < \infty$. A non-negative sequence ρ_k belongs to the class $A_p(\mathbb{Z})$ if

$$\frac{1}{(n-m+1)}\Big(\sum_{k=m}^{n} \rho_k\Big)\Big(\sum_{k=m}^{n} \rho_k^{1-p'}\Big)^{p-1} \le c$$

for every m and n, $m \le n$ $(m, n \in \mathbb{Z})$. Moreover, the sequence $\{\rho_k\}$ belongs to the class $A_1(\mathbb{Z})$ if

$$\frac{1}{n-m+1}\sum_{k=m}^{n} \rho_k \le c \min_{m\le k\le n} \rho_k$$

for every m, n, $m \le n$.

The one-weight problem for a discrete Hilbert transform has been solved in their work [6] by R. Hunt, B. Muckenhoupt and R. Wheeden. In particular, the following theorem is valid.

Theorem A. *Let $1 < p < \infty$. In order for the inequality*

$$\sum_{k=-\infty}^{+\infty} |(T\beta)_k|^p \rho_k \le c \sum_{k=-\infty}^{+\infty} |\beta_k|^p \rho_k$$

with the positive constant c independent of $\{\beta_k\}$, to be valid, it is necessary and sufficient that $\{\rho_k\} \in A_p(\mathbb{Z})$.

Let us now consider a discrete analogue of Hardy's theorem. For the classical Hardy operator $Hf(x) = \int_0^x f(t)dt$ the similar theorem has been proved by V. Kokilashvili [7], B. Muckenhoupt [8], J. Bradley [9] and V. Maz'ya [10].

Theorem 1. *Let $1 < p \le q < \infty$, $\{a_n\}$, $\{b_n\}$ be positive sequences. In order for the inequality*

$$\Big(\sum_{n=0}^{\infty}\Big|\sum_{k=0}^{n} f_k\Big|^q a_n^q\Big)^{1/q} \le c\Big(\sum_{n=0}^{\infty} |f_n|^p b_n^p\Big)^{1/p} \tag{1}$$

with the positive constant c independent of $\{f_k\}$, to be valid, it is necessary and sufficient that the condition

$$B=\sup_{n\geq 0}\Big(\sum_{k=n}^{\infty}a_k^q\Big)^{1/q}\Big(\sum_{k=0}^{n}b_k^{-p'}\Big)^{1/p'}<\infty\quad\Big(p'=\frac{p}{p-1}\Big)$$

be fulfilled; note that if c is the best constant in (1)*, then*

$$B\leq c\leq Bq^{\frac{1}{q}}\Big(\frac{q}{q-1}\Big)^{(p-1)/p}.$$

Proof. Let $\alpha_n=(\sum\limits_{k=0}^{n}b_k^{-p'})^{\frac{1}{qp'}}$. Due to Hölder's inequality we have

$$\Big(\sum_{n=0}^{\infty}a_n^q\Big|\sum_{k=0}^{n}f_k\Big|^q\Big)^{p/q}\leq$$
$$\leq\Bigg(\sum_{n=0}^{\infty}a_n^q\Big(\sum_{k=0}^{n}|f_k\alpha_kb_k|^p\Big)^{q/p}\Big(\sum_{k=0}^{n}(\alpha_kb_k)^{-p'}\Big)^{\frac{q}{p'}}\Bigg)^{p/q}=\Big(\sum_{n=0}^{\infty}A_n\Big)^{\frac{p}{q}}.$$

By Minkowsky's inequality ($\frac{q}{p}\geq 1$) we obtain

$$\Big(\sum_{n=0}^{\infty}A_n\Big)^{p/q}=\Bigg(\sum_{n=0}^{\infty}\Bigg(\sum_{k=0}^{n}|f_k\alpha_kb_k|^pa_n^p\Big(\sum_{k=0}^{n}(\alpha_kb_k)^{-p'}\Big)^{\frac{p}{p'}}\Bigg)^{\frac{q}{p}}\Bigg)^{\frac{p}{q}}\leq$$
$$\leq\sum_{k=0}^{\infty}|f_k\alpha_kb_k|^p\Bigg(\sum_{n=k}^{\infty}a_n^q\Big(\sum_{k=0}^{n}(\alpha_kb_k)^{-p'}\Big)^{\frac{q}{p'}}\Bigg)^{\frac{p}{q}}=\sum_k D_k.$$

For the intrinsic sum we have

$$\sum_{n=k}^{\infty}a_n^q\Big(\sum_{k=0}^{n}(\alpha_kb_k)^{-p'}\Big)^{\frac{q}{p'}}=\sum_{n=k}^{\infty}a_n^q\Bigg(\sum_{k=0}^{n}\Big(\sum_{i=0}^{k}b_k^{-p'}\Big)^{-\frac{1}{q}}b_k^{-p'}\Bigg)^{\frac{q}{p'}}.$$

