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REMARKS ON BICENTRIC QUADRILATERALS

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Abstract. We deal with planar moduli spaces of polygonal linkages arising from a poristic family of bicentric polygons. For bicentric quadrilaterals, we describe the topological types of moduli spaces in poristic families and find the absolute maximum and minimum of oriented area in the union of moduli spaces. Similar results are obtained for poristic quadrilaterals associated with a pair of confocal ellipses. In conclusion we outline some research perspectives suggested by our results.

რეზიუმე. შესწავლილია სახსრული ბიცენტრული მრავალკუთხედების კონფიგურაციული სივრცეები. ბიცენტრული ოთხკუთხედების შემთხვევაში აღწერილია კონფიგურაციული სივრცეების ტოპოლოგიური სტრუქტურა და გამოთვლილია ორიენტირებული ფართობის ექსტრემუმები. ანალოგიური შედეგები მიღებულია კონფოკალური ელიფსების შემთხვევაში. აღწერილია აგრეთვე ამ შედეგების ზოგიერთი შესაძლო განზოგადება.

1. Introduction

We present a number of results concerned with the planar moduli spaces $M(L_t)$ of one-dimensional family of polygonal linkages L_t arising from a poristic family $\{P_t, t \in S^1\}$ of bicentric polygons. In such a situation, the family of moduli spaces $M(L_t)$ can be considered as a fibration over the unit circle S^1 parameterizing poristic polygons. In this context it is natural to investigate the topological structure of fibers $M(L_t)$ with a view towards describing the topology of the total space E of arising fibration.

Another natural setting is concerned with consideration of various functions on E. For example, the oriented area function is defined on each $M(L_t)$, which yields a function A on the total space of fibration E. The known results on the extremal values and critical points of oriented area

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function on planar moduli spaces [6], [8] suggest that it might be interesting to obtain similar results for the function A on E.

These observations indicate a certain direction of research concerned with bicentric polygons. In particular, the following two questions naturally arise in this context.

(Q1) What is the topology of planar moduli spaces $M(L_t)$ in a given family of poristic polygons P_t?

(Q2) What are the extremal values of oriented area function as a function on E?

In the present paper we answer these questions in the case of bicentric quadrilaterals and present some related results for confocal ellipses.

2. Bicentric Polygons and Poncelet Theorem

To give a precise description of the setting under consideration and our results we begin with the necessary definitions.

Definition 2.1. A polygon P in the plane is called *bicentric* if there exist two circles C_1, C_2 with C_2 strictly inside C_1 such that all vertices of P lie on C_1 and each side of P is tangent to C_2 at a certain inner point of this side. The pair of circles (C_1, C_2) is called the *frame* of bicentric polygon P. Their centers O_1 and O_2 are called the *circumcenter* and *incenter* of P, respectively.

For example, each triangle $\triangle$ is bicentric with C_1 being the circumscribed circle (circumcircle) of $\triangle$ and C_2 the inscribed circle (incircle) of $\triangle$. Each regular polygon is also bicentric. Notice that this definition does not require of P to be convex. So any regular *star-shaped* polygon is also bicentric.

Many results on bicentric polygons can be found in the literature. The first detailed paper on properties of bicentric polygons was published by N.Fuss [**?**]. For this reason, a pair of circles constituting the frame of a bicentric k-gon will be called a *Fuss pair of circles of order k* or simply a *Fuss pair of order k*. Up to a motion of the plane, a Fuss pair of circles is completely determined by a triple of non-negative numbers (R, r, d), where $R > 0$ is the radius of circumcircle, $r > 0$ is the radius of incircle, and $d \geq 0$ is the distance between the incenter and circumcenter.

It is well-known that the triple (R, r, d) of a Fuss pair of order k satisfies an algebraic relation. For $k = 3$, it is the classical *Euler triangle formula* [1]: $R^2 - d^2 = 2Rr$.

