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PROBLEMS OF STATICS OF TWO-COMPONENT ELASTIC MIXTURES FOR A HALF-SPACE

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Abstract. In this paper we consider boundary value problems of statics of two-component elastic mixtures for a half-space, when limiting values of the tangential components of partial displacement vectors and normal components of partial stress vectors are given on the boundary. We consider also BVP_s, when limiting values of normal components of partial displacement vectors and tangent components of partial stress vectors are given on the boundary. We develop a new approach which is based on explicit forms of solutions of the Dirichlet and Neumann boundary value problems for the Laplace equation for a half-space.The uniqueness theorems are proved. Solutions are represented in quadratures.

რეზიუმე. ნაშრომში განხილულია ორკომპონენტიანი დრეკადი ნარევის სასაზღვრო ამოცანები ნახევარსივრცისთვის, როდესაც საზღვარზე მოცემულია კერძო გადაადგილებების მხები მდგენელებისა და კერძო ძაბვების ნორმალური მდგენელების ზღვრული მნიშვნელობები. ჩვენ ასევე ვიხილავთ სასაზღვრო ამოცანას, როდესაც საზღვარზე მოცემულია კერძო გადაადგილებების ნორმალური მდგენელებისა და კერძო ძაბვების მხები მდგენელების ზღვრული მნიშვნელობები. ჩვენ ვსარგებლობთ ლაპლასის განტოლებისათვის, ნახევარსივრცის შემთხვევაში, დირიხლესა და ნეიმანის სასაზღვრო ამოცანებისა ამოხსნებით. დამტკიცებულია ერთადერთობის თეორემები. ამოხსნები წარმოდგენილია კვადრატურებში.

1. Introduction

In the early 60_s of the last century, C. Truesdell and R. Toupin formulated in [25] the fundamental mechanical principles of a new model of a deformable elastic medium with a complex inner structure and thereby

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laid the foundation for the continual theory of elastic mixtures. In subsequent years this theory was generalized and developed in different directions. Based on kinematic and thermodynamic principles, theories were created for two- and many-component mixtures such as fluid-fluid (Crochet and Naghdi [9], Atkin [2],Green and Naghdi [13], [14], Green and Steel [12], and solid body-solid body (Crochet and Naghdi [9], Atkin [2], Green and Steel [12], Khoroshun and Soltanov [16], Hill [15]).

In the paper by Natrosvili, Jaghmaidze and Svanadze [20], static and dynamic problems on the linear theory of a mixture of two isotropic elastic components are investigated by the method of a potential and singular integral equations. Atkin, Chadvick and Steel [3] and Knops and Steel [17] deal with the uniqueness theorems for various linearized dynamic problems of the theory of anizotropic mixtures.

The questions on the existence and uniqueness of weak solutions of mixed static linear problems for mixtures of two nonhomogeneous anisotropic components were considered by Aron [1] and Borrelli and Patria [6], in the former work, the problem was studied by the method of functional analysis, while in the latter, by the variational method. In Khoroshun and Soltanov's monograph [16], along with theoretical questions, quite interesting concrete problems of thermoelasticity were considered for two-component mixtures.

For a wider overview of the subject (half-space) area of applications we refer to the references due to J. Barber [4], M. Basheleishvili, L. Bitsadze [5], D. Burchuladze, M. Kharashvili, K. Skhvitaridze [7], E. Constantin, N. Pavel [8], L.Giorgashvili, K. Skhvitaridze, M. Kharashvili [10], L. Giorgashvili, E. Elerdashvili, M. Kharashvili, K. Skhvitaridze [11], R. Kumar, T. Chadha [18], H. Sherief, H.Saleh [21], B. Singh, R. Kumar [22], K. Skhvitaridze, M. Kharashvili [23].

In this paper we consider the boundary value problems III and IV of statics of two-component elastic mixtures for a half-space. We develop a new approach to the Dirichlet and Neumann BVP_s for the Laplace equation for a half-space. Solutions are presented in quadratures.

2. Statement of Boundary Value Problems. Uniqueneous Theorems

In the three-dimensional linear theory of elastic two-component mixtures, a system of homogeneous differential equations of statics is written in the form [13]

$$a_1\Delta u' + b_1 \operatorname{grad}\operatorname{dv} u' + c\Delta u'' + d \operatorname{grad}\operatorname{dv} u'' = 0, \tag{2.1}$$

$$c\Delta u' + d \operatorname{grad}\operatorname{dv} u' + a_2\Delta u'' + b_2 \operatorname{grad}\operatorname{dv} u'' = 0, \tag{2.2}$$

where $u' = (u_1', u_2', u_3')^\top$, $u'' = (u_1'', u_2'', u_3'')^\top$ are partial displacement vectors, $\top$ is the transposition symbol, Δ is three-dimensional Laplace operator,

$$a_1 = \mu_1 - \lambda_5, \quad b_1 = \mu_1 + \lambda_5 + \lambda_1 - \frac{\rho_2}{\rho}\alpha', \quad a_2 = \mu_2 - \lambda_5,$$
$$b_2 = \mu_2 + \lambda_5 + \lambda_2 + \frac{\rho_1}{\rho}\alpha', \quad c = \mu_3 + \lambda_5, \quad \alpha' = \lambda_3 - \lambda_4,$$
$$d = \mu_3 + \lambda_3 - \lambda_5 - \frac{\rho_1}{\rho}\alpha', \quad \rho = \rho_1 + \rho_2,$$

ρ_1, ρ_2 are partial densities of the mixture; $\lambda_1, \lambda_2, \dots, \lambda_5$, μ_1, μ_2, μ_3 are the elastic moduli characterizing the mechanical properties of the mixture, which satisfy the conditions [20]

$$\begin{gathered} \mu_1 > 0, \ \ \mu_1\mu_2 - \mu_3^2 > 0, \quad \lambda_5 < 0, \quad \lambda_1 + \frac{2}{3}\mu_1 - \frac{\rho_2}{\rho}\alpha' > 0, \\ \Big(\lambda_1 + \frac{2}{3}\mu_1 - \frac{\rho_2}{\rho}\alpha'\Big)\Big(\lambda_2 + \frac{2}{3}\mu_2 + \frac{\rho_1}{\rho}\alpha'\Big) > \Big(\lambda_3 + \frac{2}{3}\mu_3 - \frac{\rho_1}{\rho}\alpha'\Big)^2. \end{gathered} \tag{2.3}$$

From these inequalities it follows that [20]

$$\begin{gathered} d_1 := a_1a_2 - c^2 > 0, \quad d_2 := (a_1 + b_1)(a_2 + b_2) - (c + d)^2 > 0, \\ a_1 > 0, \quad a_1 + b_1 > 0. \end{gathered} \tag{2.4}$$

The stress vector is written in the from [20]

$$T(\partial, n)U = \Big[P^{(1)}(\partial, n)U, \ \ P^{(2)}(\partial, n)U\Big]^\top,$$

where

$$\begin{aligned}
&P^{(1)}(\partial, n)U = T^{(1)}(\partial, n)u' + T^{(2)}(\partial, n)u'', \\
&P^{(2)}(\partial, n)U = T^{(3)}(\partial, n)u' + T^{(4)}(\partial, n)u'', \\
&T^{(1)}(\partial, n)u' = 2\mu_1\frac{\partial u'}{\partial n} + \Big(\lambda_1 - \frac{\rho_2}{\rho}\alpha'\Big)n \operatorname{dv} u' + (\mu_1 + \lambda_5)[n \times \operatorname{rot} u'], \\
&T^{(2)}(\partial, n)u'' = 2\mu_3\frac{\partial u''}{\partial n} + \Big(\lambda_3 - \frac{\rho_1}{\rho}\alpha'\Big)n \operatorname{dv} u'' + (\mu_3 - \lambda_5)[n \times \operatorname{rot} u''], \\
&T^{(3)}(\partial, n)u' = 2\mu_3\frac{\partial u'}{\partial n} + \Big(\lambda_3 - \frac{\rho_1}{\rho}\alpha'\Big)n \operatorname{dv} u' + (\mu_3 - \lambda_5)[n \times \operatorname{rot} u'], \\
&T^{(4)}(\partial, n)u'' = 2\mu_2\frac{\partial u''}{\partial n} + \Big(\lambda_2 + \frac{\rho_1}{\rho}\alpha'\Big)n \operatorname{dv} u'' + (\mu_2 + \lambda_5)[n \times \operatorname{rot} u''],
\end{aligned}$$

$n = (n_1, \ n_2, \ n_3)^\top$is a unit vector, $\frac{\partial}{\partial n} = \sum\limits_{j=1}^{3} n_j \frac{\partial}{\partial x_j}$is a derivative with respect to the vector n, the symbol $[a \times b]$ denotes vector products of two vectors in $\mathbb{R}^3$.