Moreover, using the easily verifiable inequality

$$\sum_{k=0}^{n}b_k^{-p'}\Big(\sum_{i=0}^{k}b_i^{-p'}\Big)^{-\frac{1}{q}}\leq q'\Big(\sum_{k=0}^{n}b_k^{-p'}\Big)^{1/q'},$$

we arrive at

$$\sum_{n=k}^{\infty}a_n^q\Big(\sum_{i=0}^{n}(\alpha_ib_i)^{-p'}\Big)^{\frac{q}{p'}}=(q')^{\frac{q}{p'}}\sum_{n=k}^{\infty}a_n^q\Big(\sum_{i=0}^{n}b_i^{-p'}\Big)^{\frac{q}{q'p'}},$$

and the last sum does not exceed

$$\lambda = B^{\frac{q}{q'}}(q')^{\frac{q}{p'}} \sum_{n=k}^{\infty} a_n^q \Big(\sum_{i=n}^{\infty} a_i^q \Big)^{-\frac{1}{q'}}.$$

The inequality

$$\sum_{n=k}^{\infty} a_n^q \Big(\sum_{i=n}^{\infty} a_i^q \Big)^{-\frac{1}{q'}} \le q \Big(\sum_{i=k}^{\infty} a_i^q \Big)^{1/q},$$

can also be easily verified. Therefore we obtain

$$\lambda \le q B^{\frac{q}{q'}}(q')^{\frac{q}{p'}} \Big(\sum_{i=k}^{\infty} a_i^q \Big)^{1/q} \le$$

$$\le qB \cdot B^{\frac{q}{q'}}(q')^{\frac{q}{p'}} \Big(\sum_{i=0}^{k} b_i^{-p'} \Big)^{-\frac{1}{p'}} = qB^q (q')^{\frac{q}{p'}} \alpha_k^{-q}.$$

Finally we have

$$\sum_k D_k \le q^{\frac{p}{q}} B^p (q')^{\frac{p}{p'}} \sum_{k=0}^{\infty} |f_k \alpha_k b_k|^p \alpha_k^{-p} =$$

$$= q^{\frac{p}{q}} B^p (q')^{\frac{p}{p'}} \sum_{k=0}^{\infty} |f_k|^p \beta_k^p.$$

Show now the necessity. Let inequality (1) be fulfilled, and let

$$f_k = \begin{cases} \beta_k^{-p'}, & 0 \le k \le n, \\ 0, & k > n. \end{cases}$$

Then we have

$$\Big(\sum_{i=0}^{\infty} \Big(\sum_{k=0}^{i} f_k \Big)^q a_i^q \Big)^{1/q} \ge \Big(\sum_{i=n}^{\infty} \Big(\sum_{k=0}^{i} \beta_k^{-p'} \Big)^q a_i^q \Big)^{1/q} \ge$$

$$\ge \Big(\sum_{i=n}^{\infty} a_i^q \Big)^{1/q} \Big(\sum_{k=0}^{n} \beta_k^{-p'} \Big).$$

On the other hand,

$$\Big(\sum_{k=0}^{\infty} |f_k|^p \beta_k^p \Big)^{1/p} = \Big(\sum_{k=0}^{n} \beta_k^{-p'} \Big)^{1/p}$$

and finally

$$B < \infty. \quad \blacksquare$$

Remark 1. The sequence a_n in Theorem 1 may be assumed to be non-negative.

Corollary 1. *Let $1 < p \le q < \infty$, $\{a_n\}$, $\{b_n\}$ be positive sequences. In order for the inequality*

$$\Big(\sum_{n=0}^{\infty} a_n\Big|\sum_{k=0}^{n} f_k\Big|^q\Big)^{1/q} \le c\Big(\sum_{n=0}^{\infty} b_n|f_n|^p\Big)^{1/p}$$

with the positive constant c independent of $\{f_n\}$, to be fulfilled, it is necessary and sufficient that

$$\sup_{n\ge 0}\Big(\sum_{k=n}^{\infty} a_k\Big)^{1/q}\Big(\sum_{k=0}^{n} b_k^{1-p'}\Big)^{1/p'} < \infty.$$

According to Corollary 1, from the duality arguments we obtain

Theorem 2. *Let $1 < p \le q < \infty$, $\{a_n\}$, $\{b_n\}$ be positive sequences. In order for the inequality*

$$\Big(\sum_{n=0}^{\infty} a_n\Big|\sum_{k=n}^{\infty} f_k\Big|^q\Big)^{1/q} \le c\Big(\sum_{n=0}^{\infty} b_n|f_n|^p\Big)^{1/p}$$

with the positive constant independent of $\{f_n\}$, to be fulfilled, it is necessary and sufficient that the condition

$$\sup_{n\ge 0}\Big(\sum_{k=0}^{n} a_k\Big)^{1/q}\Big(\sum_{k=n}^{\infty} b_k^{1-p'}\Big)^{1/p'} < \infty$$

be fulfilled.

We will now proceed to the Hilbert transform. Let $\mathbb{Z}^+ = \{0, 1, \dots\}$.

Theorem 3. *Let $1 < p < \infty$, $\{\sigma_k\}$, $\{b_k\}$ and $\{\rho_k\}$ be positive sequences on $\mathbb{Z}^+$, $\sigma_0 = 0$, $\{\sigma_k\}$ be an increasing sequence $\{\rho_{|k|}\} \in A_p(\mathbb{Z})$ $a_k = \sigma_k\rho_k$ and let the following conditions be fulfilled:*

(1) *there exists a positive constant c_1 such that for every positive integer the inequality*

$$\sigma_{2k+1}\rho_k \le c_1 b_k$$

holds;

(2) $$\sup_{k\ge 0}\Big(\sum_{n=k+1}^{\infty}\frac{a_n}{n^p}\Big)\Big(\sum_{n=0}^{k} b_n^{1-p'}\Big)^{p-1} < \infty.$$

Then there exists a positive constant c such that for every positive $\{\beta_k\}$ the inequality

$$\sum_{n=-\infty}^{+\infty} |(T\beta)_n|^p a_{|n|} \le c \sum_{n=-\infty}^{+\infty} |\beta_n|^p b_{|n|} \tag{2}$$

holds.

