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ON STATISTICAL STRUCTURES IN A POLISH NON-LOCALLY-COMPACT GROUP ADMITTING AN INVARIANT METRIC

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Abstract. By using the notion of a Haar ambivalent set, introduced by Balka, Buczolich and Elekes in 2012, essentially new classes of statistical structures having objective and strong objective estimates of an unknown parameter are considered in a Polish non-locally-compact group admitting an invariant metric, and relations between them are studied. An example of such a weakly separated statistical structure is constructed for which a question *whether there exists a consistent estimate of an unknown parameter* remains unsolvable within the theory (ZE) & (DC). These results extend those obtained recently by Pantsulaia and Kintsurashvili in 2014.

რეზიუმე. 2012 წელს ბალკა, ბუკზოლიჩისა და ელეკეშის მიერ შემოტანილი ჰაარის ემბივალენტი სიმრავლის ცნების საშუალებით ინვარიანტული მეტრიკით აღჭურვილ არალოკალურად კომპაქტურ პოლონურ ჯგუფზე განხილულია სტატისტიკური სტრუქტურების არსებითად ახალი კლასები, რომელთაც გააჩნიათ უცნობი პარამეტრის ობიექტური და ძლიერად ობიექტური ძალდებული შეფასებები და შესწავლილია მიმართება მათ შორის. აგებულია მაგალითი ისეთი სუსტად განცალებადი სტატისტიკური სტრუქტურისა რომლისთვისაც უცნობი პარამეტრის ძალდებული შეფასების არსებობის ამოცანა ამოუხსნადია (ZF) & (DC) თეორიაში. ეს შედეგები აძლიერებენ ფანცულაიას და კინწურაშვილის მიერ 2014 წელს მიღებულ შედეგებს.

1. Introduction

The notion of a Haar null set introduced by Christensen [2] and reintroduced in the context of dynamical systems by Hunt, Sauer and Yorke [4],

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has been used by Pantsulaia and Kintsurashvili [9] to introduce concepts of the so-called objective and strong objective infinite sample consistent estimates of a useful signal in the linear one-dimensional stochastic model. The purpose of the present paper is to extend these concepts to all Polish non-locally-compact groups admitting an invariant metric. Notice that a suitable extension of the property of being of a Haar null set in abelian Polish groups [2] to all non-abelian Polish groups was given by Topsøe and Hoffmann-Jørgens [14] and Mycielski [6] such that this class again constitutes an σ-ideal and coincides with Christensen's class of Haar null sets for abelian Polish groups.

The rest of the paper is organized as follows.

In Section 2, by virtue of the notion of a Haar ambivalent set introduced by Balka, Buczolich and Elekes in [1], essentially new classes of statistical structures having objective and strong objective estimates of an unknown parameter are introduced in a Polish non-locally-compact group admitting an invariant metric.

In Section 3, by using celebrated results of Mycielski and Swierczkowski[5], Solovay [13] and Skorokhod [15], we study relations between these statistical structures. These results extend those obtained recently by Pantsulaia and Kintsurashvili in [9], [10], [11]. By using wonderful results of Solecki [12] and Dougherty [3], in the same group we present some constructions of statistical structures having objective estimates of an unknown parameter.

2. On a Certain Classification of Statistical Structures on Polish Non-Locally-Compact Groups Admitting an Invariant Metric

Let G be a Polish group, by which we mean a separable group with a complete invariant metric ρ (i.e., $\rho(fh_1g, fh_2g) = \rho(h_1, h_2)$ for each $f, g, h_1, h_2 \in G$) for which the transformation (from $G \times G$ onto G) sending (x, y) into $x^{-1}y$ is continuous. Let $\mathcal{B}(G)$ denote the σ-algebra of Borel subsets of G.

Definition 2.1 ([6])**.** A Borel set $X \subseteq G$ is called shy, if there exists a Borel probability measure μ over G such that $\mu(fXg) = 0$ for all $f, g \in G$. A measure μ is called a testing measure for a set X. A subset of a Borel shy set is called shy, as well. The complement of a shy set is called a prevalent set.