Denote by Ω^- a half-space $x_3 > 0$ and let $\partial\Omega$ be the plane $x_3 = 0$.

Problem. Find, in the domain Ω^-, a regular solution $U \in C^2(\Omega^-) \cap C^1(\overline{\Omega^-})$ of system (2.1)-(2.2) such that on the boundary $\partial\Omega$ one of the following boundary conditions is fulfilled:

- **Problem (III)$^-$**

$$\begin{gathered}
\{n(z) \cdot u'(z)\}^- = f_3'(z), \\
\{P^{(1)}(\partial, n)U(z) - n(z)(n(z) \cdot P^{(1)}(\partial, n)U(z))\}^- = F'(z), \\
\{n(z) \cdot u''(z)\}^- = f_3''(z), \\
\{P^{(2)}(\partial, n)U(z) - n(z)(n(z) \cdot P^{(2)}(\partial, n)U(z))\}^- = F''(z), \quad z \in \partial\Omega,
\end{gathered} \tag{2.5}$$

- **Problem (IV)$^-$**

$$\begin{aligned}
\big\{n(z) \cdot P^{(1)}(\partial, n)U(z)\big\}^- &= f_3'(z), \\
\big\{u'(z) - n(z)(n(z) \cdot u'(z))\big\}^- &= F'(z), \\
\big\{n(z) \cdot P^{(2)}(\partial, n)U(z)\big\}^- &= f_3''(z), \\
\big\{u''(z) - n(z)(n(z) \cdot u''(z))\big\}^- &= F''(z), \quad z \in \partial\Omega,
\end{aligned} \tag{2.6}$$

in the neighborhood of a point at infinity the vect $U(x)$ satisfies the following conditions:

$$\begin{aligned}
&a)\ U_j(x) = O\big(|x|^{-1}\big), \qquad \partial_k U_j(x) = O\big(|x|^{-2}\big), \\
&\qquad\qquad k = 1,2,3, \ \ j = 1,2,\dots,6, \ \ x_3 > 0, \ \ |x| \to \infty, \\
&b)\ U_j(x) = o(1), \qquad \partial_k U_j(x) = o\big(|x|^{-1}\big), \\
&\qquad\qquad k = 1,2, \ \ j = 1,2,\dots,6, \ \ x_3 = 0, \ \ |x| \to \infty,
\end{aligned} \tag{2.7}$$

where $F' = (f_1', f_2', F_3')^\top$, $F'' = (f_1'', f_2'', F_3'')^\top$, f_j', f_j'', $j = 1,2,3$, F_3', F_3'' are the function given on the boundary, $n(z)$ is the internal normal unit vector passing at a point $z \in \partial\Omega$ in the domain Ω^-, $x = (x_1, x_2, x_3)$, $z = (z_1, z_2, 0)$.

Theorem 2.1. *If problems* (III)$^-$ *and* (IV)$^-$ *have solutions, then these solutions are unique.*

Proof. The theorem will be proved if we show that the homogeneous problems ($f_j' = 0$, $f_j'' = 0$, $j = 1,2,3$, $F_3' = 0$, $F_3'' = 0$) have only a trivial solution.

Denote by $\Omega_R := \Omega^- \cap B(O,R)$, where $B(O,R)$ is the ball with center at the origin and radius R. Denote by $\partial\Omega_R$ that part of the boundary of the ball $B(O,R)$ which lies of the domain $x_3 > 0$, by $S(O,R)$ the circle with center at the origin of radius R which lies on the plane $x_3 = 0$.

We introduce the matrix differential operator $M(\partial)$:

$$
\begin{aligned}
&M(\partial) := \begin{bmatrix} M^{(1)}(\partial) & M^{(2)}(\partial) \\ M^{(3)}(\partial) & M^{(4)}(\partial) \end{bmatrix}_{6\times 6}, \\
&M^{(l)}(\partial) := \left[M^{(l)}_{kj}(\partial) \right]_{3\times 3}, \quad l = 1,2,3,4, \\
&M^{(1)}_{kj}(\partial) := a_1 \delta_{kj} \Delta + b_1 \partial_k \partial_j, \\
&M^{(l)}_{kj}(\partial) := c \delta_{kj} \Delta + d \partial_k \partial_j, \quad l = 2,3, \\
&M^{(4)}_{kj}(\partial) := a_2 \delta_{kj} \Delta + b_2 \partial_k \partial_j,
\end{aligned}
\tag{2.8}
$$

where δ_{kj} is the Kronecker's symbol $\partial_j = \frac{\partial}{\partial x_j}$ $j = 1,2,3$, $\partial = (\partial_1, \partial_2, \partial_3)$.

Due to (2.8), system (2.1)–(2.2) can be reduced to

$$
M(\partial) U(x) = 0,
$$

where $U = (u', u'')^\top$.

For the domain Ω_R , we write the Green's formula [20]

$$
\begin{aligned}
\int\limits_{\Omega_R} U(x) \cdot M(\partial) U(x) dx = &- \int\limits_{\partial \Omega_R} U(x) \cdot T(\partial, n) U(x) ds - \\
&- \int\limits_{S(O,R)} \{U(z)\}^- \cdot \{T(\partial, n) U(z)\}^- ds - \int\limits_{\Omega_R} E(U,U) dx,
\end{aligned}
\tag{2.9}
$$

where [20]

$$
\begin{aligned}
E(U,U) =& (a_1 + b_1)(\operatorname{dv} u')^2 + (a_2 + b_2)(\operatorname{dv} u'')^2 + 2(c+d) \operatorname{dv} u' \operatorname{dv} u'' + \\
&+ \frac{\mu_1}{2} \sum_{k \neq j = 1}^{3} \left(\frac{\partial u'_k}{\partial x_j} + \frac{\partial u'_j}{\partial x_k} \right)^2 + \frac{\mu_2}{2} \sum_{k \neq j = 1}^{3} \left(\frac{\partial u''_k}{\partial x_j} + \frac{\partial u''_j}{\partial x_k} \right)^2 + \\
&+ \mu_3 \sum_{k \neq j = 1}^{3} \left(\frac{\partial u'_k}{\partial x_j} + \frac{\partial u'_j}{\partial x_k} \right) \left(\frac{\partial u''_k}{\partial x_j} + \frac{\partial u''_j}{\partial x_k} \right) - \\
&- \frac{\lambda_5}{2} \sum_{k,j=1}^{3} \left(\frac{\partial u'_k}{\partial x_j} - \frac{\partial u'_j}{\partial x_k} - \frac{\partial u''_k}{\partial x_j} + \frac{\partial u''_j}{\partial x_k} \right)^2 .
\end{aligned}
\tag{2.10}
$$

According to inequalities (2.3) and (2.4), we have $E(U,U) \geq 0$, $x \in \Omega^-$.