Proof. Represent $\{\sigma_k\}_{k=1}^{\infty}$ as follows:

$$\sigma_k = \sum_{j=1}^{k}(\sigma_j - \sigma_{j-1}) = \sum_{j=1}^{k} \sigma'_j \quad (\sigma'_1 = \sigma_1 \quad \text{since} \quad \sigma_0 = 0).$$

We obtain

$$\sum_{n=-\infty}^{+\infty} \Big|\Big(T\beta\Big)_n\Big|^p a_{|n|} = \sum_{n\neq 0} \Big|\Big(T\beta\Big)_n\Big|^p \rho_{|n|}\Big(\sum_{k=1}^{|n|}\sigma'_k\Big) =$$

$$= \sum_{k=1}^{\infty} \sigma'_k\Big(\sum_{|n|\ge k} |(T\beta)_n|^p \rho_{|n|}\Big) \le c_2 \sum_{k=1}^{\infty} \sigma'_k\Big(\sum_{|n|\ge k} \rho_{|n|}\Big|\sum_{|i|>[\frac{k}{2}]} \frac{\beta_i}{n-i}\Big|^p\Big) +$$

$$+ c_2 \sum_{k=1}^{\infty} \sigma'_k\Big(\sum_{|n|\ge k} \rho_{|n|}\Big|\sum_{|i|\le[\frac{k}{2}]} \frac{\beta_i}{n-i}\Big|^p\Big) = S_1 + S_2.$$

By virtue of Theorem A we find that

$$S_1 \le c_3 \sum_{k=1}^{\infty} \sigma'_k\Big(\sum_{|i|>[\frac{k}{2}]} |\beta_i|^p \rho_{|i|}\Big) \le c_3 \sum_{k=1}^{\infty} \sigma'_k\Big(\sum_{|i|\ge[\frac{k}{2}]} |\beta_i|^p \rho_{|i|}\Big) =$$

$$= c_4\Big(\sigma'_1 \sum_{|i|\ge 1} |\beta_i|^p \rho_{|i|} + \sum_{m=2,4,6,\dots} (\sigma'_m + \sigma'_{m+1}) \sum_{|i|\ge \frac{m}{2}} |\beta_i|^p \rho_{|i|}\Big) =$$

$$= c_4\Big(\sigma'_1 \sum_{|i|\ge 1} |\beta_i|^p \rho_{|i|} + \sum_{k=1}^{\infty}(\sigma'_{2k} + \sigma'_{2k+1}) \sum_{|i|\ge k} |\beta_i|^p \rho_{|i|}\Big) =$$

$$= c_4 \sum_{|i|\ge 1} |\beta_i|^p \rho_{|i|}(\sigma'_1 + \sigma'_2 + \dots + \sigma'_{2|i|+1}) =$$

$$= c_4 \sum_{|i|\ge 1} |\beta_i|^p \rho_{|i|}\sigma_{2|i|+1} \le c_4 \sum_{i=-\infty}^{+\infty} |\beta_i|^p \rho_{|i|}\sigma_{2|i|+1} \le c_5 \sum_{i=-\infty}^{+\infty} |\beta_i|^p b_{|i|}.$$

Estimate now S_2.

If $|n| \geq k$ and $|i| \leq [\frac{k}{2}]$, then

$$|n| \leq |n-i| + |i| \leq |n-i| + \left[\frac{k}{2}\right] \leq |n-i| + \left[\frac{|n|}{2}\right] \leq |n-i| + \frac{|n|}{2}$$

and $\frac{|n|}{2} \leq |n-i|$. Whence we obtain

$$S_2 \leq c_6 \sum_{k=1}^{\infty} \sigma'_k \Big(\sum_{|n| \geq k} \frac{\rho_{|n|}}{|n|^p} \Big) \Big(\sum_{|i| \leq [\frac{k}{2}]} |\beta_i| \Big)^p \leq$$

$$\leq c_7 \sum_{k=1}^{\infty} \sigma'_k \Big(\sum_{n=k}^{\infty} \frac{\rho_n}{n^p} \Big) \Big(\sum_{i=0}^{k-1} \Big(|\beta_i| + |\beta_{-i}| \Big) \Big)^p =$$

$$= c_7 \sum_{k=0}^{\infty} \sigma'_{k+1} \Big(\sum_{n=k+1}^{\infty} \frac{\rho_n}{n^p} \Big) \Big(\sum_{i=0}^{k} \Big(|\beta_i| + |\beta_{-i}| \Big) \Big)^p.$$

Moreover, we have

$$\sum_{k=m}^{\infty} \sigma'_{k+1} \Big(\sum_{n=k+1}^{\infty} \frac{\rho_n}{n^p} \Big) = \sum_{k=m+1}^{\infty} \sigma'_k \Big(\sum_{n=k}^{\infty} \frac{\rho_n}{n^p} \Big) =$$

$$= \sum_{n=m+1}^{\infty} \frac{\rho_n}{n^p} \Big(\sum_{k=m+1}^{n} \sigma'_k \Big) \leq \sum_{n=m+1}^{\infty} \frac{a_n}{n^p}$$

and therefore for every $m \geq 0$.