Definition 2.2 ([1])**.** A Borel set is called a Haar ambivalent set if it is neither shy nor prevalent.

Remark 2.3. Notice that if $X \subseteq G$ is shy, then there exists such a testing measure μ for a set X with a compact carrier $K \subseteq G$(i.e. $\mu(G \setminus K) = 0$). The collection of shy sets constitutes an σ-ideal, and in the case where G is locally compact, a set is shy iff it has Haar measure zero.

Definition 2.4. If G is a Polish group and $\{\mu_\theta : \theta \in \Theta\}$ is a family of Borel probability measures on G, then the family of triplets $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$, where Θ is a non-empty set equipped with the minimal σ-algebra $L(\Theta)$ of subsets of Θ generated by all singletons of Θ, is called a statistical structure. A set Θ is called a set of parameters.

Definition 2.5. ($\mathcal{O}$) The statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is called orthogonal if the measures μ_{θ_1} and μ_{θ_2} are orthogonal for each different parameters θ_1 and θ_2.

Definition 2.6. ($\mathcal{WS}$) The statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is called weakly separated if there exists a family of Borel subsets $\{X_\theta : \theta \in \Theta\}$ such that $\mu_{\theta_1}(X_{\theta_2}) = \delta(\theta_1, \theta_2)$, where δ denotes Kronecker's function defined on the Cartesian square $\Theta \times \Theta$ of the set Θ.

Definition 2.7. ($\mathcal{SS}$) The statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is called strong separated (or strictly separated) if there exists a partition of the group G into a family of Borel subsets $\{X_\theta : \theta \in \Theta\}$ such that $\mu_\theta(X_\theta) = 1$ for each $\theta \in \Theta$.

Definition 2.8. ($\mathcal{CE}$) A $(\mathcal{B}(G), L(\Theta))$-measurable mapping $T : G \to \Theta$ is called a consistent estimate of an unknown parameter $\theta \in \Theta$ for the statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ if the condition $\mu_\theta(T^{-1}(\theta)) = 1$ holds true for each $\theta \in \Theta$.

Definition 2.9. ($\mathcal{OCE}$) A $(\mathcal{B}(G), L(\Theta))$-measurable mapping $T : G \to \Theta$ is called an objective consistent estimate of an unknown parameter $\theta \in \Theta$ for the statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ if the following two conditions hold:

(i) $\mu_\theta(T^{-1}(\theta)) = 1$ for each $\theta \in \Theta$;

(ii) $T^{-1}(\theta)$ is a Haar ambivalent set for each $\theta \in \Theta$.

If the condition (i) holds but the condition (ii) fails, then T is called a subjective consistent estimate of an unknown parameter $\theta \in \Theta$ for the statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$.

Definition 2.10. ($\mathcal{SOCE}$) An objective consistent estimate $T : G \to \Theta$ of an unknown parameter $\theta \in \Theta$ for the statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is called *strong* if for each $\theta_1, \theta_2 \in \Theta$ there exists an isometrical Borel measurable bijection $A_{(\theta_1,\theta_2)} : G \to G$ such that the set $A_{(\theta_1,\theta_2)}(T^{-1}(\theta_1)) \times \Delta T^{-1}(\theta_2)$ is shy in G.

3. Relations Between Statistical Structures in Polish Non-Locally-Compact Groups Admitting an Invariant Metric

Remark 3.1. Let G be a Polish non-locally-compact group admitting an invariant metric. The relations between statistical structures introduced in

Section 2 for such a group can be presented by the following diagram:

$$\mathcal{SOCE} \rightarrow \mathcal{OCE} \rightarrow \mathcal{CE} \leftrightarrow \mathcal{SS} \rightarrow \mathcal{WS} \rightarrow \mathcal{O}$$

To show that the converse implications sometimes fail, we consider the following examples.