Applying the boundary conditions of problems $(\mathrm{III})_0^-$ and $(\mathrm{IV})_0^-$, we obtain

$$
\begin{aligned}
\Big\{ U(z) \cdot T(\partial, n) U(z) \Big\}^- &= \{ n(z) \cdot u'(z) \}^- \Big\{ n(z) \cdot P^{(1)}(\partial, n) U(z) \Big\}^- + \\
&+ \{ n(z) \cdot u''(z) \}^- \Big\{ n(z) \cdot P^{(2)}(\partial, n) U(z) \Big\}^- +
\end{aligned}
$$

$$
\begin{aligned}
&+\Big\{u'(z)-n(z)(n(z)\cdot u'(z))\Big\}^-\times\\
&\quad\times\Big\{P^{(1)}(\partial,n)U(z)-n(z)\big(n(z)\cdot P^{(1)}(\partial,n)U(z)\big)\Big\}^-+\\
&+\Big\{u''(z)-n(z)\big(n(z)\cdot u''(z)\big)\Big\}^-\times\\
&\quad\times\Big\{P^{(2)}(\partial,n)U(z)-n(z)\big(n(z)\cdot P^{(2)}(\partial,n)U(z)\big)\Big\}=0.
\end{aligned}
$$

Using this equality in (2.9), we have

$$\int\limits_{\Omega_R} E(U,U)dx+\int\limits_{\partial\Omega_R} U(x)\cdot T(\partial,n)U(x)ds=0.$$

Passing to the limit on both sides of equality (2.10) as $R\to+\infty$ and taking into consideration the asymptotic representations (2.7), we obtain

$$\int\limits_{\Omega^-} E(U,U)dx=0. \tag{2.11}$$

According to inequalities (2.3) and (2.4), from (2.10) it follows that $E(U,U)\geq 0$. By virtue of this fact, (2.11) implies

$$E(U,U)=0,\quad x\in\Omega^-.$$

Hence, taking into account (2.10), we obtain

$$
\begin{gathered}
\operatorname{dv} u'(x)=0,\quad \operatorname{dv} u''(x)=0,\\
\frac{\partial u_k'(x)}{\partial x_j}+\frac{\partial u_j'(x)}{\partial x_k}=0,\quad \frac{\partial u_k''(x)}{\partial x_j}+\frac{\partial u_j''(x)}{\partial x_k}=0,\\
\frac{\partial u_k'(x)}{\partial x_j}-\frac{\partial u_j'(x)}{\partial x_k}-\frac{\partial u_k''(x)}{\partial x_j}+\frac{\partial u_j''(x)}{\partial x_k}=0,\quad x\in\Omega^-.
\end{gathered}
$$

A solution of this system has the form

$$u'(x)=[a'\times x]+b',\quad u''(x)=[a'\times x]+b'',\ \ x\in\Omega^-,$$

where a',b',b'' are the three-component constant vectors.

By asymptotic (2.7), we have $a'=b'=b''=0$, i.e. $U(x)=0$, $x\in\Omega^-$. □

3. Solution of the Problem $(\mathrm{III})^-$

If in the boundary conditions (2.5) we assume that $n(z)=(0,0,1)^\top$, then these boundary conditions can be rewritten as follows:

$$
\begin{gathered}
\{u_3'(z)\}^-=f_3'(z),\quad \{u_3''(z)\}^-=f_3''(z),\\
\left\{\frac{\partial u_j'(z)}{\partial x_3}\right\}^-=f_j^{(1)}(z),\quad \left\{\frac{\partial u_j''(z)}{\partial x_3}\right\}^-=f_j^{(2)}(z),\quad z\in\partial\Omega,
\end{gathered}
\tag{3.1}
$$

where

$$f_j^{(1)}(z) = \frac{1}{d_1}\left[a_2 f_j'(z) - c f_j''(y) - \alpha_1 \frac{\partial f_3'(z)}{\partial z_j} + 2\lambda_5(\mu_2+\mu_3)\frac{\partial f_3''(z)}{\partial z_j}\right],$$

$$f_j^{(2)}(z) = \frac{1}{d_1}\left[a_1 f_j''(z) - c f_j'(z) + 2\lambda_5(\mu_1+\mu_3)\frac{\partial f_3'(z)}{\partial z_j} - \alpha_2 \frac{\partial f_3''(z)}{\partial z_j}\right],$$

$$j = 1, 2,$$

$$\alpha_1 = \mu_1\mu_2 - \mu_3^2 + (\mu_2 - \mu_1)\lambda_5, \quad \alpha_2 = \mu_1\mu_2 - \mu_3^2 - (\mu_2 - \mu_1)\lambda_5.$$

From equations (2.1) and (2.2), we have

$$\Delta \operatorname{rot} u'(x) = 0, \quad \Delta \operatorname{rot} u''(x) = 0, \quad x \in \Omega^-. \tag{3.2}$$

From the boundary conditions (3.1), we obtained

$$\big\{\operatorname{rot} u'(z)\big\}_j^- = f_j^{(3)}(z), \quad \{\operatorname{rot} u''(z)\}_j^- = f_j^{(4)}(z), \quad j = 1, 2, \tag{3.3}$$

$$\left\{\frac{\partial}{\partial x_3}\operatorname{rot} u'(z)\right\}_3^- = f_3^{(3)}(z), \quad \left\{\frac{\partial}{\partial x_3}\operatorname{rot} u''(z)\right\}_3^- = f_3^{(4)}(z), \quad z \in \partial\Omega, \tag{3.4}$$

where

$$f_1^{(3)}(z) = \frac{\partial f_3'(z)}{\partial z_2} - f_2^{(1)}(z), \qquad f_2^{(3)}(z) = -\frac{\partial f_3'(z)}{\partial z_1} + f_1^{(1)}(z),$$

$$f_1^{(4)}(z) = \frac{\partial f_3''(z)}{\partial z_2} - f_2^{(2)}(z), \qquad f_2^{(4)}(z) = -\frac{\partial f_3''(z)}{\partial y_1} + f_1^{(2)}(z),$$

$$f_3^{(3)}(z) = \frac{\partial f_2^{(1)}(z)}{\partial z_1} - \frac{\partial f_1^{(1)}(z)}{\partial z_2}, \qquad f_3^{(4)}(z) = \frac{\partial f_2^{(2)}(z)}{\partial z_1} - \frac{\partial f_1^{(2)}(z)}{\partial z_2}.$$

$$\big[\operatorname{rot} u'(x)\big]_j = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\,\frac{1}{r}\, f_j^{(3)}(y)dy, \quad j = 1, 2,$$

$$\big[\operatorname{rot} u''(x)\big]_j = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\,\frac{1}{r}\, f_j^{(4)}(y)dy, \quad j = 1, 2,$$

$$\big[\operatorname{rot} u'(x)\big]_3 = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{1}{r}\, f_3^{(3)}(y)dy,$$

$$\big[\operatorname{rot} u''(x)\big]_3 = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{1}{r}\, f_3^{(4)}(y)dy, \quad r = |x - y|.$$

from the above equalities, we obtain

$$\big[x\times \operatorname{rot} u'(x)\big]_j=\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{x_1\delta_{2j}-x_2\delta_{1j}}{r}\,f_3^{(3)}(y)dy+$$
$$+\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}x_3\frac{\partial}{\partial x_3}\frac{1}{r}\big(\delta_{1j}f_2^{(3)}(y)-\delta_{2j}f_1^{(3)}(y)\big)dy,\quad j=1,2,$$
$$\big[x\times \operatorname{rot} u''(x)\big]_j=\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{x_1\delta_{2j}-x_2\delta_{1j}}{r}\,f_3^{(4)}(y)dy+$$
$$+\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}x_3\frac{\partial}{\partial x_3}\frac{1}{r}\big(\delta_{1j}f_2^{(4)}(y)-\delta_{2j}f_1^{(4)}(y)\big)dy,\quad j=1,2,$$

$$\begin{aligned}\big[x\times \operatorname{rot} u'(x)\big]_3&=\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{\partial}{\partial x_3}\frac{1}{r}\big(x_2f_1^{(3)}(y)-x_1f_2^{(3)}(y)\big)dy,\\ \big[x\times \operatorname{rot} u''(x)\big]_3&=\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\frac{\partial}{\partial x_3}\frac{1}{r}\big(x_2f_1^{(4)}(y)-x_1f_2^{(4)}(y)\big)dy.\end{aligned}\tag{3.5}$$