$$\Big(\sum_{k=m}^{\infty} \sigma'_{k+1} \Big(\sum_{n=k+1}^{\infty} \frac{\rho_n}{n^p} \Big) \Big) \Big(\sum_{n=0}^{m} b_n^{1-p'} \Big)^{p-1} \leq$$

$$\leq \Big(\sum_{n=m+1}^{\infty} \frac{a_n}{n^p} \Big) \Big(\sum_{n=0}^{m} b_n^{1-p'} \Big)^{p-1}.$$

By virtue of Corollary 1 we obtain

$$S_2 \leq c_{10} \sum_{k=0}^{+\infty} \Big(|\beta_k| + |\beta_{-k}| \Big)^p b_k \leq c_{11} \sum_{k=0}^{+\infty} \Big(|\beta_k|^p + |\beta_{-k}|^p \Big) b_k \leq$$

$$\leq c_{12} \sum_{k=-\infty}^{+\infty} |\beta_k|^p b_{|k|}. \quad \blacksquare$$

Lemma 1. *Let $1 < p < \infty$, b_1, b_2, b_3 and b_4 be positive integers satisfying the condition $0 \leq b_1 < b_2 \leq b_3 < b_4 < \infty$, and let the sequence $\{\rho_k\}$ be defined on $\mathbb{Z}^+$, and $\{\rho_{|k|}\} \in A_p(\mathbb{Z})$.*

Then there exist a constant $c > 0$ such that for every $n \in \mathbb{N}$ the inequality

$$\sum_{k=b_3 n}^{b_4 n} \rho_k \le c \sum_{k=b_1 n}^{b_2 n} \rho_k$$

holds.

Proof. Let $n \in \mathbb{N}$ and

$$\Gamma = \{k : b_1 n \le k \le b_2 n\},$$
$$\Gamma_1 = \{k : b_3 n \le k \le b_4 n\},$$
$$B = \{k \in \mathbb{Z}^+; \ \ k \le b_4 n\}.$$

For the sequence β_k, define a maximal function

$$\beta_k^* = \sup_{\substack{n,m \\ n \le k \le m}} \Big(\frac{1}{m-n+1} \sum_{k=n}^{m} |\beta_k| \Big).$$

We have

$$\beta_k^* \ge \frac{1}{b_4 n + 1} \Big(\sum_{i=0}^{b_4 n} |\beta_i| \Big) (\chi_{\Gamma_1})_k \ge$$
$$\ge \frac{1}{(b_4+1)n} \Big(\sum_{i=0}^{b_4 n} |\beta_i| \Big) (\chi_{\Gamma_1})_k \ge \frac{b_2 - b_1}{b_4 + 1} \frac{1}{(b_2 - b_1)n} \Big(\sum_{i=b_1 n}^{b_2 n} |\beta_i| \Big) (\chi_{\Gamma_1})_k \ge$$
$$\ge \frac{b_2 - b_1}{b_4 + 1} \frac{1}{(b_2 - b_1)n + 1} \Big(\sum_{i=b_1 n}^{b_2 n} |\beta_i| \Big) (\chi_{\Gamma_1})_k,$$

where

$$(\chi_{\Gamma_1})_k = \begin{cases} 0, & k \notin \Gamma_1, \\ 1, & k \in \Gamma_1. \end{cases}$$

Because the operator β^* is bounded in l_{p,ρ_k} (see [11]), we have

$$\sum_{k=b_3 n}^{b_4 n} \frac{1}{((b_2 - b_1)n + 1)^p} \Big(\sum_{i=b_1 n}^{b_2 n} |\beta_i| \Big)^p_{\rho_k} \le c_1 \sum_{k=-\infty}^{+\infty} |\beta_k|^p \rho_{|k|}.$$

Taking $\beta_k = (\chi_\Gamma)_k$, we obtain

$$\sum_{k=b_3 n}^{b_4 n} \rho_k \le c_1 \sum_{k=b_1 n}^{b_2 n} \rho_k. \quad \blacksquare$$

Theorem 4. *Let* $1 < p < \infty$, $\{\sigma_k\}$ *and* $\{u_k\}$ *be positive, increasing on* $\mathbb{Z}^+$ *sequences and let* $\sigma_0 = 0$, $\{\rho_k\}$ *be a positive on* $\mathbb{Z}^+$ *sequence satisfying the condition* $\{\rho_{|k|}\} \in A_p(\mathbb{Z})$. $a_k = \sigma_k\rho_k$, $b_k = u_k\rho_k$ *and the condition*

$$\sup_{n\geq 0}\Big(\sum_{k=n+1}^{\infty}\frac{a_k}{k^p}\Big)\Big(\sum_{k=0}^{n} b_k^{1-p'}\Big)^{p-1} < \infty \tag{3}$$

is fulfilled. Then there exists a positive constant c *such that for every* $\{\beta_n\}$ *inequality* (2) *holds.*

Proof. By virtue of Theorem 3 it suffices to show that for every $n \in \mathbb{Z}^+$ the inequality

$$\sigma_{2n+1} \leq c_1 u_n$$

holds.

