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MULTIVARIATE HARDY-TYPE INEQUALITIES ON TIME SCALES VIA SUPERQUADRACITY

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Abstract. A new Jensen inequality for multivariate superquadratic functions is proved. The derived Jensen inequality is then employed to obtain the general Hardy-type inequality for superquadratic and subquadratic functions of several variables.

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1. Introduction

In 1920, Hardy [14] (see also [13]) proved that if $p > 1$ and $\{a_k\}_{k=1}^{\infty}$ is a sequence of nonnegative real numbers, then

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k\right)^p \leq \left(\frac{p}{p-1}\right)^p \sum_{n=1}^{n} a_n^p. \tag{1.1}$$

Furthermore, Hardy [14] announced (without proof) that if $p > 1$ and the function f is nonnegative and integrable over the interval $(0, x)$ then

$$\int_0^{\infty}\left(\frac{1}{x}\int_0^{x} f(t)dt\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^{\infty} f^p(x)dx. \tag{1.2}$$

Inequality (1.2) was finally proved by Hardy [12] in 1925. Thus, inequality (1.2)is usually referred to in the literature as the classical Hardy integral inequality while inequality (1.1) is its discrete analogue. The constant $\left(\frac{p}{p-1}\right)^p$ on the right hand sides of both inequalities (1.1) and (1.2) is the best possible.

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In the last five decades, the Hardy inequality (1.2) has been extensively studied and generalized. A lot of information as regards it applications, alternative proofs, variants, generalizations and refinements abound in the literature (see the books [13, 19, 20] and the references cited therein).

In particular, Krulić et al. [18] studied a more general class of Hardy-type inequalities using convexity approach and obtained that

$$\left(\int_{\Omega_1} u(x)\Psi^{\frac{q}{p}}(A_k f(x))d\mu_1(x)\right)^{\frac{1}{q}} \le \left(\int_{\Omega_2} v(t)\Psi(f(t))d\mu_2(t)\right)^{\frac{1}{p}} \tag{1.3}$$

holds for all nonnegative convex functions Ψ defined on a convex set $I \subseteq \Re$ and for all measurable functions $f : \Omega_2 \to \Re$ such that $f(\Omega_2) \subseteq I$, where

$$A_k f(x) := \frac{1}{K(x)} \int_{\Omega_2} k(x,t) f(t) d\mu_2(t), \ x \in \Omega_1$$

$$K(x) := \int_{\Omega_2} k(x,t) d\mu_2(t) > 0, \ x \in \Omega_1$$

and

$$v(t) := \left[\int_{\Omega_1} u(x) \left(\frac{k(x,t)}{K(x)}\right)^{\frac{q}{p}} d\mu_1(x)\right]^{\frac{p}{q}} < \infty, \ t \in \Omega_2.$$

Observe that by setting $\Omega_1 = \Omega_2 = \Re_+ = (0,\infty), d\mu_1(x) = dx, d\mu_1(y) = dy, u(x) = \frac{1}{x}, k(x,y) = \frac{1}{x}\chi_{0<x<y}(x,y), \Psi(x) = x^p, f(y) = f(y^{\frac{p}{p-1}})y^{\frac{1}{p-1}}$ in (1.3) yields inequality (1.2).

In his PhD thesis, Stefan Hilger [15] (see also [16, 17]) initiated the calculus of time scales in order to create a theory that will unify discrete and continuous analysis. This new concept has inspired researchers to study Hardy inequalities on time scales. The first known work in this direction is probably due to Řehák [24] who obtained Hardy integral inequality on time scale. Indeed, he showed that

$$\int_a^{\infty} \left(\frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} f(t)\Delta t\right)^p \Delta x < \left(\frac{p}{p-1}\right)^p \int_a^{\infty} f^p(x)\Delta x,$$

where $a > 0, p > 1$ and f is a nonnegative function.

In 2001, Agarwal et al. [4] obtained the following Jensen's inequality on time scales

$$\Phi\left(\frac{1}{b-a} \int_a^b f(x)\Delta x\right) \le \frac{1}{b-a} \int_a^b \Phi(f(x))\Delta x.$$

Anwar et al. [5] obtained the Jensen inequality for convex functions in several variables on time scale. Also they deduced the Jensen functionals and established some of its basic properties for multivariate convex functions on an arbitrary time scale. Specifically, the following result is established.