Example 3.2. $\rceil(\mathcal{WS} \leftarrow \mathcal{O})$ Let $F \subset G$ be a closed subset of the cardinality $2^{\aleph_0}$. Let $\phi : [0,1] \rightarrow F$ be a Borel isomorphism of $[0,1]$ onto F. We set $\mu(X) = \lambda(\phi^{-1}(X \cap F))$ for $X \in \mathcal{B}(G)$, where λ denotes a linear Lebesgue measure on $[0,1]$. We put $\Theta = F$. Let fix $\theta_0 \in \Theta$ and put: $\mu_\theta = \mu$ if $\theta = \theta_0$, and $\mu_\theta = \delta_\theta|_{\mathcal{B}(G)}$, otherwise, where δ_θ denotes a Dirac measure on G concentrated at the point θ, and $\delta_\theta|_{\mathcal{B}(G)}$ denotes the restriction of the δ_θ to the class $\mathcal{B}(G)$. Then the statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ stands for $\mathcal{O}$ which is not $\mathcal{WS}$.

Example 3.3. (SM) $\rceil(\mathcal{SS} \leftarrow \mathcal{WS})$ Following [7] (see, Theorem 1, p. 335), in the system of axioms (ZFC) the following three conditions are equivalent:

1) the Continuum Hypothesis ($c = 2^{\aleph_0} = \aleph_1$);

2) for an arbitrary probability space $(E; S; \mu)$, the μ-measure of the union of any family $(E_i)_{i \in I}$ of μ-measure zero subsets such that $\mathrm{card}(I) < c$, is equal to zero;

3) an arbitrary weakly separated family of probability measures, of cardinality continuum, is strictly separated.

The latter relation means that under the Continuum Hypothesis in ZFC we have $\mathcal{SS} \leftarrow \mathcal{WS}$. This is just Skorohod well known result(see, [15]). Moreover, following [7] (see Theorem 2, p. 339), if (F, ρ) is a Radon metric space and $(\mu_i)_{i \in I}$ is a weakly separated family of Borel probability measures with $\mathrm{card}(I) \leq c$, then in the system of axioms $(ZFC)\&(MA)$, the family $(\mu_i)_{i \in I}$ is strictly separated.

Let us consider a counter-example to the implication $\mathcal{SS} \leftarrow \mathcal{WS}$ in the Solovay model (SM) [13] which is the following system of axioms: $(ZF)+$ $(DC)+$ "every subset of the real axis $\mathbf{R}$ is Lebesgue measurable", where (ZF) denotes the Zermelo-Fraenkel set theory and (DC) denotes the axiom of Dependent Choices.

For $\theta \in (0;1)$, let b_θ be a linear classical Borel measure defined on the set $\{\theta\} \times (0;1)$. For $\theta \in (1.2)$, let b_θ be a linear classical Borel measure defined on the set $(0;1) \times \{\theta - 1\}$. By λ_θ we denote a Borel probability measure on $(0;1) \times (0;1)$ produced by b_θ, i.e.,

$$(\forall X)(\forall \theta_1)(\forall \theta_2)(X \in \mathcal{B}((0;1) \times (0;1)) \ \& \ \theta_1 \in (0;1) \ \& \ \theta_2 \in (1;2) \rightarrow$$
$$\lambda_{\theta_1}(X) = b_{\theta_1}((\{\theta_1\} \times (0;1)) \cap X) \ \& \ \lambda_{\theta_2}(X) = b_{\theta_2}(((0;1) \times \{\theta_1 - 1\}) \cap X)).$$

If we put $\theta = (0;1) \cup (1;2)$, then we get a statistical structure $((0;1) \times (0;1), \mathcal{B}((0;1) \times (0;1)), \lambda_\theta)_{\theta \in \Theta}$.