By Hölder's inequality and Lemma 1 we find that for every positive integer $n > 1$

$$\begin{aligned}\frac{\sigma_{3n}}{u_n} \leq \frac{\sigma_{3n}}{u_n}\frac{1}{(3n+1)^p}\Big(\sum_{k=3n}^{6n}\rho_k\Big)\Big(\sum_{k=3n}^{6n}\rho_k^{1-p'}\Big)^{p-1} \leq \\ \leq c\frac{\sigma_{3n}}{u_n}\frac{2^p}{(6n)^p}\Big(\sum_{k=3n}^{6n}\rho_k\Big)\Big(\sum_{k=0}^{n}\rho_k^{1-p'}\Big)^{p-1} \leq \\ \leq c_1\Big(\sum_{k=3n}^{6n}\frac{a_k}{k^p}\Big)\Big(\sum_{k=0}^{n} b_k^{1-p'}\Big)^{p-1} \leq c_1\Big(\sum_{k=n+1}^{\infty}\frac{a_k}{k^p}\Big)\Big(\sum_{k=0}^{n} b_k^{1-p'}\Big)^{p-1} \leq c_2.\end{aligned}$$

(Here the use is made of the fact that $\rho_{|k|}^{1-p'} \in A_{p'}(\mathbb{Z})$.) ■

Analogous theorem for Calderon-Zygmund singular integrals has been proved in [3],[5].

Theorem 5. *Let* $1 < p < \infty$, $\{a_k\}$, $\{b_k\}$ *be positive on* $\mathbb{Z}^+$ *sequences and let inequality* (2) *holds. Then for* $\{a_k\}$ *and* $\{b_k\}$ *condition* (3) *is fulfilled.*

Proof. Let

$$\beta_k = \begin{cases} b_k^{1-p'}, & 0 \leq k \leq m,\\ 0, & \text{otherwise},\end{cases}$$

where $m \in \mathbb{Z}^+$. Then

$$\sum_{k=-\infty}^{+\infty} |(T\beta)_k|^p a_{|k|} \leq c\sum_{k=0}^{m} b_k^{1-p'}.$$

On the other hand,

$$\sum_{k=-\infty}^{+\infty} |(T\beta)_k|^p a_{|k|} \geq \sum_{k=m+1}^{\infty} a_k \Big(\sum_{i=0}^{m} \frac{b_i^{1-p'}}{k-i} \Big)^p \geq$$
$$\geq c_1 \Big(\sum_{k=m+1}^{\infty} \frac{a_k}{k^p} \Big) \Big(\sum_{i=0}^{m} b_i^{1-p'} \Big)^p$$

and we obtain condition (3). (Here the use is made of the inequalities $k-i \leq k+i \leq k+m < k+m+1 \leq 2k$ for $k \geq m+1$, $0 \leq i \leq m$). ■

Theorem 6. *Let* $1 < p < \infty$, $\{a_k\}$, $\{b_k\}$ *be positive on* $\mathbb{Z}^+$ *sequences and let inequality* (2) *be fulfilled. Then* $\{a_k\}$ *and* $\{b_k\}$ *satisfy the condition*

$$\sup_{n\geq 0} \Big(\sum_{k=0}^{n} a_n \Big) \Big(\sum_{k=n+1}^{\infty} k^{-p'} b_k^{1-p'} \Big)^{p-1} < \infty.$$

Proof. Let $n \geq 0$ and

$$\beta_k = \begin{cases} (kb_k)^{1-p'}, & \text{for} \quad n+1 \leq k \leq m, \\ 0, & \text{otherwise,} \end{cases}$$

We obtain

$$\sum_{k=-\infty}^{+\infty} |(T\beta)_k|^p a_{|k|} \leq c \sum_{k=n+1}^{m} (kb_k)^{(1-p')p} b_k = c \sum_{k=n+1}^{m} k^{-p'} b_k^{1-p'}.$$

On the other hand, we have

$$\sum_{k=-\infty}^{+\infty} |(T\beta)_k|^p a_{|k|} \geq \sum_{k=0}^{n} a_k \Big(\sum_{j=n+1}^{m} \frac{b_j^{1-p'} j^{1-p'}}{j-k} \Big)^p.$$

Moreover, $j-k \leq j+k \leq j+n < j+j = 2j$ for $0 \leq k \leq n$ and $j \geq n+1$. Therefore the last sum is greater than

$$\frac{1}{2^p} \Big(\sum_{k=0}^{n} a_k \Big) \Big(\sum_{j=n+1}^{m} \frac{b_j^{1-p'}}{j^{p'}} \Big)^p \quad \text{for} \quad m \geq n+1$$

and

$$\Big(\sum_{k=0}^{n} a_k \Big) \Big(\sum_{j=n+1}^{m} \frac{b_j^{1-p'}}{j^{p'}} \Big)^{p-1} \leq c, \quad \text{for} \ \ m \geq n+1 \ \ \text{and} \ \ \forall n \in \mathbb{N}.$$