Theorem 1.1. *Let* $(\Omega_1, \Sigma_1, \mu_\Delta)$ *and* $(\Omega_2, \Sigma_2, \lambda_\Delta)$ *be two time scale measure spaces. Suppose that* $U \subset \mathbb{R}^n$ *is a closed convex set and* $\Phi \in C(U, \mathbb{R})$ *is convex. Moreover, let* $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be nonnegative such that* $k(x, .)$ *is* $\lambda_\Delta -$ *integrable. Then*

$$\Phi\left(\frac{\int_{\Omega_2} k(x,y)\mathbf{f}(y)\Delta y}{\int_{\Omega_2} k(x,y)\Delta y}\right) \leq \frac{\int_{\Omega_2} k(x,y)\Phi(\mathbf{f}(y))\Delta y}{\int_{\Omega_2} k(x,y)\Delta y} \tag{1.4}$$

holds for all functions $\mathbf{f} : \Omega_2 \to U$*, where* $f_j(y)$ *are* μ_{Δ_2}*-integrable for all* $j \in \{1, 2, \ldots, n\}$*, and* $\int_{\Omega_2} k(x,y)\mathbf{f}(y)\Delta(y)$ *denotes the* n*-tuple*

$$\left(\int\limits_{\Omega_2} k(x,y)f_1(y)\Delta(y), \int\limits_{\Omega_2} k(x,y)f_2(y)\Delta(y), \ldots, \int\limits_{\Omega_2} k(x,y)f_n(y)\Delta(y)\right).$$

Donchev et al. [10] employed the above result to derive the following Hardy-type inequality involving multivariate convex functions on time scales:

Theorem 1.2. *If* $K : \Omega_1 \to \mathbb{R}$ *is defined by* $K(x) := \int_{\Omega_2} k(x,y)\Delta y < \infty$*,* $x \in \Omega_1$ *and* $\zeta : \Omega_1 \to \mathbb{R}$ *is such that*

$$w(y) := \int\limits_{\Omega_1} \left(\frac{k(x,y)\zeta(x)}{K(x)}\right) \Delta x, \; y \in \Omega_2,$$

then

$$\int\limits_{\Omega_1} \zeta(x)\Phi\left(\frac{1}{K(x)} \int\limits_{\Omega_2} k(x,y)\boldsymbol{f}(y)\Delta y\right)\Delta x \leq \int\limits_{\Omega_2} w(y)\Phi(\boldsymbol{f}(y))\Delta y \tag{1.5}$$

holds for all λ_Δ*-integrable functions* $\boldsymbol{f} : \Omega_2 \to \mathbb{R}^n$ *such that* $\boldsymbol{f}(\Omega_2) \subset U$*.*

Recently, Abramovich et al. [2] introduced the concept of superquadratic functions in one variable as a generalization of the class of convex functions. In particular, they define the one variable superquadratic and subquadratic functions as follows:

Definition 1.3 ([2], Definition 2.1)**.** A function $\Phi : [0, \infty) \to \Re$ is said to be superquadratic provided that for all $x \geq 0$ there exists a constant $C_x \in R$ such that

$$\Phi(y) - \Phi(x) - C_x(y - x) - \Phi(|y - x|) \quad \geq \quad 0$$

for all $y \geq 0$. Φ is subquadratic if $-\Phi$ is superquadratic.

Instead of using convex functions if we use superquadratic functions, then some similar refined Hardy-type inequalities can be derived. The first result in this direction can be found in [21] (see also [22] for the multidimensional analogue). Furthermore, Abramovich et al. [3] obtained a generalization of this concept for superquadratic functions in several variables.

Barić et al. [6] obtained a one variable Jensen's inequality on an arbitrary time scales $\mathbb{T}$ using superquadracity argument. In particular, the following result was derived and proved:
Let $a, b \in \mathbb{T}$ and $f : [a,b]_{\mathbb{T}^k} \to [0,\infty)$ is rd-continuous and $\Phi : [0,\infty) \to \Re$ is continuous and superquadratic. Then

$$\Phi\left(\frac{1}{b-a}\int_a^b f(t)\Delta t\right) \leq$$
$$\leq \frac{1}{b-a}\int_a^b \left[\Phi(f(s)) - \Phi\left(\left|f(s) - \frac{1}{b-a}\int_a^b f(t)\Delta t\right|\right)\right]\Delta s. \tag{1.6}$$

In a recent paper, Oguntuase and Persson [23] obtained some new Hardy-type inequalities on time scales using the concept of superquadratic functions. In particular the following result is obtained:

Theorem 1.4. *Let* $(\Omega_1, \Sigma_1, \mu_{\Delta_1})$ *and* $(\Omega_2, \Sigma_2, \mu_{\Delta_2})$ *be two time scale measure spaces with positive* σ*-finite measures and let* $u : \Omega_1 \to \mathbb{R}$ *and* $k : \Omega_1 \times \Omega_2 \to \mathbb{R}$ *be nonnegative functions such that* $k(x,.)$ *is* μ_{Δ_2}*-integrable for* $x \in \Omega_1$. *Furthermore suppose that* $K : \Omega_1 \to \mathbb{R}$ *is defined by*

$$K(x) := \int_{\Omega_2} k(x,y)\Delta\mu_2(y) > 0, \ x \in \Omega_1$$

and

$$v(y) := \int_{\Omega_1} \left(\frac{k(x,y)u(x)}{K(x)}\right)\Delta\mu_1(x) < \infty, \ y \in \Omega_2.$$

If $\Phi : [a,\infty) \to \mathbb{R}$ $(a \geq 0)$ *is a nonnegative superquadratic function,then the inequality*

$$\int_{\Omega_1} u(x)\Phi(A_k f(x))\Delta\mu_1(x)+$$
$$+\int_{\Omega_2}\int_{\Omega_1} u(x)\frac{k(x,y)}{K(x)}\Phi\left(|f(y) - A_k f(x)|\right)\Delta\mu_1(x)\Delta\mu_2(y) \leq$$
$$\leq \int_{\Omega_2} v(x)\Phi(f(x))\Delta\mu_2(x), \tag{1.7}$$

holds for all nonnegative μ_{Δ_2}*-integrable function* $f : \Omega_2 \to \mathbb{R}$*, and for* $A_k f : \Omega_1 \to \mathbb{R}$ *defined by,*

$$A_k f(x) = \frac{1}{K(x)} \int\limits_{\Omega_2} k(x,y) f(y) \Delta\mu_2(y), \; x \in \Omega_1.$$

If Φ *is subquadratic, then the inequality sign in* (1.7) *is reversed.*

Motivated by the above results, our main aim in this paper is to establish a new Jensen inequality for multivariate superquadratic functions and then employed it to derive the general Hardy-type inequalities for multivariate superquadratic functions involving a more general kernel on an arbitrary time scales.

The paper is organized as follows: In Section 2 we recall some basic notions, definitions and results on multivariate superquadratic functions on time scales. In Section 3 we prove our results and give some remarks.

2. Preliminaries, Definitions and Some Basic Results

First, we recall that a time scale (or measure chain) $\mathbb{T}$ is an arbitrary nonempty closed subset of the real line $\mathbb{R}$ with the topology of the subspace $\mathbb{R}$. Examples of time scales are the real numbers $\mathbb{R}$ and the discrete time scale $\mathbb{Z}$. Since time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators. Let $t \in \mathbb{T}$, we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$$

and the backward jump operator by

$$\rho(t) = \inf\{s \in \mathbb{T} : s < t\}.$$

If $\sigma(t) > t$, we say that t is right-scattered and if $\rho(t) < t$ we say that t is left-scattered. The points that are right-scattered and left-scattered at time are called isolated. If $\sigma(t) = t$ then t is said to be right-dense, and if $\rho(t) = t$ then t is said to be left-dense. The points that are simultaneously right-dense and left-dense are called dense. The mapping $\mu : \mathbb{T} \to [0, \infty)$ defined by

$$\mu(t) = \sigma(t) - t$$

is called the graininess function. If $\mathbb{T}$ has a left-scattered maximum M, then define $\mathbb{T}^k = \mathbb{T} \backslash \{M\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. Let $f : \mathbb{T} \to \mathbb{R}$ be a function. Then we define the function $f^\sigma : \mathbb{T} = \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$. Also, for a function $f : \mathbb{T} \to \Re$, the delta derivative is defined by

$$f^\Delta(t) := \lim_{s \to t, \sigma(s) \neq t} \frac{f^\sigma(s) - f(t)}{\sigma(s) - t}.$$