For $k \geq 4$, this relation is called *Fuss's relation* and reads as:

$$\frac{1}{(R+d)^2} + \frac{1}{(R-d)^2} = \frac{1}{r^2}. \tag{1}$$

Analogous relations for $k > 4$ are called *generalized Fuss's relations.* They are explicitly known for many values of k and suggest a number of interesting and difficult problems.

Another important aspect of bicentric polygons is their relation to the so-called *Poncelet porism* (PP) [1]. Recall that Poncelet porism states that if a pair of ellipses (E_1, E_2) is such that there exists a k-gon inscribed in E_1 and circumscribed about E_2, then for each point p of E_1, there exists such a k-gon having p as its vertex (see, e.g., [1]).

Definition 2.2. A pair of ellipses (E_1, E_2) such that there exists a k-gon inscribed in E_1 and circumscribed about E_2 is called a *Poncelet pair of ellipses of order* k. The set of all such k-gons is called *Poncelet family of* k*-gons* $\mathcal{P}(E_1, E_2)$ defined by (E_1, E_2).

Thus a Fuss pair of circles of order k is a particular case of the notion of Poncelet pair of ellipses. It follows that if P is a bicentric k-gon with the frame (C_1, C_2), then there exists a whole one-dimensional family of bicentric k-gons P_t with the same frame. In fact, each point of C_1 is a vertex of such a bicentric k-gon. This family of polygons will be called the poristic family $\mathcal{P}(C_1, C_2)$ of a Fuss pair (C_1, C_2).

Notice that the two problems formulated above for bicentric polygons also make sense in the context of a general Poncelet family $\mathcal{P}(E_1, E_2)$, where (E_1, E_2) is a pair of ellipses. With a view towards further developments, in Section 4 we establish some auxiliary results in this general context and then apply them to the special case of confocal ellipses.

Our main results are concerned with the Fuss pairs of circles of order four and the Poncelet pairs of confocal ellipses of order four. In particular, we find the absolute maximum and minimum of A on E for Fuss pairs of order 4 (Theorems 5.3) and for Poncelet pairs of confocal ellipses (Theorem 6.4).

3. Planar Moduli Spaces of Bicentric Quadrilaterals

We begin with definitions and results concerned with polygonal linkages [5]. Recall that a polygonal linkage $L(l)$ is defined by a collection of positive numbers l called sidelengths of $L(l)$. In particular, any polygon P defines a polygonal linkage $L(P)$ with the sidelengths equal to the lengths of the sides of P. In the sequel, we will consider a family of linkages $L_t = L(P_t)$ generated by a poristic family of polygons P_t introduced in the previous section.

For any polygonal linkage L, its planar moduli space $M(L)$ is defined as the set of its realizations in $\mathbb{R}^2$ taken modulo the group of orientation preserving isometries of $\mathbb{R}^2$ [5]. The moduli spaces $M(L)$ have natural structures of compact real algebraic varieties. For a generic sidelength vector l, $M(L)$ is a smooth compact manifold.

In particular, a quadrilateral (4-bar) linkage $Q = Q(l)$ is defined by a quadruple of positive numbers $l = (a, b, c, d) \in \mathbb{R}^4_+$ and its planar moduli space $M(L)$ is a one-dimensional algebraic manifold, hence a collection of algebraic arcs. The list of possible topological types of planar moduli spaces of 4-bar linkages is well known (see, e.g., [5]).

Proposition 3.1. *The complete list of homeomorphy types of planar moduli spaces of a* 4*-bar linkages is: circle, disjoint union of two circles, bouquet of two circles, union of two circles with two common points, union of two circles with three common points.*

Moreover, the topological type of moduli space $M(Q)$ can be easily read off its sidelengths [5]. Moduli space is non-singular if and only if the sidelengths satisfy the so-called *Grashof condition*, i.e. for any choice of signs, the sum $a \pm b \pm c \pm d$ does not vanish. This means that Q does not have aligned configurations. A non-singular moduli space is homeomorphic to a circle or disjoint union of two circles.