Passing to the limit as $m \to \infty$, we get the desired result. ■

Theorem 7. *Let* $1 < p < \infty$, $\{\sigma_k\}$; $\{u_k\}$ *be positive, increasing on* $\mathbb{Z}^+$ *sequences,* $\sigma_0 = 0$. *If the inequality*

$$\sum_{k=-\infty}^{+\infty} |(T\beta)_k|^p \sigma_{|k|} \le c \sum_{k=-\infty}^{+\infty} |\beta_k|^p u_{|k|} \tag{4}$$

is fulfilled, where the constant c, $c > 0$, *does not depend on* β_k, *then for* $\{\rho_k\}$, $\{\rho_{|k|}\} \in A_1(\mathbb{Z})$, *we have*

$$\sum_{k=-\infty}^{+\infty} |(T\beta)_k|^p \sigma_{|k|}\rho_{|k|} \le c \sum_{k=-\infty}^{+\infty} |\beta_k|^p u_{|k|}\rho_{|k|}. \tag{5}$$

Proof. Let inequality (4) be fulfilled. Then by Theorem 5 we have

$$\sup_{n\ge 0} \Big(\sum_{k=n+1}^{\infty} \frac{\sigma_k}{k^p}\Big)\Big(\sum_{k=0}^{n} u_k^{1-p'}\Big)^{p-1} < \infty.$$

Whence

$$\Big(\sum_{k=n}^{\infty} \frac{\sigma_k\rho_k}{k^p}\Big)\Big(\sum_{k=0}^{n} u_k^{1-p'}\rho_k^{1-p'}\Big)^{p-1} \le \sum_{j=0}^{\infty}\Big(\sum_{k=2^j n}^{2^{j+1}n-1} \frac{\sigma_k\rho_k}{k^p}\Big)\Big(\sum_{k=0}^{n} u_k^{1-p'}\rho_k^{1-p'}\Big)^{p-1} \le$$

$$\le \sum_{j=0}^{\infty} \sigma_{2^{j+1}n}\frac{1}{(2^j n)^p}\Big(\sum_{k=2^j n}^{2^{j+1}n-1}\rho_k\Big)\Big(\max_{0\le k\le 2^{j+1}n-1}\frac{1}{\rho_k}\Big)\Big(\sum_{k=0}^{n} u_k^{1-p'}\Big)^{p-1} \le$$

$$\le c_1 \sum_{j=0}^{\infty} \frac{\sigma_{2^{j+1}n}}{(2^j n)^{p-1}}\Big(\sum_{k=0}^{n} u_k^{1-p'}\Big)^{p-1} \le$$

$$\le c_2 \sum_{j=0}^{\infty} \sigma_{2^{j+1}n}\Big(\sum_{k=2^{j+1}n}^{2^{j+2}n-1}\frac{1}{k^p}\Big)\Big(\sum_{k=0}^{n} u_k^{1-p'}\Big)^{p-1} \le$$

$$\le c_2 \sum_{j=0}^{\infty}\Big(\sum_{k=2^{j+1}n}^{2^{j+2}n-1}\frac{\sigma_k}{k^p}\Big)\Big(\sum_{k=0}^{n} u_k^{1-p'}\Big)^{p-1} = c_2\Big(\sum_{k=2n}^{\infty}\frac{\sigma_k}{k^p}\Big)\Big(\sum_{k=0}^{n} u_k^{1-p'}\Big)^{p-1} \le$$

$$\le c_2\Big(\sum_{k=2n}^{\infty}\frac{\sigma_k}{k^p}\Big)\Big(\sum_{k=0}^{2n-1} u_k^{1-p'}\Big)^{p-1} < \infty$$

for every n, $n \in N$. Thus, owing to Theorem 4, we obtain inequality (5). ■

Theorem 8. *Let* $1 < p < \infty$, $\{\sigma_k\}$, $\{u_k\}$, $\{\rho_k\}$ *be positive on* $\mathbb{Z}^+$ *sequences,* $\sigma_0 = 0$, $\{\rho_{|k|}\} \in A_p(\mathbb{Z})$, $a_k = \sigma_k\rho_k$, $b_k = u_k\rho_k$ *and let the following conditions be fulfilled:*

1) *there exists a positive constant* η *such that for every* j, $j \in \mathbb{N}$ *the following inequality holds*

$$\max_{2^{j-1}\le k\le 2^{j+2}} \sigma_k \le \eta \min_{2^{j-1}\le k\le 2^{j+2}} \sigma_k;$$

$$2) \quad \sup_{n\ge 0}\Big(\sum_{k=n}^{\infty}\frac{a_k}{k^p}\Big)\Big(\sum_{k=0}^{n} b_k^{1-p'}\Big)^{p-1} < \infty;$$

$$3) \quad \sup_{n\ge 0}\Big(\sum_{k=0}^{n} a_k\Big)\Big(\sum_{k=n}^{\infty}\frac{b_k^{1-p'}}{k^{p'}}\Big)^{p-1} < \infty.$$

Then there exists the positive constant c such that for every $\{\beta_k\}$ *inequality* (2) *holds.*