Setting $X_\theta = \{\theta\} \times (0;1)$ for $\theta \in (0;1)$, and $X_\theta = (0;1) \times \{\theta - 1\}$ for $\theta \in (1.2)$, we observe that for the family of Borel subsets $\{X_\theta : \theta \in \Theta\}$ we have $\lambda_{\theta_1}(X_{\theta_2}) = \delta(\theta_1, \theta_2)$, where δ denotes Kronecker's function defined on the Cartesian square $\Theta \times \Theta$ of the set Θ. In other words, $(\lambda_\theta)_{\theta\in\Theta}$ is weakly separated. Now assume that this family is strong separated. Then there will be a partition $\{Y_\theta : \theta \in \Theta\}$ of the $(0;1) \times (0;1)$ into Borel subsets $(Y_\theta)_{\theta\in\Theta}$ such that $\lambda_\theta(Y_\theta) = 1$ for each $\theta \in \Theta$. If we consider $A = \cup_{\theta\in(0;1)} Y_\theta$ and $B = \cup_{\theta\in(1;2)} Y_\theta$, we observe by Fubini's theorem that $\ell_2(A) = 1$ and $\ell_2(B) = 1$, where ℓ_2 denotes the 2-dimensional Lebesgue measure defined on $(0;1)\times(0;1)$. This is the contradiction and we have proved that $(\lambda_\theta)_{\theta\in\Theta}$ is not strictly separated. An existence of a Borel isomorphism g between $(0;1) \times (0;1)$ and G allows us to construct a family $(\mu_\theta)_{\theta\in\Theta}$ in G as follows: $\mu_\theta(X) = \lambda_\theta(g^{-1}(X))$ for each $X \in \mathcal{B}(G)$ and $\theta \in \Theta$ which is $\mathcal{WS}$, but no $\mathcal{SS}$(equivalently, $\mathcal{CE}$).

By virtue the celebrated result of Mycielski and Swierczkowski (see, [5]) asserted that under the Axiom of Determinacy (AD) every subset of the real axis $\mathbf{R}$ is Lebesgue measurable, the same example can be used as a counter-example to the implication $\mathcal{SS} \leftarrow \mathcal{WS}$ in the theory $(ZF) + (DC) + (AD)$.

Since the answer to the question *whether* $(\mu_\theta)_{\theta\in\Theta}$ *has a consistent estimate* is *yes* in the theory (ZFC) & (AC), and *no* in the theory $(ZF) + (DC) + (AD)$, we deduce that this question is not solvable within the theory $(ZF) + (DC)$.

Example 3.4. $\rceil(\mathcal{OCE} \leftarrow \mathcal{CE})$ Setting $\Theta = G$ and $\mu_\theta = \delta_\theta|\mathcal{B}(G)$ for $\theta \in \Theta$, where δ_θ denotes a Dirac measure in G concentrated at the point θ and $\delta_\theta|\mathcal{B}(G)$ denotes its restriction to $\mathcal{B}(G)$, we get a statistical structure $(G, \mathcal{B}(G), \mu_\theta)_{\theta\in\Theta}$. Let $L(\Theta)$ denote the minimal σ-algebra of subsets of Θ generated by all singletons of Θ. Setting $T(g) = g$ for $g \in G$, we get a consistent estimate of an unknown parameter θ for the family $(\mu_\theta)_{\theta\in\Theta}$. Notice that there does not exist an objective consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta\in\Theta}$. Indeed, if we assume the contrary and T_1 be such an estimate, then we get that $T_1^{-1}(\theta)$ is a Haar ambivalent set for each $\theta \in \Theta$. Since T_1 is a consistent estimate of an unknown parameter θ for each $\theta \in \Theta$, we get that the condition $\mu_\theta(T_1^{-1}(\theta)) = 1$ holds true which implies that $\theta \in T_1^{-1}(\theta)$ for each $\theta \in \Theta$. Let us fix any parameter $\theta_0 \in \Theta$. Since $T_1^{-1}(\theta_0)$ is a Haar ambivalent set, there is $\theta_1 \in T_1^{-1}(\theta_0)$ which differs from θ_0. Then $T_1^{-1}(\theta_0)$ and $T_1^{-1}(\theta_1)$ are not disjoint because $\theta_1 \in T_1^{-1}(\theta_0) \cap T_1^{-1}(\theta_1)$, and we get the contradiction.