A function $f : \mathbb{T} \to \Re$ is called rd-continuous provided it is continuous at all right-dense points in $\mathbb{T}$ and its left-sided limits exists (finite) at all left-dense points in $\mathbb{T}$. We refer to the books ([8, 9]) for more details on the calculus of time scales. Note that we have

$$\sigma(t) = t, \ \mu(t) = 0, \ f^{\Delta} = f^{'}, \ \int_a^b f(t)\Delta t = \int_a^b f(t)dt, \text{ when } \mathbb{T} = \Re,$$

$$\sigma(t) = t + 1, \ \mu(t) = 1, \ f^{\Delta} = \Delta f, \ \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t), \text{ when } \mathbb{T} = \mathbb{Z},$$

In the sequel, we let $n \in \mathbb{N}$, we define n-dimensional time scale by the Cartesian product of given time scales $\mathbb{T}_i, i \in \{1, \ldots, n\}$, as $\Omega^n = \{\mathbf{x} = (x_1, x_2, \ldots, x_n) : x_i \in \mathbb{T}_i, i \in \{1, \ldots, n\}\}$. Clearly, Ω^n equipped with the usual inner product $\langle \cdot, \cdot \rangle$ and the norm $|| \cdot ||$ defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^{n} x_i y_i$$

and

$$|| \mathbf{x} ||= \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

respectively is a complete metric space with distance defined as follows

$$d(\mathbf{x}, \mathbf{y}) =|| \mathbf{x} - \mathbf{y} ||, \mathbf{x}, \mathbf{y} \in \Omega^n.$$

Furthermore, for $\mathbf{a}, \mathbf{b} \in \Omega^n$, $[\mathbf{a}, \ \mathbf{b})$ means the set $[a_1, b_1) \times [a_2, b_2) \times \cdots \times [a_n, b_n)$ and we write $\mathbf{a} < \mathbf{b}$ if componentwise $a_i < b_i, \ i = 1, 2, \ldots, n$. Moreover, we define the subsets K_n and $K_n^+ \in \Re^n$ as

$$K_n = [0, \infty)^n = \{\mathbf{x} \in \Re^n : \mathbf{a} \leq \mathbf{x}\}$$

and

$$K_n^+ = (0, \infty)^n = \{\mathbf{x} \in \Re^n : \mathbf{a} < \mathbf{x}\},$$

where $\mathbf{a} \geq \mathbf{0}$.

We refer to [10] for the construction of Lebesgue measure and Lebesgue Δ-integrals on Ω^n and also to [8, 9, 11] for theory of measure spaces and measurable functions on time scales.

Now we recall some essentials about partial derivatives on time scales. For given time scales $\mathbb{T}_i, i \in \{1, 2, \ldots, n\}$, let $\sigma_i, \ \rho_i$ and Δ_i denote the jump operator, the backward operator, and the delta differential operator, respectively. Let f be a real-valued function on Ω^n. At point $\mathbf{x} = \{x_1, x_2, \ldots, x_n\} \in \Omega^n$, we say that f has a "$\Delta_1$ partial derivative" $\Delta_1 f(\mathbf{x})$ (with respect to x_1) if for each $\varepsilon > 0$ there exists a neighborhood U_{x_1} of x_1

such that

$$|f(\sigma_1(x_1), x_2, \ldots, x_n) - f(s^1, x_2, \ldots, x_n) - \\ -\Delta_1 f(\mathbf{x})(\sigma_1(x_1) - s^1)| \leq \varepsilon(\sigma_1(x_1) - s^1)$$

for all $s^1 \in U_{x_1}$. Generally, we say that f has a Δ_j partial derivative $\Delta_j f(\mathbf{x})$ (with respect to x_j) if for each $\varepsilon > 0$, there exists a neighborhood U_{x_j} of x_j such that

$$|f(x_1, x_2, \ldots, \sigma_j(x_j), \ldots, x_n) - f(x_1, x_2, \ldots, s^j, \ldots, x_n) - \\ -\Delta_j f(\mathbf{x})(\sigma_j(x_j) - s^j)| \leq \varepsilon(\sigma_j(x_j) - s^j)$$

for all $s^j \in U_{x_j}$.

Definition 2.1. ([3], Definition 1). A function $\phi : K_n \to \Re$ is said to be superquadratic if for every $\mathbf{x} \in K_n$ there exists a vector $c(\mathbf{x}) \in \Re^n$ such that

$$\phi(\mathbf{y}) \quad \geq \quad \phi(\mathbf{x}) + \langle c(\mathbf{x}), \mathbf{y} - \mathbf{x} \rangle + \phi(|\mathbf{y} - \mathbf{x}|) \tag{2.1}$$

holds for all $\mathbf{y} \in K_n$. ϕ is said to be strictly superquadratic if (2.1) is strict for all $\mathbf{x} \neq \mathbf{y}$.
Furthermore, we say that ϕ is subquadratic if $-\phi$ is superquadratic.