Proof. We have

$$\sum_{k=-\infty}^{+\infty}|(T\beta)_k|^p a_{|k|} = \sum_{|k|\ne 0}|(T\beta)_k|^p a_{|k|} =$$

$$=\sum_{j=0}^{\infty}\Big(\sum_{2^j\le|k|<2^{j+1}}|(T\beta)_k|^p a_k\Big)\le c_1\sum_{j=0}^{\infty}\Big(\sum_{2^j\le|k|<2^{j+1}}\Big|\sum_{|n|<2^{j-1}}\frac{\beta_n}{k-n}\Big|^p a_{|k|}\Big)+$$

$$+c_1\sum_{j=0}^{\infty}\Big(\sum_{2^j\le|k|<2^{j+1}}\Big|\sum_{2^{j-1}\le|n|\le 2^{j+2}}\frac{\beta_n}{k-n}\Big|^p a_{|k|}\Big)+$$

$$+c_1\sum_{j=0}^{\infty}\Big(\sum_{2^j\le|k|<2^{j+1}}\Big|\sum_{|n|>2^{j+2}}\frac{\beta_n}{k-n}\Big|^p a_{|k|}\Big) = S_1+S_2+S_3.$$

Estimate S_1. If $|n|<2^{j-1}$ and $2^j\le|k|<2^{j+1}$, then $|k|\le|k-n|+|n|<|k-n|+2^{j-1}=|k-n|+\frac{2^j}{2}\le|k-n|+\frac{|k|}{2}\Rightarrow\frac{|k|}{2}<|k-n|$. Moreover $|n|<2^{j-1}<2^j\le|k|$. By Corollary 1 we obtain

$$S_1\le\sum_{j=0}^{\infty}\Big(\sum_{2^j\le|k|<2^{j+1}}\Big(\sum_{|n|<2^{j-1}}\frac{|\beta_n|}{|n-k|}\Big)^p a_{|k|}\Big)\le$$

$$\le c_2\sum_{j=0}^{\infty}\Big(\sum_{2^j\le|k|<2^{j+1}}\frac{a_{|k|}}{|k|^p}\Big(\sum_{|n|\le|k|}|\beta_n|\Big)^p\Big)=$$

$$=c_2\sum_{j=0}^{\infty}\Big(\sum_{2^j\le|k|<2^{j+1}}\frac{a_{|k|}}{|k|^p}\Big(\sum_{n=0}^{|k|}(|\beta_n|+|\beta_{-n}|)\Big)^p\Big)=$$

$$= c_2 \sum_{j=0}^{\infty} \bigg(\sum_{k=2^j}^{2^{j+1}-1} \frac{a_k}{k^p} \bigg(\sum_{n=0}^{k} (|\beta_n| + |\beta_{-n}|) \bigg)^p \bigg) =$$

$$= c_2 \sum_{k=1}^{\infty} \frac{a_k}{k^p} \bigg(\sum_{n=0}^{k} (|\beta_n| + |\beta_{-n}|) \bigg)^p \le c_3 \sum_{k=0}^{\infty} b_k (|\beta_k| + |\beta_{-k}|)^p \le$$

$$\le c_4 \sum_{k=-\infty}^{+\infty} b_k |\beta_k|^p.$$

Estimate now S_3. If $|n| > 2^{j+2}$ and $2^j \le |k| < 2^{j+1}$, then $|k| < 2^{j+1} < 2^{j+2} < |n|$ and $|n| \le |n-k| + |k| < |n-k| + 2^{n+1} < |n-k| + \frac{|n|}{2}$. Whence we get $\frac{|n|}{2} < |n-k|$, and for S_3 we have

$$S_3 \le c_5 \sum_{j=0}^{\infty} \bigg(\sum_{2^j \le |k| < 2^{j+1}} a_{|k|} \bigg(\sum_{|n| \ge |k|} \frac{|\beta_n|}{|n|} \bigg)^p \bigg) =$$

$$= c_5 \sum_{j=0}^{\infty} \bigg(\sum_{2^j \le |k| < 2^{j+1}} a_{|k|} \bigg(\sum_{n=|k|}^{\infty} \frac{(|\beta_n| + |\beta_{-n}|)}{n} \bigg)^p \bigg) =$$

$$= c_6 \sum_{j=0}^{\infty} \bigg(\sum_{k=2^j}^{2^{j+1}-1} a_k \bigg(\sum_{n=k}^{\infty} \frac{(|\beta_n| + |\beta_{-n}|)}{n} \bigg)^p \bigg) \le$$

$$\le c_7 \sum_{k=0}^{\infty} \Big(|\beta_k| + |\beta_{-k}| \Big)^p b_k \le c_8 \sum_{k=-\infty}^{+\infty} |\beta_k|^p b_{|k|}.$$

Here we used Theorem 3.

Estimate S_2. Using the fact that $\{\rho_{|k|}\} \in A_p(\mathbb{Z})$, we get

$$S_2 \le c_1 \sum_{j=0}^{\infty} \Big(\max_{2^j \le |k| < 2^{j+1}} \sigma_{|k|} \Big) \sum_{2^j \le |k| < 2^{j+1}} \bigg| \sum_{2^{j-1} \le |n| \le 2^{j+2}} \frac{\beta_n}{n-k} \bigg|^p \rho_{|k|} \le$$

$$\le c_1 \sum_{j=0}^{\infty} \max_{2^j \le |k| < 2^{j+1}} \sigma_{|k|} \bigg(\sum_{k=-\infty}^{+\infty} \bigg| \sum_{2^{j-1} \le |n| \le 2^{j+2}} \frac{\beta_n}{n-k} \bigg|^p \rho_{|k|} \bigg) \le$$

$$\le c_8 \sum_{j=0}^{\infty} \max_{2^j \le |k| < 2^{j+1}} \sigma_{|k|} \bigg(\sum_{2^{j-1} \le |k| \le 2^{j+2}} |\beta_k|^p \rho_{|k|} \bigg) \le$$

$$\le c_8 \sum_{j=0}^{\infty} \sum_{2^{j-1} \le |k| \le 2^{j+2}} |\beta_k|^p \rho_{|k|} \Big(\max_{2^{j-1} \le |k| \le 2^{j+2}} \sigma_{|k|} \Big) \le$$

$$\leq c_9 \sum_{j=0}^{\infty} \Big(\sum_{2^{j-1} \leq |k| \leq 2^{j+2}} |\beta_k|^p \rho_{|k|} u_{|k|} \Big) \leq c_{10} \sum_{k=-\infty}^{+\infty} |\beta_k|^p b_{|k|}.$$