In our setting we have to deal with singular moduli spaces. To this end, recall that a *kite* is defined as a quadrilateral with sidelengths of the form (a, a, b, b). A kite with $a = b$ is called a rhombus. The following results on singular moduli spaces are given in [5].

Proposition 3.2. *The planar moduli space of a kite is homeomorphic to a union of two circles having two different points in common. The planar moduli space of a rhomboid is homeomorphic to a union of two circles having three different points in common. For any quadrilateral linkage which is not a kite but does not satisfy Grashof condition, the planar moduli space is homeomorphic to a bouquet of two circles.*

To answer question (Q1) for quadrilateral linkages we need a few additional observations, the first of which is a well-known result of elementary geometry.

Proposition 3.3. *The sums of opposite sides of tangential quadrilateral are equal.*

According to the above-said, this means that sidelengths of bicentric quadrilaterals do not satisfy the Grashof condition, which implies the following conclusion.

Corollary 3.4. *The planar moduli spaces of bicentric quadrilaterals contain singular points.*

We also use two obvious observations concerned with poristic quadrilaterals:

(1) if the circles of a Fuss pair of order four are concentric, then all poristic quadrilaterals are congruent to a square;

(2) each poristic family of bicentric quadrilaterals contains a kite.

Combining these results and observations presented above, we immediately obtain an answer to question (Q1).

Theorem 3.5. *For a concentric pair of Fuss circles of order four, all moduli spaces of poristic quadrilaterals are homeomorphic to a union of two circles having three different points in common. The planar moduli space of a bicentric quadrilateral with $d \neq 0$ is homeomorphic either to a bouquet of two circles, or to two circles with two common points.*

In fact, it is easy to show that each poristic family of quadrilaterals contains exactly two kites, so the last topological type mentioned in the theorem appears twice.

Thus question (Q1) has a quite satisfactory answer for bicentric quadrilaterals and it is now natural to wonder if similar results are available for bicentric polygons with more than four sides. However, the situation becomes much more complicated already in the case of bicentric pentagons. In particular, Corollary 3.4 is specific for quadrilaterals and need not hold for bicentric polygons with the number of sides bigger than four.

To see this, consider a concentric Fuss pair of order five formed by the incircle and circumcircle of a regular pentagon. In this case, all poristic pentagons P_t are congruent to a regular pentagon. As is well-known, the planar moduli space of a regular pentagon is non-singular and diffeomorphic to a two-sphere with four handles, [5] so an analog of Corollary 3.4 does not, in general, hold for pentagons.

Further comments on bicentric polygons having more than four sides are presented in the last section. In the rest of this paper we concentrate on question (Q2) for bicentric quadrilaterals and then present analogous results for poristic quadrilaterals arising from Poncelet pair of confocal ellipses of order four.

4. Auxiliary Results on Poncelet Porism

Question (Q2) requires some analytic considerations, so we present now some auxiliary formulas which will be used in the sequel. Let (E_1, E_2) be a Poncelet pair of ellipses with $E_2 \subset \operatorname{int} E_1$. Without loss of generality, we can assume that their equations are given in canonical form as

$$E_1 = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}, \;\; E_2 = \left\{ \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1 \right\}, \;\; a > c, \;\; b > d. \qquad (2)$$

For further use we give an analytical description of one step of Poncelet process. To this end, we consider a point $p(u, v) \in E_1$, denote by $T_+(p, E_2)$ the tangent line to E_2 through p going in positive (counterclockwise) direction, and by $q(s, t) = T_+(p, E_2) \cap E_2$ the point of tangency. In other words,

$T_qE_2 = T_+(p, E_2)$. Let $r(z, w)$ denote the second point of intersection of $T_+(p, E_2)$ with E_1.

Our goal is to express s, t, z, w through u, v. This can be done as follows. Let (x, y) denote Cartesian coordinates of a variable point in the plane. Then by elementary analytic geometry, equation of T_qE_2 is $\frac{sx}{c^2} + \frac{tv}{d^2} = 1$. Since $q \in E_2$, we have the second equation on (s, t) of the form $\frac{s^2}{c^2} + \frac{t^2}{d^2} = 1$.