Remark 3.5. Notice that if (Θ, ρ) is a metric space and if in Definition 2.9 the requirement of a $(\mathcal{B}(G), L(\Theta))$-measurability will be replaced with a $(\mathcal{B}(G), \mathcal{B}(\Theta))$-measurability, then the implication $\mathcal{SS} \to \mathcal{CE}$ may be false. Indeed, let G be a Polish group and $f : G \leftarrow \Theta(:= G)$ be a non-measurable (in the Borel sense) bijection. For each $\theta \in \Theta$ denote by μ_θ the restriction of the Dirac measure $\delta_{f(\theta)}$ to the σ-algebra of Borel subsets of the group G. It is clear that the statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ is strictly separated. Let us show that there does not exist a consistent estimate for that statistical structure. Indeed, let $T : G \to \Theta$ be $(\mathcal{B}(G), \mathcal{B}(\Theta))$-measurable mapping such that $\mu_\theta(\{x : T(x) = \theta\}) = 1$ for each $\theta \in \Theta$. Since the measure μ_θ is concentrated at the point $f(\theta)$, we find that $f(\theta) \in \{x : T(x) = \theta\}$ for each $\theta \in \Theta$ which implies that $T(f(\theta)) = \theta$ for each $\theta \in \Theta$. The latter relation means that $T = f^{-1}$. Since f is not $(\mathcal{B}(G), \mathcal{B}(\Theta))$-measurable, we claim that $f^{-1} = T$ is not also $(\mathcal{B}(G), \mathcal{B}(\Theta))$-measurable, and we get the contradiction.

There naturally arises a question whether there exists such a statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in a Polish non-locally-compact group admitting an invariant metric which has an objective consistent estimate of a parameter θ. To answer positively to this question, we need the following two lemmas.

Lemma 3.6 ([12], Theorem, p. 206)**.** *Assume G is a Polish, non-locally-compact group admitting an invariant metric. Then there exists a closed set $F \subseteq G$ and a continuous function $\phi : F \to 2^N$ such that for any $x \in 2^N$ and any compact set $K \subseteq G$ there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$.*

Lemma 3.7 ([3] Proposition 12, p. 87)**.** *Let G be a non-locally-compact Polish group with an invariant metric. Then any compact subset* (*and hence any K_σ subset*) *of G is shy.*

Remark 3.8. In [10](see proof of Theorem 4.1, Step 2) has been constructed a partition $\Phi = \{A_\theta : \theta \in [0,1]\}$ of the $\mathbf{R}^N$ into Haar ambivalent sets such that for each $\theta_1, \theta_2 \in [0,1]$ there exists an isometric (with respect to Tychonoff metric which is invariant under translates) Borel measurable bijection $A_{(\theta_1,\theta_2)}$ of $\mathbf{R}^N$ such that $A_{(\theta_1,\theta_2)}(A_{\theta_1}) \Delta A_{\theta_2}$ is shy. In this context and concerning with Lemma 3.6 it is natural to ask whether an arbitrary Polish non-locally-compact group with an invariant metric admits a similar partition in Haar ambivalent sets. Notice that we have no any information in this direction.

Theorem 3.9. *Let G be a Polish non-locally-compact group admitting an invariant metric. Then there exists a statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in G which has an objective consistent estimate of a parameter θ such that:*

(i) *$\Theta \subseteq G$ and card$(\Theta) = 2^{\aleph_0}$;*

(ii) *μ_θ is the restriction of the Dirac measure concentrated at the point θ to the Borel σ-algebra $\mathcal{B}(G)$ for each $\theta \in \Theta$.*

Proof. By virtue of Lemma 3.6, there exists a closed set $F \subseteq G$ and a continuous function $\phi : F \to 2^N$ such that for any $x \in 2^N$ and any compact set $K \subseteq G$ there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$.