If in equations (2.1) and (2.2) we take into account the equality grad dv $u=\Delta u+\operatorname{rot}\operatorname{rot} u$, we get

$$\begin{aligned}&(a_1+b_1)\Delta u'(x)+b_1\operatorname{rot}\operatorname{rot} u'(x)+(c+d)\Delta u''(x)+\\ &\quad+d\operatorname{rot}\operatorname{rot} u''(x)=0,\\ &(c+d)\Delta u'(x)+d\operatorname{rot}\operatorname{rot} u'(x)+(a_2+b_2)\Delta u''(x)+\\ &\quad+b_2\operatorname{rot}\operatorname{rot} u''(x)=0,\quad x\in\Omega^-.\end{aligned}\tag{3.6}$$

On the other hand,

$$\Delta\big[x\times \operatorname{rot} u'(x)\big]=2\operatorname{rot}\operatorname{rot} u'(x),\quad \Delta\big[x\times \operatorname{rot} u''(x)\big]=2\operatorname{rot}\operatorname{rot} u''(x),$$

therefore equations (3.6) will be as rewritten

$$\Delta v'(x)=0,\quad \Delta v''(x)=0,\quad x\in\Omega^-,\tag{3.7}$$

where

$$\begin{aligned}v'(x)&=2(a_1+b_1)u'(x)+2(c+d)u''(x)+b_1[x\times \operatorname{rot} u'(x)]+\\ &\quad+d[x\times \operatorname{rot} u''(x)],\\ v''(x)&=2(c+d)u'(x)+2(a_2+b_2)u''(x)+d[x\times \operatorname{rot} u'(x)]+\\ &\quad+b_2[x\times \operatorname{rot} u''(x)].\end{aligned}\tag{3.8}$$

If consider the boundary conditions (3.1),(3.3)-(3.4), we will get

$$\{v_3'(z)\}^- = f_3^{(5)}(z), \quad \{v_3''(z)\}^- = f_3^{(6)}(z), \tag{3.9}$$

$$\left\{\frac{\partial}{\partial x_3} v_j'(z)\right\}^- = f_j^{(5)}(z), \quad \left\{\frac{\partial}{\partial x_3} v_j''(z)\right\}^- = f_j^{(6)}(z), \tag{3.10}$$
$$j = 1, 2, \quad z \in \partial\Omega,$$

where

$$\begin{aligned}
f_3^{(5)}(z) =&2(a_1+b_1)f_3'(z) + 2(c+d)f_3''(z) + z_1\left[b_1 f_2^{(3)}(z) + df_2^{(4)}(z)\right] - \\
&- z_2\left[b_1 f_1^{(3)}(z) + df_1^{(4)}(z)\right], \\
f_3^{(6)}(z) =&2(c+d)f_3'(z) + 2(a_2+b_2)f_3''(z) + z_1\left[df_2^{(3)}(z) + b_2 f_2^{(4)}(z)\right] - \\
&- z_2\left[df_1^{(3)}(z) + b_2 f_1^{(4)}(z)\right], \\
f_j^{(5)}(z) =&2(a_1+b_1)f_j^{(1)}(z) + 2(c+d)f_j^{(2)}(z) - b_1(-1)^j\left[z_{3-j}f_3^{(3)}(z) - \right. \\
&\left. -f_{3-j}^{(3)}(z)\right] - d(-1)^j\left[z_{3-j}f_3^{(4)}(z) - f_{3-j}^{(4)}(y)\right], \quad j = 1, 2, \\
f_j^{(6)}(z) =&2(c+d)f_j^{(1)}(z) + 2(a_2+b_2)f_j^{(2)}(z) - d(-1)^j\left[z_{3-j}f_3^{(3)}(z) - \right. \\
&\left. -f_{3-j}^{(3)}(z)\right] - b_2(-1)^j\left[z_{3-j}f_3^{(4)}(z) - f_{3-j}^{(4)}(z)\right], \quad j = 1, 2.
\end{aligned}$$

The Dirichlet problems (3.7),(3.9) and the Neumann problems(3.7),(3.10) have the following solution

$$\begin{aligned}
v_3'(x) &= -\frac{1}{2\pi}\iint_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r}\, f_3^{(5)}(y)dy, \\
v_3''(x) &= -\frac{1}{2\pi}\iint_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r}\, f_3^{(6)}(y)dy;
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
v_j'(x) &= -\frac{1}{2\pi}\iint_{-\infty}^{+\infty} \frac{1}{r}\, f_j^{(5)}(y)dy, \quad j = 1, 2, \\
v_j''(x) &= -\frac{1}{2\pi}\iint_{-\infty}^{+\infty} \frac{1}{r}\, f_j^{(6)}(y)dy, \quad j = 1, 2.
\end{aligned} \tag{3.12}$$

From (3.8), we have

$$\begin{aligned}
u'(x) &= \zeta_2 v'(x) - \zeta_3 v''(x) + \zeta_4\left[x \times \operatorname{rot} u'(x)\right] + \zeta_5\left[x \times \operatorname{rot} u''(x)\right], \\
u''(x) &= \zeta_1 v''(x) - \zeta_3 v'(x) + \zeta_6\left[x \times \operatorname{rot} u'(x)\right] + \zeta_7\left[x \times \operatorname{rot} u''(x)\right],
\end{aligned} \tag{3.13}$$

where

$$\zeta_1=(a_1+b_1)/2d_2,\quad \zeta_2=(a_2+b_2)/2d_2,\quad \zeta_3=(c+d)/2d_2,$$
$$\zeta_4=\left(d(c+d)-b_1(a_2+b_2)\right)/2d_2,\quad \zeta_5=\left(b_2(c+d)-d(a_2+b_2)\right)/2d_2,$$
$$\zeta_6=\left(b_1(c+d)-d(a_1+b_1)\right)/2d_2,\quad \zeta_7=\left(d(c+d)-b_2(a_1+b_1)\right)/2d_2.$$