Since the first equation is linear in (s, t), we can eliminate one of them and substitute in the second equation which reduces the problem to solving a quadratic equation. In this way we obtain

$$s = \frac{a^2b^4c^2u + \sqrt{a^4b^8c^4u^2 - (b^4c^2u^2 + a^4d^2v^2)(a^4b^4c^2 - a^4d^2c^2v^2)}}{b^4c^2u^2 + a^4d^2v^2}, \quad (3)$$

$$t = \frac{a^4b^2d^2v + \sqrt{a^8b^4d^4v^2 - (a^4d^2v^2 + b^4c^2u^2)(a^4b^4d^2 - b^4c^2d^2u^2)}}{a^4d^2v^2 + b^4c^2u^2}. \quad (4)$$

To find coordinates of r, one has to solve the system

$$\frac{z^2}{a^2} + \frac{w^2}{b^2} = 1, \quad \frac{sx}{c^2} + \frac{tv}{d^2} = 1,$$

with values of (s, t) given by formulas (3), (4). This system has the same form as the previous one, so it can be solved for (z, w) in a similar way, and we finally obtain

$$z = \frac{a^2b^4c^2u + \sqrt{a^4b^8c^4u^2 - (b^4c^2u^2 + a^4d^2v^2)(a^4b^4c^2 - a^4d^2c^2v^2)}}{b^4c^2u^2 + a^4d^2v^2}, \quad (5)$$

$$w = \frac{a^4b^2d^2v + \sqrt{a^8b^4d^4v^2 - (a^4d^2v^2 + b^4c^2u^2)(a^4b^4d^2 - b^4c^2d^2u^2)}}{a^4d^2v^2 + b^4c^2u^2}, \quad (6)$$

where s and t are given by (3), (4).

In many cases these formulas become simpler if one chooses the starting point properly. In particular, this happens if a pair of Poncelet ellipses has some symmetry. Such a symmetry exists in two special cases: (1) when (E_1, E_2) is a Fuss pair of circles, and (2) when (E_1, E_2) is a Poncelet pair of confocal ellipses. In both these cases the above formulas will enable us to derive useful conclusions. For a Fuss pair of circles (C_1, C_2), the above formulas yield useful relations between metric elements of bicentric polygons with frame (C_1, C_2) one of which is presented below.

Let Q be a point of C_1. Then there are two tangent lines l_1 and l_2 to C_2 passing through Q. Let $T_1 = l_1 \cap C_2$ and $T_2 = l_2 \cap C_2$ be the corresponding points of tangency. Obviously, the two segments $[Q, T_1]$ and $[Q, T_2]$ have the same length which will be denoted t_Q. Let B_1 (B_2) denote the second intersection point of l_1 (l_2) with C_2. Denote by t_2 the biggest of lengths of two segments $[T_1, B_1]$ and $[T_2, B_2]$ and t_3 the other length. In this setting, formulas (3)–(6) yield the following result which was obtained by a different method in [10].

Proposition 4.1. *With these assumptions and notation one has:*

$$t_2 = \frac{2Rrt + \sqrt{D}}{r^2 + t^2}, \quad t_3 = \frac{2Rrt - \sqrt{D}}{r^2 + t^2},$$

where $t = t_Q$*,* $D = 4R^2r^2t^2 - r^2(r^2 + t^2)(2Rr + r^2 + t^2)$*.*

The form of the above expressions suggests that the two lengths t_2, t_3 are the roots of a certain quadratic equation. It is easy to show that this is really the case and write down its coefficients. It is also easy to show that the maximal and minimal value of t_Q are

$$t_M = \sqrt{(R+d)^2 - r^2}, \quad t_m = \sqrt{(R-d)^2 - r^2}.$$

In other words, the range of t_A is the segment $[t_m, t_M]$.