For $x \in 2^N \setminus \{(0,0,\dots)\}$, we put $X_x = \phi^{-1}(x)$. We set $X_{(0,0,\dots)} = \phi^{-1}((0,0,\dots)) \cup (G \setminus F)$. Thus we have a partition $\{X_x : x \in 2^N\}$ of G into Borel subsets such that each element of the partition is Borel measurable and a Haar ambivalent set. Let $\{\theta_x : x \in 2^N\}$ be any selector. We put $\Theta = \{\theta : \theta = \theta_x \text{ for some } x \in 2^N\}$ and denote by μ_θ the restriction of the Dirac measure concentrated at the point θ to the σ-algebra $\mathcal{B}(G)$. Thus we have constructed a statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in G. We put $T(g) = \theta$ for each $g \in X_\theta$. Now it is obvious that T is the objective consistent estimate of a parameter θ for the statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in G such that the conditions (i)-(ii) are fulfilled. □

Theorem 3.10. *Let G be a Polish non-locally-compact group admitting an invariant metric. Let μ be a Borel probability measure whose carrier is a compact set K_0(i.e., $\mu(G \setminus K_0) = 0$). Then there exists a statistical structure $\{(G, \mathcal{B}, \mu_\theta) : \theta \in \Theta\}$ in G which has an objective consistent estimate of a parameter θ such that*

(i) $\Theta \subseteq G$ *and* $card(\Theta) = 2^{\aleph_0}$;

(ii) μ_θ *is a θ-shift of the measure μ (i.e. $\mu_\theta(X) = \mu(\theta^{-1}X)$ for $X \in \mathcal{B}(G)$ and $\theta \in \Theta$).*

Proof. By virtue of Lemma 3.6, there exists a closed set $F \subseteq G$ and a continuous function $\phi : F \to 2^N$ such that for any $x \in 2^N$ and any compact set $K \subseteq G$ there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$.

For $x \in 2^N \setminus \{(0,0,\dots)\}$, we put $X_x = \phi^{-1}(x)$. We set $X_{(0,0,\dots)} = \phi^{-1}((0,0,\dots)) \cup (G \setminus F)$. Thus we have a partition $\{X_x : x \in 2^N\}$ of G into Borel subsets such that each element of the partition is Borel measurable, a Haar ambivalent set and for any $x \in 2^N$ and any compact set $K \subseteq G$ there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$.

If we take under K a set K_0, then for any $x \in 2^N$ there is $g(K_0, x) \in G$ with $g(K_0, x)K_0 \subseteq X_x$.

We put $\Theta = \{\theta : \theta = g(K_0, x) \;\&\; x \in 2^N\}$. For each $\theta \in \Theta$ and $X \in \mathcal{B}(G)$, we put $\mu_\theta(X) = \mu(\theta^{-1}X)$.

For $g \in X_x$ we put $T(g) = g(K_0, x)$. Let us show that $T : G \to \Theta$ is an objective consistent estimate of a parameter θ. Indeed, on the one hand, for each $\theta \in \Theta$, we have

$$\mu_\theta(T^{-1}(\theta)) = \mu_{g(K_0,x)}(T^{-1}(g(K_0,x))) = \mu_{g(K_0,x)}(X_x) =$$
$$= \mu(g(K_0,x)^{-1}X_x) \geq \mu(g(K_0,x)^{-1}g(K_0,x)K_0) = \mu(K_0) = 1,$$

which means that $T : G \to \Theta$ is a consistent estimate of a parameter θ.