If we take into account equalities (3.5), (3.11)-(3.12) in (3.13), we get

$$U(x)=\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty}\mathbf{K}(x,y)f(y)dy, \tag{3.14}$$

where

$$\mathbf{K}(x,y)=\begin{bmatrix}\mathbf{K}^{(1)}(x,y) & \mathbf{K}^{(2)}(x,y)\\ \mathbf{K}^{(3)}(x,y) & \mathbf{K}^{(4)}(x,y)\end{bmatrix}_{6\times 6},\quad \mathbf{K}^{(p)}(x,y)=\left[\mathbf{K}^{(p)}_{lj}(x,y)\right]_{3\times 3},$$
$$p=1,2,3,4,\quad f=(f',f'')^\top,\quad f'=(f'_1,f'_2,f'_3)^\top,\quad f''=(f''_1,f''_2,f''_3)^\top,$$
$$\begin{aligned}\mathbf{K}^{(p)}_{lj}(x,y)=&(1-\delta_{l3})(1-\delta_{3j})\left[\frac{1}{d_1}\left(-a_2\delta_{1p}+c(\delta_{2p}+\delta_{3p})-a_1\delta_{4p}\right)\delta_{lj}\frac{1}{r}-\right.\\&\left.-\beta'_p\frac{\partial^2 r}{\partial x_l\partial x_j}\right]+\delta_{3j}(1-\delta_{l3})\left[-\alpha''_p\frac{\partial}{\partial x_l}\frac{1}{r}+\beta''_p x_3\frac{\partial^2}{\partial x_l\partial x_3}\frac{1}{r}\right]-\\&-\delta_{l3}(1-\delta_{3j})\beta'_p x_3\frac{\partial}{\partial x_j}\frac{1}{r}-\delta_{l3}\delta_{3j}\left[(\delta_{1p}+\delta_{4p})\frac{\partial}{\partial x_3}\frac{1}{r}-\right.\\&\left.-\beta''_p x_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r}\right],\quad p=1,2,3,4,\end{aligned}$$
$$\beta'_l=\frac{1}{d_1}\left(a_2\zeta_{l+3}-c\zeta_{l+4}\right),\quad l=1,3,$$
$$\beta'_l=\frac{1}{d_1}\left(a_1\zeta_{l+3}-c\zeta_{l+2}\right),\quad l=2,4,$$
$$\beta''_l=\frac{1}{d_1}\left[(\alpha_2-d_1)\zeta_{l+4}-(\alpha_1+d_1)\zeta_{l+3}\right],\quad l=1,3,$$
$$\alpha''_l=\beta''_l+(-1)^l\frac{\alpha_1}{d_1}-\delta_{2l},\quad l=1,2,$$
$$\beta''_l=\frac{1}{d_1}\left[(\alpha_1-d_1)\zeta_{l+2}-(\alpha_2+d_1)\zeta_{l+3}\right],\quad l=2,4,$$
$$\alpha''_l=\beta''_l-(-1)^l\frac{\alpha_2}{d_1}-\delta_{3l},\quad l=3,4.$$

Here we have used the identites

$$\iint\limits_{-\infty}^{+\infty}\frac{1}{r}\frac{\partial f_j(y)}{\partial y_j}dy=\iint\limits_{-\infty}^{+\infty}\frac{\partial}{\partial x_j}\frac{1}{r}f_j(y)dy,\quad j=1,2,$$

$$\iint\limits_{-\infty}^{+\infty} (x_l - y_l)\frac{\partial}{\partial x_3}\frac{1}{r}\frac{\partial f_j(y)}{\partial y_j}dy = \iint\limits_{-\infty}^{+\infty} \frac{\partial^3 r}{\partial x_l \partial x_j \partial x_3} f_j(y)dy, \quad l,j = 1,2.$$

Calculating the stress vector $T(\partial, n)U(x)$ by (3.14), we get

$$T(\partial, n)U(x) = \frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \mathbf{L}(x,y)f(y)dy, \tag{3.15}$$

where

$$\mathbf{L}(x,y) = \begin{bmatrix} \mathbf{L}^{(1)}(x,y) & \mathbf{L}^{(2)}(x,y) \\ \mathbf{L}^{(3)}(x,y) & \mathbf{L}^{(4)}(x,y) \end{bmatrix}_{6\times 6},$$

$$\mathbf{L}^{(l)}(x,y) = \Big[\mathbf{L}^{(l)}_{kj}(x,y)\Big]_{3\times 3}, \quad l = 1,2,3,4,$$

$$\begin{aligned}
\mathbf{L}^{(l)}_{kj}(x,y) =& (1-\delta_{k3})(1-\delta_{3j})\left[-(\delta_{1l}+\delta_{4l})\delta_{kj}\frac{\partial}{\partial x_3}\frac{1}{r} - \gamma_l x_3 \frac{\partial^2}{\partial x_k \partial x_j}\frac{1}{r}\right] + \\
&+ \delta_{3j}(1-\delta_{k3})\delta_l x_3 \frac{\partial^3}{\partial x_k \partial x_3^2}\frac{1}{r} + \\
&+ \delta_{k3}(1-\delta_{3j})\left[(\gamma_l + \eta_l)\frac{\partial}{\partial x_j}\frac{1}{r} - \gamma_l x_3 \frac{\partial^2}{\partial x_j \partial x_3}\frac{1}{r}\right] + \\
&+ \delta_{k3}\delta_{3j}\left[\left(\frac{4\lambda_5 d_3}{d_1}\zeta'_l - \delta_l\right)\frac{\partial^2}{\partial x_3^2}\frac{1}{r} + \delta_l x_3 \frac{\partial^3}{\partial_3^3}\frac{1}{r}\right], \quad l = 1,2,3,4,
\end{aligned}$$

$$\begin{aligned}
&\gamma_l = 2(\mu_1\beta'_l + \mu_3\beta'_{l+2}), \quad l = 1,2, \quad \gamma_l = 2(\mu_3\beta'_{l-2} + \mu_2\beta'_l), \quad l = 3,4,\\
&\delta_l = 2(\mu_1\beta''_l + \mu_3\beta''_{l+2}), \quad l = 1,2, \quad \delta_l = 2(\mu_3\beta''_{l-2} + \mu_2\beta''_l), \quad l = 3,4,\\
&\eta_l = \alpha_1\delta_{1l} + (d_1 - \alpha_2)\delta_{2l} + (d_1 - \alpha_1)\delta_{3l} + \alpha_2\delta_{4l},\\
&\zeta'_l = \delta_{1l} + \delta_{4l} - \delta_{2l} - \delta_{3l}, \quad l = 1,2,3,4,
\end{aligned}$$

$d_3 = \mu_1\mu_2 - \mu_3^2$, δ_{kj} is the Kronecker symbol.

Assume that the functions $f'_j(z), f''_j(z) \in C^{0,\alpha}(\partial\Omega)$, $f'_3(z), f''_3(z) \in C^{1,\alpha}(\partial\Omega)$, $j = 1,2$, $0 < \alpha < 1$, then by straightforward verification we establish that the vector $U(x)$ represented in the form (3.14) is a solution of system (2.1)-(2.2) in the domain Ω^-. If in the functions $\left\{P^{(l)}(\partial, n)U(x)\right\}_j$, $l,j = 1,2$ from (3.15) and in the functions $u'_3(x), u''_3(x)$ from (3.14) we pass to the limit as $x \to z \in \partial\Omega$ ($x_3 \to 0$) and take into account [19], [24]

$$\lim_{x\to z}\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r}f(y)dy = -f(z), \quad z \in \partial\Omega, \tag{3.16}$$

we obtain that the vector $U(x)$ represented in the form (3.14) satisfies the boundary conditions (2.5).