For example, the function

$$\phi(\mathbf{x}) = -||\mathbf{x}||_p = -\left(\sum_{i=1}^n x_i^p\right)^{1/p}, \ \mathbf{x} \in K_n$$

is superquadratic for $p \geq 1$. We refer to [3] for more examples of superquadratic functions.

Moreover, in this paper, for a given function $\phi : X \subset \Omega^n \to \Re$, we use the notation

$$\Delta\phi(\mathbf{x}) = (\Delta_1\phi(\mathbf{x}), \Delta_2\phi(\mathbf{x}), \ldots, \Delta_n\phi(\mathbf{x}))$$

to denote Δ - gradient of ϕ at a point $\mathbf{x} \in X$, where $\Delta_j\phi(\mathbf{x})$ denotes the Δ_j-partial derivative of ϕ with respect to j:th variable at a point $\mathbf{x}$.

The following lemma shows that nonnegative superquadratic functions are indeed convex functions.

Lemma 2.2 ([3], Lemma 1)**.** *Let ϕ be a superquadratic function and* $c(\mathbf{x}) = (c_1(\mathbf{x}), c_2(\mathbf{x}), \ldots, c_n\mathbf{x})$ *be as in Definition* 2.1*. Then*

(1) $\phi(\mathbf{0}) \leq 0$ *and* $c_j(\mathbf{0}) \leq 0$ *for all* $j \in \{1, 2, \ldots, n\}$.
(2) *If* $\phi(\mathbf{0}) = 0$ *and* $\Delta\phi(\mathbf{0}) = \mathbf{0}$, *then* $c_j(\mathbf{x}) = \Delta_j\phi(\mathbf{x})$ *whenever* $\Delta_j\phi(\mathbf{x})$ *exists for some index* $j \in \{1, 2, \ldots., n\}$ *at* $\mathbf{x} \in K_n$.
(3) *If* $\phi \geq 0$, *then* ϕ *is convex and* $\phi(\mathbf{0}) = 0$ *and* $\Delta\phi(\mathbf{0}) = \mathbf{0}$.

The following Fubini's theorem on time scale in [7] will be needed in the proof of our results in Section 3:

Lemma 2.3. *Let* $(\Omega, \mathcal{M}, \mu_\Delta)$ *and* $(\Lambda, \mathcal{L}, \lambda_\Delta)$, *be two finite dimensional time scale measures spaces. If* $f : \Omega \times \Lambda \to \Re$ *is a* $\mu_\Delta \times \lambda_\Delta$*-integrable functions and define the function* $\phi(y) = \int_\Omega f(x,y)\Delta x$ *for a.e.* $y \in \Lambda$ *and* $\varphi(x) = \int_\lambda f(x,y)\Delta y$ *for a.e.* $x \in \Omega$*, then* ϕ *is* $\lambda_\Delta-$*integrable on* Λ, φ *is* μ_Δ*-integrable on* Ω *and*

$$\int\limits_\Omega \Delta x \int\limits_\Lambda f(x,y)\Delta y = \int\limits_\Lambda \Delta y \int\limits_\Omega f(x,y)\Delta x. \tag{2.2}$$

3. Multivariate Hardy-type Inequalities for Superquadratic Functions on Time Scales

Let $\mathbf{f}(y) = (f_1(y), f_2(y), \ldots, f_n(y))$ be n-tuple of functions such that $f_j(y)$ are μ_{Δ_2}-integrable for all $j \in \{1, 2, \ldots, n\}$. Then $\int_\Lambda \mathbf{f}(y)\Delta\mu_2(y)$ denotes the n-tuple

$$\left(\int\limits_\Lambda f_1(y)\Delta\mu_2(y), \int\limits_\Lambda f_2(y)\Delta\mu_2(y), \ldots, \int\limits_\Lambda f_n(y)\Delta\mu_2(y) \right);$$

That is, $\Delta-$ integral acts on each component of $\mathbf{f}(y)$. Then, we present our first result on Jensen inequality for functions of several variables on time scale as follows:

Theorem 3.1. *Let* $(\Omega, \Sigma_1, \mu_{\Delta_1})$ *and* $(\Lambda, \Sigma_2, \mu_{\Delta_2})$ *be two time scale measure spaces with a* σ *finite measures. Suppose that* $U \subset \Re^n$ *is a closed convex set and* $\Phi \in C(U, \Re)$ *is superquadratic and* $\boldsymbol{f}(\Lambda) \subset U$*. Moreover, let* $\Phi : K_n \to \Re$ *be continuous and superquadratic,* $k : \Omega \times \Lambda \to \Re$ *be nonnegative such that k(x,.) is* $\mu_{\Delta_2}-$*integrable. Then the inequality*

$$\Phi\left(\frac{\int_\Lambda k(x,y)\boldsymbol{f}(y)\Delta\mu_2(y)}{\int_\Lambda k(x,y)\Delta\mu_2(y)} \right) \leq$$
$$\leq \int\limits_\Lambda \frac{k(x,y)}{\int_\Lambda k(x,y)\Delta\mu_2(y)} \left(\Phi(\mathbf{f}(y)) - \Phi(|\boldsymbol{f}(y) - A_k\mathbf{f}(x)|) \right) \Delta\mu_2(y) \tag{3.1}$$

holds for all functions $\boldsymbol{f} : \Lambda \to K_n$*. If* Φ *is subquadratic, then the inequality* (3.1) *is reversed.*

Proof. Suppose that $k(x,y)$ and $f_j(y)$ are μ_{Δ_2}-integrable. Then, for each fixed $x \in \Omega$, the functions

$$\begin{aligned} K(x) &:= \int\limits_\Lambda k(x,y)\Delta\mu_2(y), \\ A_k f_j(x) &:= \frac{1}{K(x)} \int\limits_\Lambda k(x,y) f_j(y)\Delta\mu_2(y) \text{ for all } j \in \{1, 2, \ldots, n\}, \end{aligned} \tag{3.2}$$

and

$$A_k\mathbf{f}(x) := (A_kf_1(x), A_kf_2(x), \ldots, A_kf_n(x)) \tag{3.3}$$

are well defined. By Definition 2.1, there exists an n-tuple constant $\mathbf{C} = (C_1, C_2, \ldots, C_n)$ such that

$$C\left(A_k\mathbf{f}(x)\right) := (C_1(A_k\mathbf{f}(x)), C_2(A_k\mathbf{f}(x)), \ldots, C_n(A_k\mathbf{f}(x)))$$

and that

$$\Phi(\mathbf{y}) \geq \Phi(A_k\mathbf{f}(x)) + \langle C(A_k\mathbf{f}(x)), \mathbf{y} - A_k\mathbf{f}(x)\rangle + \Phi(|\mathbf{y} - A_k\mathbf{f}(x)|), \tag{3.4}$$

since $A_k\mathbf{f}(x) \in K_n$. Replace $\mathbf{y}$ by $\mathbf{f}(y)$ in inequality (3.4) and then we have

$$\Phi(\mathbf{f}(y)) - \Phi(|\mathbf{f}(y) - A_k\mathbf{f}(x)|) - \Phi(A_k\mathbf{f}(x)) \geq \langle C(A_k\mathbf{f}(x)), \mathbf{f}(y) - A_k\mathbf{f}(x)\rangle =$$

$$= \sum_{j=1}^{n} C_j(A_k\mathbf{f}(x))\left(f_j(y) - A_kf_j(x)\right). \tag{3.5}$$

By using the continuity of Φ, we have that $\Phi\circ\mathbf{f}$ is $\Delta\mu_2(y)-$ integrable. Thus, by multiplying inequality (3.5) by $k(x, y)$ and integrating with respect to $\Delta\mu_2(y)$ over the set Λ yields

$$\int_\Lambda k(x,y)\left(\Phi(\mathbf{f}(y)) - \Phi(|\mathbf{f}(y) - A_k\mathbf{f}(x)|)\right)\Delta\mu_2(y) - \phi(A_k\mathbf{f}(x))\int_\Lambda k(x,y)\Delta\mu_2(y) \geq$$

$$\geq \sum_{j=1}^{n} C_j(A_k\mathbf{f}(x))\Bigg(\int_\Lambda k(x,y)(f_j(y) - A_kf_j(x))\Delta\mu_2(y)\Bigg).$$