In the last inequality the use is made of the inequality

$$\sum_{j=0}^{\infty} \chi_{\{n: 2^{j-1} \leq |n| \leq 2^{j+2}\}}(k) \leq 5. \quad \blacksquare$$

Remark 2. In the proof of theorem 8. $2^{j-1} = 1$ for $j = 0$.

Corollary 2. *Let $1 < p < \infty$, $\{a_k\}$ and $\{b_k\}$ be positive, increasing on $\mathbb{Z}^+$ sequences with the condition $a_0 = 0$, and let the condition*

$$\sup_{n \geq 0} \Big(\sum_{k=n}^{\infty} \frac{a_k}{k^{p'}} \Big) \Big(\sum_{k=0}^{n} b_k^{1-p'} \Big)^{p-1} < \infty$$

be fulfilled. Then there exists a positive constant c such that for every $\{\beta_k\}$ inequality (2) *holds.*

Proof. According to the theorem, it suffices to show that

$$1) a_{2^{j+2}} \leq \lambda b_{2^{j-1}}, \quad j \in \mathbb{N};$$

$$2) \sup_{n \geq 0} \Big(\sum_{k=0}^{n} a_k \Big) \Big(\sum_{k=n}^{\infty} \frac{b_k^{1-p'}}{k^{p'}} \Big)^{p-1} < \infty.$$

Show that condition (2) is fulfilled. Let $n \geq 1$. We obtain

$$\Big(\sum_{k=0}^{n+1} a_k \Big) \Big(\sum_{k=n+1}^{\infty} \frac{b_k^{1-p'}}{k^{p'}} \Big)^{p-1} \leq \frac{a_{n+1}}{b_{n+1}} (n+2) \Big(\sum_{k=n+1}^{\infty} \frac{1}{k^{p'}} \Big)^{p-1} \leq$$

$$\leq \frac{a_{n+1}}{b_{n+1}} (n+2) \Big(\int_{n}^{\infty} \frac{dx}{x^{p'}} \Big)^{p-1} \leq c_1 \Big(\sum_{k=n+1}^{2n+1} \frac{a_k}{k^p} \Big) \Big(\sum_{k=0}^{n+1} b_k^{1-p'} \Big)^{p-1} \leq$$

$$\leq c_1 \Big(\sum_{k=n+1}^{\infty} \frac{a_k}{k^p} \Big) \Big(\sum_{k=0}^{n+1} b_k^{1-p'} \Big)^{p-1} \leq c_2.$$

Thus

$$B_1 = \sup_{n \geq 2} \Big(\sum_{k=0}^{n} a_k \Big) \Big(\sum_{k=n}^{\infty} \frac{b_k^{1-p'}}{k^{p'}} \Big)^{p-1} < \infty.$$

Let

$$B_2 = \Big(\sum_{k=0}^{1} a_k \Big) \Big(\sum_{k=1}^{\infty} \frac{b_k^{1-p'}}{k^{p'}} \Big)^{p-1}.$$

Obviously, $B_2 < \infty$, and if $B = \max\{B_1, B_2\}$, then

$$\sup_{n\geq 0}\Big(\sum_{k=0}^{n} a_k\Big)\Big(\sum_{k=n}^{\infty}\frac{b_k^{1-p'}}{k^{p'}}\Big)^{p-1} \leq B < \infty.$$

Let us now show that condition (1) is fulfilled. We have,

$$\frac{a_{2^{j+2}}}{b_{2^{j-1}}} \leq c_1 \frac{a_{2^{j+2}}}{b_{2^{j-1}}}\Big(\sum_{k=2^{j+2}}^{2^{j+3}}\frac{1}{k^p}\Big)\Big(\sum_{k=1}^{2^{j-1}} 1\Big)^{p-1} \leq c_1\Big(\sum_{k=2^{j-1}}^{\infty}\frac{a_k}{k^p}\Big)\Big(\sum_{k=1}^{2^{j-1}} b_k^{1-p'}\Big)^{p-1} \leq c_2$$

for every j, $j \in \mathbb{N}$. ■

Example. Let $a_n = n^{p-1}\ln^{\gamma}(n+1)$, $b_n = (n+1)^{p-1}\ln^{\beta}(n+2)$, where $0 < \beta < p-1$, $\gamma < \beta - p$. Then the pair (a_n, b_n) satisfies the condition

$$\sup_{k\geq 0}\Big(\sum_{n=k+1}^{\infty}\frac{a_n}{n^p}\Big)\Big(\sum_{n=0}^{k} b_n^{1-p'}\Big)^{p-1} < \infty,$$

and thus, by virtue of Theorem 4, inequality (2) holds.

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