If the boundary vector-function satisfies the conditions

$$\big|f_j'(z)\big| < \frac{A}{1+|z|^2}, \quad \big|f_j''(z)\big| < \frac{A}{1+|z|^2}, \quad j=1,2,$$
$$|f_3'(z)| < \frac{A}{1+|z|}, \quad |f_3''(z)| < \frac{A}{1+|z|}, \quad z\in\partial\Omega, \quad A=\text{const}>0,$$

then the vector $U(x)$ represented by formula (3.14) is a regular solution of problem $(\mathrm{III})^-$ which satisfies the following decay conditions at infinity

$$\begin{aligned} u_j'(x), u_j''(x) &= O\left(|x|^{-1}\ln|x|\right), \quad j=1,2,\\ u_3'(x), u_3''(x) &= O\left(|x|^{-1}\right),\\ \partial_k u_j'(x), \partial_k u_j''(x) &= O\left(|x|^{-2}\right), \quad j=1,2,\\ \partial_k u_3'(x), \partial_k u_3''(x) &= O\left(|x|^{-2}\ln|x|\right), \quad k=1,2,3. \end{aligned}$$

4. Solution of the Problem $(\mathrm{IV})^-$

If in the boundary conditions (2.6) we assume that $n(z)=(0,0,1)^\top$, then these boundary conditions can be rewritten as follows:

$$\begin{gathered} \big\{u_j'(z)\big\}^- = f_j'(z), \quad \big\{u_j''(z)\big\}^- = f_j''(z), \quad j=1,2,\\ \left\{\frac{\partial u_3'(z)}{\partial x_3}\right\}^- = f_3^{(1)}(z), \quad \left\{\frac{\partial u_3''(z)}{\partial x_3}\right\}^- = f_3^{(2)}(z), \quad z\in\partial\Omega, \end{gathered} \tag{4.1}$$

where

$$\begin{aligned} f_3^{(1)}(z) = \frac{1}{d_2}\bigg[&(a_2+b_2)f_3'(z)-(c+d)f_3''(z)-\alpha_1'\left(\frac{\partial f_1'(z)}{\partial z_1}+\frac{\partial f_2'(z)}{\partial z_2}\right)+\\ &+\alpha_1''\left(\frac{\partial f_1''(z)}{\partial z_1}+\frac{\partial f_2''(z)}{\partial z_2}\right)\bigg],\\ f_3^{(2)}(z) = \frac{1}{d_2}\bigg[&(a_1+b_1)f_3''(z)-(c+d)f_3'(z)+\alpha_2'\left(\frac{\partial f_1'(z)}{\partial z_1}+\frac{\partial f_2'(z)}{\partial z_2}\right)-\\ &-\alpha_2''\left(\frac{\partial f_1''(z)}{\partial z_1}+\frac{\partial f_2''(z)}{\partial z_2}\right)\bigg],\\ \alpha_1' &= d_2-2\mu_1(a_2+b_2)+2\mu_3(c+d),\\ \alpha_1'' &= 2\mu_3(a_2+b_2)-2\mu_2(c+d),\\ \alpha_2' &= 2\mu_3(a_1+b_1)-2\mu_1(c+d),\\ \alpha_2'' &= d_2-2\mu_2(a_1+b_1)+2\mu_3(c+d). \end{aligned}$$

From equations (2.1) and (2.2), we have

$$\Delta\,\mathrm{dv}\,u'(x)=0, \quad \Delta\,\mathrm{dv}\,u''(x)=0, \quad x\in\Omega^-. \tag{4.2}$$

From the boundary conditions (4.1), we obtained

$$\{\mathrm{dv}\,u'(z)\}^- = f_3^{(3)}(z), \quad \{\mathrm{dv}\,u''(z)\}^- = f_3^{(4)}(z), \quad z\in\partial\Omega, \tag{4.3}$$

where

$$f_3^{(3)}(z) = \frac{1}{d_2}\left[(a_2+b_2)f_3'(z) - (c+d)f_3''(z) + \right.$$
$$\left. +(d_2-\alpha_1')\left(\frac{\partial f_1'(z)}{\partial z_1} + \frac{\partial f_2'(z)}{\partial z_2}\right) + \alpha_1''\left(\frac{\partial f_1''(z)}{\partial z_1} + \frac{\partial f_2''(z)}{\partial z_2}\right)\right],$$

$$f_3^{(4)}(z) = \frac{1}{d_2}\left[(a_1+b_1)f_3''(z) - (c+d)f_3'(z) + \alpha_2'\left(\frac{\partial f_1'(z)}{\partial z_1} + \frac{\partial f_2'(z)}{\partial z_2}\right) + \right.$$
$$\left. + (d_2-\alpha_2'')\left(\frac{\partial f_1''(z)}{\partial z_1} + \frac{\partial f_2''(z)}{\partial z_2}\right)\right].$$

The Dirichlet problems (4.2)-(4.3) have the following solution

$$\begin{aligned} \operatorname{dv} u'(x) &= -\frac{1}{2\pi}\iint_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r} f_3^{(3)}(y)dy, \quad x \in \Omega^-, \\ \operatorname{dv} u''(x) &= -\frac{1}{2\pi}\iint_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r} f_3^{(4)}(y)dy, \quad x \in \Omega^-, \end{aligned} \tag{4.4}$$

where $r = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + x_3^2}$.

If the equalities

$$\Delta(x \operatorname{dv} u'(x)) = 2 \operatorname{grad} \operatorname{dv} u'(x), \quad \Delta(x \operatorname{dv} u''(x)) = 2 \operatorname{grad} \operatorname{dv} u''(x),$$

are taken into eccount in equations (2.1)-(2.2), then they will be rewritten as follows:

$$\Delta v'(x) = 0, \quad \Delta v''(x) = 0, \quad x \in \Omega^-, \tag{4.5}$$

where

$$\begin{aligned} v'(x) &= 2a_1 u'(x) + 2cu''(x) + b_1 x \operatorname{dv} u'(x) + dx \operatorname{dv} u''(x), \\ v''(x) &= 2cu'(x) + 2a_2 u''(x) + dx \operatorname{dv} u'(x) + b_2 x \operatorname{dv} u''(x). \end{aligned} \tag{4.6}$$

By projecting vectorial equality (4.5) on the Ox_j -axis, we qet

$$\Delta v_j'(x) = 0, \quad \Delta v_j''(x) = 0, \quad j = 1,2, \quad x \in \Omega^-. \tag{4.7}$$

On the other hand, bearing in mind the boundary conditions (4.1) and (4.3) , we find that

$$\{v_j'(z)\}^- = f_j^{(5)}(z), \quad \{v_j''(z)\}^- = f_j^{(6)}(z), \quad j = 1,2, \quad z \in \partial\Omega, \tag{4.8}$$

where

$$\begin{aligned} f_j^{(5)}(z) &= 2a_1 f_j'(z) + 2cf_j''(z) + b_1 z_j f_3^{(3)}(z) + dz_j f_3^{(4)}(z), \\ f_j^{(6)}(z) &= 2cf_j'(z) + 2af_j''(z) + dz_j f_3^{(3)}(z) + b_2 z_j f_3^{(4)}(z). \end{aligned}$$

The solution of the Dirichlet problems (4.7)-(4.8) has the form

$$
\begin{aligned}
v_j'(x) &= -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r} f_j^{(5)}(y)dy, \quad j=1,2, \ \ x\in\Omega^-, \\
v_j''(x) &= -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{\partial}{\partial x_3}\frac{1}{r} f_j^{(6)}(y)dy, \quad j=1,2, \ \ x\in\Omega^-.
\end{aligned}
\tag{4.9}
$$

By projecting vectorial equality (4.5) on the Ox_3 -axis, we have

$$
\Delta v_3'(x)=0, \quad \Delta v_3''(x)=0, \quad x\in\Omega^-. \tag{4.10}
$$

In view of the boundary conditions (4.1) and (4.3), we get

$$
\left\{\frac{\partial v_3'(z)}{\partial x_3}\right\}^- = f_3^{(5)}(z), \quad \left\{\frac{\partial v_3''(z)}{\partial x_3}\right\}^- = f_3^{(6)}(z), \quad z\in\partial\Omega, \tag{4.11}
$$

where

$$
\begin{aligned}
f_3^{(5)}(z) &= 2a_1 f_3^{(1)}(z) + 2c f_3^{(2)}(z) + b_1 f_3^{(3)}(z) + d f_3^{(4)}(z), \\
f_3^{(6)}(z) &= 2c f_3^{(1)}(z) + 2a_2 f_3^{(2)}(z) + d f_3^{(3)}(z) + b_2 f_3^{(4)}(z).
\end{aligned}
$$