On the other hand, for each $\theta = g(K_0, x) \in \Theta$, we have that a set $T^{-1}(\theta) = T^{-1}(g(K_0, x)) = X_x$ is Borel measurable and a Haar ambivalent set which together with the latter relation implies that $T : G \to \Theta$ is an objective consistent estimate of a parameter θ. Now it is obvious to check that for the statistical structure $\{(G,\mathcal{B},\mu_\theta) : \theta \in \Theta\}$ the conditions (i)-(ii) are fulfilled. □

The next theorem shows whether one can construct objective consistent estimates by virtue of some consistent estimates in a Polish non-locally-compact group admitting an invariant metric.

Theorem 3.11. *Let G be a Polish non-locally-compact group admitting an invariant metric. Let $card(\Theta) = 2^{\aleph_0}$ and $T : G \to \Theta$ be a consistent estimate of a parameter θ for the family of Borel probability measures $(\mu_\theta)_{\theta\in\Theta}$ such that there exists $\theta_0 \in \Theta$ for which $T^{-1}(\theta_0)$ is a prevalent set. Then there exists an objective consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta\in\Theta}$.*

Proof. For $\theta \in \Theta$ we put $S_\theta = T^{-1}(\theta)$. Since S_{θ_0} is a prevalent set we deduce that

$$\cup_{\theta\in\Theta\setminus\{\theta_0\}} S_\theta = \mathbf{R}^N \setminus S_{\theta_0}$$

is shy in G.

We know that the measure μ_{θ_0} is concentrated on a union of a countable family of compact subsets $\{F_k^{(\theta_0)} : k \in N\}$. By Lemma 3.7 we know that $\cup_{k\in N} F_k^{(\theta_0)}$ is shy in G.

We put $\tilde{S}_\theta = S_\theta$ for $\theta \in \Theta \setminus \{\theta_0\}$ and $\tilde{S}_{\theta_0} = \cup_{k\in N} F_k^{(\theta_0}$. Clearly, $S = \cup_{\theta\in\Theta} \tilde{S}_\theta$ is also shy in G.

By virtue of Lemma 3.6, there exist a closed set $F \subseteq G$ and a continuous function $\phi : F \to 2^N$ such that for any $x \in 2^N$ and any compact set $K \subseteq G$ there is $g \in G$ with $gK \subseteq \phi^{-1}(x)$. Let $f : 2^N \to \Theta$ be any bijection. For $\theta \in \Theta$, we put

$$B_\theta = (\phi^{-1}(f^{-1}(\theta)) \setminus S) \cup S_\theta.$$

Notice that $(B_\theta)_{\theta\in\Theta}$ is a partition of G into Haar ambivalent sets. We put $T_1(g) = \theta$ for $g \in B_\theta(\theta \in \Theta)$. Since

$$\mu_\theta(T_1^{-1}(\theta)) = \mu_\theta(B_\theta) \geq \mu_\theta(S_\theta) = 1$$

for $\theta \in \Theta$, we claim that T_1 is a consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta\in\Theta}$.

Since $T_1^{-1}(\theta)) = B_\theta$ is a Haar ambivalent set for each $\theta \in \Theta$, we complete the proof of the theorem. □

Example 3.12. Let F be a probability distribution function on $\mathbf{R}$ such that the integral $\int_{\mathbf{R}} x dF(x)$ exists and is equal to zero. Suppose that p is a Borel probability measure on $\mathbf{R}$ defined by F. For $\theta \in \Theta(:= \mathbf{R})$, let p_θ be θ-shift of the measure p (i.e., $p_\theta(X) = p(X - \theta)$ for $X \in \mathcal{B}(\mathbf{R})$). Setting $G = \mathbf{R}^N$, for $\theta \in \Theta$ we put $\mu_\theta = p_\theta^N$, where p_θ^N denotes the infinite power of the measure p_θ. We set $T((x_k)_{k\in N} = \lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n}$, if $\lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n}$ exists, is finite and differs from the zero, and $T((x_k)_{k\in N}) = 0$, otherwise. Notice that $T : \mathbf{R}^N \to \Theta$ is a consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta\in\Theta}$ such that $T^{-1}(0)$ is a prevalent set. Indeed, by virtue the Strong Law of Large Numbers, we know that

$$\mu_\theta\left\{(x_k)_{k\in N} : \lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n} = \theta\right\} = 1$$

for $\theta \in \Theta$.