Finally, by using (3.2) and (3.3) we find that

$$\int_\Lambda k(x,y)\left(\Phi(\mathbf{f}(y)) - \Phi(|\mathbf{f}(y) - A_k\mathbf{f}(x)|)\right)\Delta\mu_2(y) - \phi(A_k\mathbf{f}(x))\int_\Lambda k(x,y)\Delta\mu_2(y) \geq$$

$$\geq \sum_{j=1}^{n} C_j(A_k\mathbf{f}(x))\Bigg(\int_\Lambda k(x,y)(f_j(y) - A_kf_j(x))\Delta\mu_2(y)\Bigg) =$$

$$= \sum_{j=1}^{n} C_j(A_k\mathbf{f}(x))\Bigg(\int_\Lambda k(x,y)f_j(y)\Delta\mu_2(y) - A_kf_j(x)\int_\Lambda k(x,y)\Delta\mu_2(y)\Bigg) =$$

$$= \sum_{j=1}^{n} C_j(A_k\mathbf{f}(x))\Bigg(\int_\Lambda k(x,y)f_j(y)\Delta\mu_2(y) - K(x)A_kf_j(x)\Bigg) =$$

$$= \sum_{j=1}^{n} C_j(A_k\mathbf{f}(x))\Bigg(\int_\Lambda k(x,y)f_j(y)\Delta\mu_2(y) - \int_\Lambda k(x,y)f_j(y)\Delta\mu_2(y)\Bigg) =$$

$$= \sum_{j=1}^{n} C_j(A_k\mathbf{f}(x))(0) = 0.$$

That is

$$\phi(A_k\mathbf{f}(x)) \leq \frac{1}{\int_\Lambda k(x,y)\Delta\mu_2(y)} \times$$

$$\times \int_\Lambda k(x,y)\left(\Phi(\mathbf{f}(y)) - \Phi(|\mathbf{f}(y) - A_k\mathbf{f}(x)|)\right)\Delta\mu_2(y) =$$

$$= \int_\Lambda \frac{k(x,y)}{\int_\Lambda k(x,y)\Delta\mu_2(y)}\left(\Phi(\mathbf{f}(y)) - \Phi(|\mathbf{f}(y) - A_k\mathbf{f}(x)|)\right)\Delta\mu_2(y),$$

which by (3.2)–(3.3), means that (3.1) holds. □

Remark 3.2. In Theorem 3.1, let Φ be nonnegative and convex and replace the time scale measure $\Delta\mu_2(y)$ by the Lebesgue scale measure Δy, then inequality (3.1) reads

$$\Phi\left(\frac{\int_\Lambda k(x,y)\mathbf{f}(y)\Delta y}{\int_\Lambda k(x,y)\Delta y}\right) \leq$$

$$\leq \int_\Lambda \frac{k(x,y)}{\int_\Lambda k(x,y)\Delta y}\left(\Phi(\mathbf{f}(y)) - \Phi(|\mathbf{f}(y) - A_k\mathbf{f}(x)|)\right)\Delta y. \tag{3.6}$$

Clearly inequality (3.6) is a refinement of inequality (1.4).

Remark 3.3. In Theorem 3.1 if we set $n = 1, k(x,y) = 1, \Lambda = [a,b)_{\mathbb{T}}$ then we obtain

$$\Phi\left(\frac{1}{b-a}\int_a^b f(y)\Delta y\right) \leq$$

$$\leq \frac{1}{b-a}\int_a^b \left(\Phi(f(y)) - \Phi\left(\left|f(y) - \frac{1}{b-a}\int_a^b f(y)\Delta y\right|\right)\right)\Delta y. \tag{3.7}$$

Observe that inequality (3.7) coincides with inequality (1.6) obtained by *Barić* et al. [6].

Next, we present our new multivariate Hardy-type inequality on time scale as follows:

Theorem 3.4. *Let $(\Omega, \Sigma_1, \mu_{\Delta_1})$ and $(\Lambda, \Sigma_1, \mu_{\Delta_2})$ be two time scale measure spaces with positive σ-finite measures and let $\zeta : \Omega \to \Re$ and $k : \Omega \times \Lambda \to \Re$ be nonnegative such that k(x,.) is a μ_{Δ_2}-integrable function. Furthermore, suppose that $K : \Omega \to \Re$ is defined by*

$$K(x) := \int_\Lambda k(x,y)\Delta\mu_2(y) > 0, \ x \in \Omega$$

and

$$\eta(y) := \int\limits_{\Omega} \frac{\zeta(x)k(x,y)}{K(x)} \Delta\mu_1(x) < \infty, \quad y \in \Lambda.$$