The solution of the Neumann problem (4.10) and (4.11) has the form

$$
\begin{aligned}
v_3'(x) &= -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{1}{r} f_3^{(5)}(y)dy, \quad x\in\Omega^-, \\
v_3''(x) &= -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \frac{1}{r} f_3^{(6)}(y)dy, \quad x\in\Omega^-.
\end{aligned}
\tag{4.12}
$$

From (4.6), we have

$$
\begin{aligned}
u'(x) &= \frac{1}{2d_1}\big[a_2 v'(x) - c v''(x) + (cd - a_2 b_1) x \operatorname{dv} u'(x) + \\
&\quad + (cb_2 - da_2) x \operatorname{dv} u''(x)\big], \\
u''(x) &= \frac{1}{2d_1}\big[a_1 v''(x) - c v'(x) + (cb_1 - da_1) x \operatorname{dv} u'(x) + \\
&\quad + (cd - a_1 b_2) x \operatorname{dv} u''(x)\big].
\end{aligned}
\tag{4.13}
$$

Taking into account equalities (4.4), (4.9), (4.12) in (4.13),we get

$$
U(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \widetilde{\mathbf{K}}(x,y) f(y)dy, \tag{4.14}
$$

where

$$
\begin{aligned}
&\widetilde{\mathbf{K}}(x,y)=\begin{bmatrix}\widetilde{\mathbf{K}}^{(1)}(x,y) & \widetilde{\mathbf{K}}^{(2)}(x,y)\\ \widetilde{\mathbf{K}}^{(3)}(x,y) & \widetilde{\mathbf{K}}^{(4)}(x,y)\end{bmatrix}_{6\times 6}, \quad \widetilde{\mathbf{K}}^{(p)}(x,y)=\Big[\widetilde{\mathbf{K}}^{(p)}_{lj}(x,y)\Big]_{3\times 3},\\
&p=1,2,3,4, \quad f=(f',f'')^\top, \quad f'=(f_1',f_2',f_3')^\top, \quad f''=(f_1'',f_2'',f_3'')^\top,
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathbf{K}}^{(p)}_{lj}(x,y)=&(1-\delta_{l3})(1-\delta_{3j})\left[(\delta_{1p}+\delta_{4p})\,\delta_{lj}\frac{\partial}{\partial x_3}\frac{1}{r}+\eta_{1p}\frac{\partial^3 r}{\partial x_l\partial x_j\partial x_3}\right]+\\
&+\delta_{3j}(1-\delta_{l3})\eta_{2p}x_3\frac{\partial}{\partial x_l}\frac{1}{r}+\\
&+\delta_{l3}(1-\delta_{3j})\left(\eta_{3p}\frac{\partial}{\partial x_j}\frac{1}{r}+\eta_{1p}x_3\frac{\partial^2}{\partial x_j\partial x_3}\frac{1}{r}\right)+\\
&+\delta_{l3}\delta_{3j}\left(\eta_{4p}\frac{1}{r}+\eta_{2p}x_3\frac{\partial}{\partial x_3}\frac{1}{r}\right), \quad p=1,2,3,4,\\
\eta_{1p}=&\beta_1'\delta_{1p}+\beta_1''\delta_{2p}+\beta_3'\delta_{3p}+\beta_3''\delta_{4p},\\
\eta_{2p}=&\beta_2'\delta_{1p}+\beta_2''\delta_{2p}+\beta_4'\delta_{3p}-\beta_4''\delta_{4p},\\
\eta_{3p}=&-\left(\beta_1'+\frac{\alpha_1'}{d_2}\right)\delta_{1p}+\left(\frac{\alpha_1''}{d_2}-\beta_1''\right)\delta_{2p}+\\
&+\left(\frac{\alpha_2'}{d_2}-\beta_3'\right)\delta_{3p}-\left(\frac{\alpha_2''}{d_2}+\beta_3''\right)\delta_{4p},\\
\eta_{4p}=&\left(\frac{a_2+b_2}{d_2}-\beta_2'\right)\delta_{1p}-\left(\frac{c+d}{d_2}+\beta_2''\right)\delta_{2p}-\\
&-\left(\frac{c+d}{d_2}+\beta_4'\right)\delta_{3p}+\left(\frac{a_1+b_1}{d_2}+\beta_4''\right)\delta_{4p},\\
\beta_1'=&\frac{1}{2d_1d_2}\big[(cd-a_2b_1)(d_2-\alpha_1')+(cb_2-da_2)\alpha_2'\big],\\
\beta_2'=&\frac{1}{2d_1d_2}\big[(cd-a_2b_1)(a_2+b_2)-(cb_2-da_2)(c+d)\big],\\
\beta_1''=&\frac{1}{2d_1d_2}\big[(cd-a_2b_1)\alpha_1''+(cb_2-da_2)(d_2-\alpha_2'')\big],\\
\beta_2''=&\frac{1}{2d_1d_2}\big[(cb_2-da_2)(a_1+b_1)-(cd-a_2b_1)(c+d)\big],\\
\beta_3'=&\frac{1}{2d_1d_2}\big[(cb_1-da_1)(d_2-\alpha_1')+(cd-a_1b_2)\alpha_2'\big],\\
\beta_4'=&\frac{1}{2d_1d_2}\big[(cb_1-da_1)(a_2+b_2)-(cd-a_1b_2)(c+d)\big],\\
\beta_3''=&\frac{1}{2d_1d_2}\big[(cb_1-da_1)\alpha_1''+(cd-a_1b_2)(d_2-\alpha_2'')\big],
\end{aligned}
$$

$$\beta_4'' = \frac{1}{2d_1d_2}\big[(cb_1-da_1)(c+d)-(cd-a_1b_2)(a_1+b_1)\big],$$

δ_{kj} is the Kronecker symbol.

Calculating the stress vector $T(\partial,n)U(x)$ by (4.14),we get

$$T(\partial,n)U(x) = -\frac{1}{2\pi}\iint\limits_{-\infty}^{+\infty} \widetilde{\mathbf{L}}(x,y)f(y)dy, \tag{4.15}$$

where

$$\widetilde{\mathbf{L}}(x,y) = \begin{bmatrix} \widetilde{\mathbf{L}}^{(1)}(x,y) & \widetilde{\mathbf{L}}^{(2)}(x,y) \\ \widetilde{\mathbf{L}}^{(3)}(x,y) & \widetilde{\mathbf{L}}^{(4)}(x,y) \end{bmatrix}_{6\times 6},$$

$$\widetilde{\mathbf{L}}^{(p)}(x,y) = \Big[\widetilde{\mathbf{L}}^{(p)}_{kj}(x,y)\Big]_{3\times 3}, \quad p=1,2,3,4,$$

$$\begin{aligned}
\widetilde{\mathbf{L}}^{(1)}_{kj}(x,y) = {} & (1-\delta_{k3})(1-\delta_{3j})\left[a_1\delta_{kj}\frac{\partial^2}{\partial x_3^2}\frac{1}{r} + \big(a_1\beta_1' + c\beta_3' + (\mu_1+\lambda_5)\eta_{3l} + \right.\\
& \left. + (\mu_3-\lambda_5)\eta_{33}\big)\frac{\partial^2}{\partial x_k \partial x_j}\frac{1}{r} + 2(\mu_1\beta_1' + \mu_3\beta_3')x_3\frac{\partial^3}{\partial x_k\partial x_j\partial x_3}\frac{1}{r}\right] + \\
& + \delta_{3j}(1-\delta_{k3})\left[\big(a_1\beta_2' + c\beta_4' + (\mu_1+\lambda_5)\eta_{41} + (\mu_3-\lambda_5)\eta_{43}\big)\frac{\partial}{\partial x_k}\frac{1}{r} + \right.\\
& \left. + 2(\mu_1\beta_2' + \mu_3\beta_4')x_3\frac{\partial^2}{\partial x_k\partial x_3}\frac{1}{r}\right] + \\
& + \delta_{k3}(1-\delta_{3j})2(\mu_1\beta_1' + \mu_3\beta_3')x_3\frac{\partial^3}{\partial x_j\partial x_3^2}\frac{1}{r} + \\
& + \delta_{k3}\delta_{3j}\left[\frac{\partial}{\partial x_3}\frac{1}{r} + 2(\mu_1\beta_2' + \mu_3\beta_4')x_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r}\right],
\end{aligned}$$