These formulas were used in [4] to investigate extremal problems for perimeters of poristic quadrilaterals. We refer to some of the results of [4] in the next section.

5. Extremal Values of Oriented Area for Bicentric Quadrilaterals

We proceed with investigation of question (Q2). As a first step, we consider an extremal problem for the area of bicentric polygons associated with a given Fuss pair of circles (C_1, C_2) with metric data (R, r, d). We will always assume that the coordinate system is such that the center of C_1 is at the origin and the center of C_2 lies on the positive semi-axis Ox. In other words, $O_1 = (0, 0)$, $O_2 = (d, 0)$. In this case we will speak of a *standard Fuss pair.*

Consider a standard Fuss pair of circles of order k. According to Definition 2.1, there exists a k-gon P with vertices on C_1 and sides tangent to the circle C_2. By the Poncelet theorem, there exists a one-dimensional family $\mathcal{P}(C_1, C_2) = \{P_t\}$ of k-gons with the same property. It is convenient to parameterize Poncelet polygons by the argument ϕ of the first point on the outer circle C_1. For simplicity we write t instead of ϕ.

For our purposes we need to find the maximum of area $A(t)$ of P_t for $t \in [0, 2\pi]$. Since in our situation $2A(t) = rp(t)$, where $p(t)$ is the perimeter of P_t, it suffices to find the maximal value of $p(t)$. As it has been shown in [4], the latter problem can be solved by using the formulas from Section 4 and some standard calculus.

According to [4], Proposition 4.1 combined with the Brahmagupta formula for the area of cyclic polygon yields the following formula for the perimeter of bicentric quadrilateral which is also given in [10].

Proposition 5.1. *If s denotes the length of tangent line to C_2 from the point $Q \in C_1$, then the perimeter of poristic quadrilateral having Q as one* *of its vertices is given by*

$$p(s) = 2\Big(s + \frac{r^2}{s} + \frac{4Rrs}{s^2 + r^2}\Big).$$

It remains to find zeros of the derivative $\frac{dp}{ds}$ and compare the values of p at critical points, which gives the following result [4].

Proposition 5.2. *The maximal value of perimeter $p(s)$ is equal to*

$$2\Big(\sqrt{R^2 - (r-d)^2} + \sqrt{R^2 - (r+d)^2}\Big).$$

We conclude that the maximum value of $A(t)$ is attained on the polygon P_0 and it is equal to $r\big(\sqrt{R^2 - (r-d)^2} + \sqrt{R^2 - (r+d)^2}\big)$. We can now give an answer to question (Q2) using the results on cyclic configurations of polygonal linkages obtained in [8]. Recall that a polygon is called *cyclic* if it has a circumscribed circle. In particular, any bicentric polygon is cyclic.

As was shown in [8], the maximal value of oriented area on the planar moduli space of polygonal linkage L is attained on a convex cyclic configuration of L. Since each P_t is bicentric, the maximal value of A on each $M(P_t)$ is attained on P_t itself. Hence the maximal value of A on E coincides with the maximum of $A(t)$ for $t \in [0, 2\pi]$. Combining these observations with Proposition 5.2, we obtain the desired answer to question (Q2).

Theorem 5.3. *For $k = 4$, the absolute maximum of A on E is attained at P_0 and the absolute minimum is attained at the same quadrilateral P_0 taken with reversed orientation. The extremal values are*

$$\pm r\Big(\sqrt{R^2 - (r-d)^2} + \sqrt{R^2 - (r+d)^2}\Big).$$

As an easy consequence, we get the following criterion.

Corollary 5.4. *The function A is constant on E if and only if the circles C_1, C_2 are concentric.*

Analogs of these results for bicentric k-gons with $k \geq 5$ are seemingly much more difficult to obtain and we make no attempt to discuss them here. However, it appeared possible to obtain analogs of these results for poristic quadrilaterals associated with a pair of confocal Poncelet ellipses, which are presented in the next section.