Following [9] (Lemma 4.14, p. 60), a set S, defined by

$$S = \left\{(x_k)_{k\in N} : \lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n} \text{ exists and is finite}\right\},$$

is a Borel shy set, which implies that $\mathbf{R}^N \setminus S$ is a prevalent set. Since $\mathbf{R}^N \setminus S \subseteq T^{-1}(0)$, we deduce that $T^{-1}(0)$ is a prevalent set. Since for the statistical structure $\{(\mathbf{R}^N, \mathcal{B}(\mathbf{R}^N), \mu_\theta) : \theta \in \Theta\}$ all conditions of Theorem 3.11 are fulfilled, we claim that there exists an objective consistent estimate of a parameter θ for the family $(\mu_\theta)_{\theta\in\Theta}$.

In [9], in the case of the linear one-dimensional stochastic model examples of *objective* and *strong objective* infinite sample consistent (well-founded) estimates ([9], $T^\star$(p. 63), T° (p. 67)) of a useful signal were constructed by using the axiom of choice and a certain partition of the non-locally compact abelian Polish group $\mathbf{R}^N$ constructed in [8]. In [11], it has been proved that infinite-sample consistent estimates of an unknown parameter effectively constructed in [16] are objective.

The next example presents a certain effective construction of the statistical structure in $\mathbf{R}^N$ which has a strong objective infinite-sample consistent estimate of an unknown parameter.

Example 3.13 ([10] Theorem 3.1, p. 117)**.** Let F be a strictly increasing continuous probability distribution function on $\mathbf{R}$, μ be a Borel probability measure on $\mathbf{R}$ defined by F, $F_\theta(x) = F(x - \theta)(x \in \mathbf{R})$ for $\theta \in \Theta := [0, 1]$ and μ_θ be a Borel probability measure on $\mathbf{R}$ defined by F_θ.

For each real number $a \in \mathbb{R}$, we denote by $\{a\}$ its fractal part in the decimal system. Suppose that the Borel probability measure λ, defined by the sequence of transformed signals $(\xi_k)_{k\in\mathbb{N}}$ coincides with $\left(\mu^{\mathbb{N}}\right)_{\theta_0}$ for some $\theta_0 \in [0, 1]$. Let $T : \mathbb{R}^{\mathbb{N}} \to [0, 1]$ be defined by: $T((x_k)_{k\in\mathbb{N}}) = \{\lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n}\}$ if $\lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n} \neq 1$, $T((x_k)_{k\in\mathbb{N}}) = 1$ if $\lim_{n\to\infty} \frac{\sum_{k=1}^n x_k}{n} = 1$, and $T((x_k)_{k\in\mathbb{N}}) = \sum_{k\in\mathbb{N}} \frac{\chi_{(0,+\infty)}(x_k)}{2^k}$, otherwise, where $\chi_{(0,+\infty)}(\cdot)$ denotes an indicator function of the set $(0, +\infty)$ defined on the real axis $\mathbb{R}$. Then T is a strong objective infinite sample consistent estimate for the statistical structure $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \mu_\theta^{\mathbb{N}})_{\theta\in[0,1]}$.

In context with Example 3.13 we state the following

Problem 3.14. Let G be a Polish non-locally-compact group admitting an invariant metric. Does there exist a statistical structure $\{(G, \mathcal{B}(G), \mu_\theta) : \theta \in \Theta\}$ with $\text{card}(\Theta) = 2^{\aleph_0}$ for which there exists a strong objective consistent estimate of a parameter θ?

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