$$\begin{aligned}
\widetilde{\mathbf{L}}^{(2)}_{kj}(x,y) = {} & (1-\delta_{k3})(1-\delta_{3j})\left[c\delta_{kj}\frac{\partial^2}{\partial x_3^2}\frac{1}{r} + \big(a_1\beta_1'' + c\beta_3'' + (\mu_1+\lambda_5)\eta_{32} + \right.\\
& \left. + (\mu_3-\lambda_5)\eta_{34}\big)\frac{\partial^2}{\partial x_k \partial x_j}\frac{1}{r} + 2(\mu_1\beta_1'' + \mu_3\beta_3'')x_3\frac{\partial^3}{\partial x_k\partial x_j\partial x_3}\frac{1}{r}\right] + \\
& + \delta_{3j}(1-\delta_{k3})\left[\big(a_1\beta_2'' - c\beta_4'' + (\mu_1+\lambda_5)\eta_{42} + (\mu_3-\lambda_5)\eta_{44}\big)\frac{\partial}{\partial x_k}\frac{1}{r} + \right.\\
& \left. + 2(\mu_1\beta_2'' - \mu_3\beta_4'')x_3\frac{\partial^2}{\partial x_k\partial x_3}\frac{1}{r}\right] + \\
& + \delta_{k3}(1-\delta_{3j})2(\mu_1\beta_1'' + \mu_3\beta_3'')x_3\frac{\partial^3}{\partial x_j\partial x_3^2}\frac{1}{r} + \\
& + \delta_{k3}\delta_{3j}2(\mu_1\beta_2'' - \mu_3\beta_4'')x_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r},
\end{aligned}$$

$$
\begin{aligned}
\widetilde{\mathbf{L}}_{kj}^{(3)}(x,y) = {} & (1-\delta_{k3})(1-\delta_{3j})\left[c\delta_{kj}\frac{\partial^2}{\partial x_3^2}\frac{1}{r} + \big(c\beta_1' + a_2\beta_3' + (\mu_3-\lambda_5)\eta_{31} + \right.\\
& \left. + (\mu_2+\lambda_5)\eta_{33}\big)\frac{\partial^2}{\partial x_k \partial x_j}\frac{1}{r} + 2(\mu_3\beta_1' + \mu_2\beta_3')x_3\frac{\partial^3}{\partial x_k\partial x_j\partial x_3}\frac{1}{r}\right] + \\
& + \delta_{3j}(1-\delta_{k3})\left[\big(c\beta_2' + a_2\beta_4' + (\mu_3-\lambda_5)\eta_{41} + (\mu_2+\lambda_5)\eta_{43}\big)\frac{\partial}{\partial x_k}\frac{1}{r} + \right.\\
& \left. + 2(\mu_3\beta_2' + \mu_2\beta_4')x_3\frac{\partial^2}{\partial x_k \partial x_3}\frac{1}{r}\right] + \\
& + \delta_{k3}(1-\delta_{3j})2(\mu_3\beta_1' + \mu_2\beta_3')x_3\frac{\partial^3}{\partial x_j\partial x_3^2}\frac{1}{r} + \\
& + \delta_{k3}\delta_{3j}2(\mu_3\beta_2' + \mu_2\beta_4')x_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r},
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathbf{L}}_{kj}^{(4)}(x,y) = {} & (1-\delta_{k3})(1-\delta_{3j})\left[a_2\delta_{kj}\frac{\partial^2}{\partial x_3^2}\frac{1}{r} + \big(c\beta_1'' + a_2\beta_3'' + (\mu_3-\lambda_5)\eta_{32} + \right.\\
& \left. + (\mu_2+\lambda_5)\eta_{34}\big)\frac{\partial^2}{\partial x_k \partial x_j}\frac{1}{r} + 2(\mu_3\beta_1'' + \mu_2\beta_3'')x_3\frac{\partial^3}{\partial x_k\partial x_j\partial x_3}\frac{1}{r}\right] + \\
& + \delta_{3j}(1-\delta_{k3})\left[\big(c\beta_2'' - a_2\beta_4'' + (\mu_3-\lambda_5)\eta_{42} + (\mu_2+\lambda_5)\eta_{44}\big)\frac{\partial}{\partial x_k}\frac{1}{r} + \right.\\
& \left. + 2(\mu_3\beta_2'' - \mu_2\beta_4'')x_3\frac{\partial^2}{\partial x_k \partial x_3}\frac{1}{r}\right] + \\
& + \delta_{k3}(1-\delta_{3j})2(\mu_3\beta_1'' + \mu_2\beta_3'')x_3\frac{\partial^3}{\partial x_j\partial x_3^2}\frac{1}{r} + \\
& + \delta_{k3}\delta_{3j}\left[\frac{\partial}{\partial x_3}\frac{1}{r} + 2(\mu_3\beta_2'' - \mu_2\beta_4'')x_3\frac{\partial^2}{\partial x_3^2}\frac{1}{r}\right].
\end{aligned}
$$

Assume that the functions $f_j'(z)$, $f_j''(z) \in C^{1,\alpha}(\partial\Omega)$, $f_3'(z)$, $f_3''(z) \in C^{0,\alpha}(\partial\Omega)$, $j=1,2$, $0<\alpha<1$, then by the straightforward verification we establish that the vector $U(x)$ represented in the form (4.14) is a solution of system (2.1)-(2.2) in the domain Ω^-. If in the functions $u_j'(x)$, $u_j''(x)$, $P^{(j)}(\partial,n)U(x)$, $j=1,2$ from (4.14)-(4.15) we pass to the limit as $x\to z\in\partial\Omega$ $(x_3\to 0)$ and take into account (3.16), we obtain that the vector $U(x)$ represented in the form (4.14) satisfies the boundary conditions (2.6).

If the boundary vector-function satisfies the conditions

$$
\begin{aligned}
& |f_3'(z)| < \frac{A}{1+|z|}, \quad |f_3''(z)| < \frac{A}{1+|z|}, \\
& \left|f_j'(z)\right| < \frac{A}{1+|z|^2}, \quad \left|f_j''(z)\right| < \frac{A}{1+|z|^2}, \quad z\in\partial\Omega, \quad A = \text{const} > 0,
\end{aligned}
$$

then the vector $U(x)$ represented by formula (4.14) is a regular solution of problem $(\mathrm{IV})^-$ which satisfies the following decay conditions at infinity

$$\begin{aligned} u_j'(x), u_j''(x) =& O\left(|x|^{-1}\right), \quad j=1,2,\\ u_3'(x), u_3''(x) =& O\left(|x|^{-1}\ln|x|\right),\\ \partial_k u_j'(x), \partial_k u_j''(x) =& O\left(|x|^{-2}\ln|x|\right),\\ \partial_k u_3'(x), \partial_k u_3''(x) =& O\left(|x|^{-2}\right), \quad j=1,2, \quad k=1,2,3. \end{aligned}$$

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