6. Poristic Quadrilaterals for Confocal Poncelet Ellipses

Let (E_1, E_2) be a pair of confocal ellipses. Without loss of generality, we can assume that their equations are given in canonical form as

$$E_1 = \Big\{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\Big\}, \quad E_2 = \Big\{\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1\Big\},$$

where $a^2 - b^2 = c^2 - d^2$ (as is well known, the latter equality expresses the confocality condition).

Let us now assume that (E_1, E_2) is a Poncelet pair of order 4. We obtain a relation between parameters (a, b, c, d) which can be considered as an analog of Fuss's relation. We also describe configurations which are extremal for the area of Poncelet quadrilaterals defined by (E_1, E_2).

To derive the desired relation between (a, b, c, d), we notice first that, by the symmetry, the Poncelet quadrilateral with the first vertex $(a, 0)$ is a rhombus with vertices $(a, 0)$, $(0, b)$, $(-a, 0)$, $(0, -b)$. Hence the line $L_1 = \{(x, y) : (a - x)b - ay = 0\}$ connecting vertices $(a, 0)$ and $(0, b)$ should be the tangent to E_2 at certain point $p_1 = (s, t)$. In other words, the intersection of L_1 and E_2 should consist of one point.

The system of two equations defining the coordinates of p_1 is:

$$(a - s)b - at = 0, \quad \frac{s^2}{c^2} + \frac{t^2}{d^2} = 1,$$

and we need to find the condition that it has exactly one real solution. Using the linearity of the first equation, we reduce this system to one quadratic equation with real coefficients and then write down the relation between (a, b, c, d) which expresses vanishing of the discriminant of the latter equation. Combining this relation with the confocality condition, we obtain the sought analog of the Fuss's fourth relation.

Reduction to quadratic equation is obtained by substituting $t = \dfrac{(a - s)b}{a}$ into equation of E_2 which gives the following quadratic equation on s:

$$(a^2d^2 + b^2c^2)s^2 - 2ab^2c^2x + a^2c^2(b^2 - d^2) = 0.$$

Vanishing of its discriminant gives the following relation between (a, b, u, v):

$$a^2b^2c^4 - a^2c^2(a^2d^2 + b^2c^2)(b^2 - d^2) = 0.$$

Combining this condition with the condition of confocality and excluding u from the arising system, we get the equation for v of the form $(a^2+b^2)d^4 - b^4d^2 = 0$, which gives $d^2 = \dfrac{b^4}{a^2 + b^2}$. From the confocality condition we also get $c^2 = \dfrac{a^4}{a^2 + b^2}$, which gives the following result.

Proposition 6.1. *The semi-axes of a pair of confocal Poncelet ellipses of order four satisfy the following relations:*

$$c = \frac{a^2}{\sqrt{a^2 + b^2}}, \quad d = \frac{b^2}{\sqrt{a^2 + b^2}}. \tag{7}$$

In fact, these equations give also a sufficient condition.

Proposition 6.2. *If the relations* (7) *are satisfied for a pair of confocal ellipses, then these ellipses form a Poncelet pair of order four.*

Indeed, under these conditions the rectangular with vertices

$$\Big(u, \sqrt{b^2\Big(1-\frac{u^2}{a^2}\Big)}\Big), \quad \Big(-u, \sqrt{b^2\Big(1-\frac{u^2}{a^2}\Big)}\Big),$$
$$\Big(-u, -\sqrt{b^2\Big(1-\frac{u^2}{a^2}\Big)}\Big), \quad \Big(u, -\sqrt{b^2\Big(1-\frac{u^2}{a^2}\Big)}\Big)$$

is tangent to the inner ellipse. Hence it is a Poncelet quadrilateral for (E_1, E_2).

These two propositions show that equations (7) should be indeed considered as an analog of Fuss's relation. In fact, it is possible to rewrite this result in a form more similar to Fuss's relation. To this end, notice that up to a motion of the plane, a pair of confocal ellipses is defined by three positive numbers $2c$ (distance between foci), $L > 2c$ (sum of distances to foci for E_1) and $l \in (2c, L)$ (sum of distances to foci for E_2). Since the sum of distances to foci is equal to doubled big semi-axis, equations 7 give the following relation between (c, L, l):

$$L^4 - L^2 l^2 + 2c^2 l^2 = 0$$

which may be considered as a direct analog of Fuss's relation.

For such a pair of ellipses, sidelengths of poristic polygons do not satisfy the relation for circumscribed quadrilateral given in Proposition 3.3. Nevertheless, it turns out that their moduli spaces are always singular and one has an analog of Theorem 3.5.

To this end, we use the results of [2] about Poncelet quadrilaterals of confocal ellipses. In particular, it was proved in [2] that in this case all poristic quadrilaterals are parallelograms (cf. also [9]). This implies that their sidelengths do not satisfy Grashof condition. Hence their planar moduli spaces are singular and one may again use the description of singular moduli spaces given in Proposition 3.1. In this way, one obtains a direct analog of Theorem 3.5.

Theorem 6.3. *For a pair of confocal Poncelet ellipses of order four, all moduli spaces of poristic quadrilaterals are singular. In each such family we have two homemorphy types of planar moduli space: a bouquet of two circles and a union of two circles having three different points in common.*

We can also answer question (Q2) for poristic quadrilaterals associated with confocal ellipses. It is known that in this situation perimeter of poristic quadrilaterals $p(t)$ is constant and equal to $4\sqrt{a^2+b^2}$ [9] (which according to [1] is four times the radius of the orthoptic circle of E_1). Let P_t be a poristic quadrilateral which is a parallelogram with sides c and d. The maximum value of oriented area A on the moduli space of $L(P_t)$ is obviously equal to cd. The sum $c + d$ is constant since it is equal to the semi-perimeter of P_t. Hence the product cd is maximal when $c = d$, i.e. when P_t is a rhombus. Notice that the poristic quadrilateral P_0 is always a rhombus, so the maximum of A on E is attained on the cyclic configuration of $L(P_0)$ which is a square with the side equal to $\sqrt{a^2+b^2}$. Thus we obtain the following analog of Theorem 5.3.

Theorem 6.4. *For a pair of confocal Poncelet ellipses of order four, the absolute maximum of A on E is equal to a^2+b^2 and is attained at the cyclic configuration of rhomboid $L(P_0)$. The absolute minimum $-(a^2+b^2)$ is attained at the same configuration taken with the opposite orientation.*

7. Concluding Remarks

An obvious perspective suggested by our results is to look for their analogs for poristic polygons with the number of sides bigger than four. This is an interesting problem already for bicentric pentagons and we add a few words about this case.

As follows from the remarks at the end of Section 3, for a concentric Fuss pair of order five the total space E of the fibration considered above is a compact smooth manifold, diffeomorphic to a direct product of circle S^1 and Riemann surface of genus four. By continuity, the homeomorphic type of E will remain unchanged for any Fuss pair of order five with sufficiently small d. However, it is unclear if the same topological type of E will be preserved for all admissible values of d. It is also unclear what topological types can arise for moduli spaces of poristic pentagons with $d \neq 0$.

For pentagons, there is good evidence that the absolute maximum of A on E is attained at poristic pentagon P_0. If E appears to be smooth, the standard topological reasoning implies that any differentiable function on E should have sufficiently many critical points and one may wish to find their amount and types. This issue becomes especially interesting if the function considered is a Morse function. In view of results of [6], [7], one may hope to prove that the oriented area function on E is a Morse function and estimate the number and types of its critical points on E.

The same comments are applicable to any regular polygon with odd number of sides. In particular, there exists considerable amount of information about homology groups of moduli spaces of regular polygons which may be used to investigate the critical points of A on E.

Several aspects of these problems can be successfully investigated using the results of this paper, but such developments obviously require a separate publication.

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