If $\Phi : K_n \to \Re$ is a continuous and superquadratic function, then the inequality

$$\begin{aligned}
&\int\limits_{\Omega} \zeta(x)\Phi\left(A_k\mathbf{f}(x)\right)\Delta\mu_1(x)+\\
&+\int\limits_{\Lambda}\int\limits_{\Omega} \frac{\zeta(x)k(x,y)}{K(x)}\Phi\left(\left|\mathbf{f}(y)-A_k\mathbf{f}(x)\right|\right)\Delta\mu_1(x)\Delta\mu_2(y) \le\\
&\le \int\limits_{\Lambda} \eta(y)\Phi(\mathbf{f}(y))\Delta\mu_2(y)
\end{aligned} \tag{3.8}$$

holds for all nonnegative integrable functions $\mathbf{f} : \Lambda \to \Re$ and for $A_k\mathbf{f} : \Lambda \to \Re$ defined by

$$A_k\mathbf{f}(x) = \frac{1}{K(x)} \int\limits_{\Lambda} k(x,y)\mathbf{f}(y)\Delta\mu_2(y), \ x \in \Omega$$

If Φ is subquadratic, then the inequality sign in (3.8) *is reversed.*

Proof. By applying the Jensen's inequality (3.1) we find that

$$\begin{aligned}
&\int\limits_{\Omega} \zeta(x)\Phi\left(A_k\mathbf{f}(x)\right)\Delta\mu_1(x) =\\
&= \int\limits_{\Omega} \zeta(x)\Phi\Bigg(\frac{1}{K(x)}\int\limits_{\Lambda} k(x,y)\mathbf{f}(y)\Delta\mu_2(y)\Bigg)\Delta\mu_1(x) \le\\
&\le \int\limits_{\Omega} \zeta(x)\int\limits_{\Lambda} \frac{k(x,y)}{K(x)}\Phi(\mathbf{f}(y))\Delta\mu_2(y)\Delta\mu_1(x)-\\
&- \int\limits_{\Omega} \zeta(x)\int\limits_{\Lambda} \frac{k(x,y)}{K(x)}\Phi\Bigg(\Bigg|\mathbf{f}(y)-\int\limits_{\Lambda} \frac{k(x,y)\mathbf{f}(y)\Delta\mu_2(y)}{K(x)}\Bigg|\Bigg)\Delta\mu_2(y)\Delta\mu_1(x).
\end{aligned}$$

Hence, by (2.2) we obtain that

$$\begin{aligned}
&\int\limits_{\Omega} \zeta(x)\Phi\left(A_k\mathbf{f}(x)\right)\Delta\mu_1(x) \le\\
&\le \int\limits_{\Lambda} \Phi(\mathbf{f}(y))\int\limits_{\Omega} \frac{\zeta(x)k(x,y)}{K(x)}\Delta\mu_1(x)\Delta\mu_2(y)-\\
&- \int\limits_{\Lambda}\int\limits_{\Omega} \frac{\zeta(x)k(x,y)}{K(x)}\Phi\Bigg(\Bigg|\mathbf{f}(y)-\frac{1}{K(x)}\int\limits_{\Lambda} k(x,y)\mathbf{f}(y)\Delta\mu_2(y)\Bigg|\Bigg)\Delta\mu_1(x)\Delta\mu_2(y) =
\end{aligned}$$

$$= \int_{\Lambda} \eta(y)\Phi(\mathbf{f}(y))\Delta\mu_2(y) -$$

$$- \int_{\Lambda}\int_{\Omega} \frac{\zeta(x)k(x,y)}{K(x)} \Phi\left(|\mathbf{f}(y) - (A_k\mathbf{f})(x)|\right) \Delta\mu_1(x)\Delta\mu_2(y).$$

The proof of the case in which Φ is subquadratic is similar the only difference is that the sign of inequalities are reversed and so the proof is complete. □

Remark 3.5. In Theorem 3.4, if we let Φ to be nonnegative and convex and if we replace the time scale measures $\mu_{\Delta_2}(y)$ and $\mu_{\Delta_1}(x)$ by the Lebesgue scale measures Δy and Δx respectively, then inequality (3.8) yields

$$\int_{\Omega} \zeta(x)\Phi\left(A_k\mathbf{f}(x)\right)\Delta x + \int_{\Lambda}\int_{\Omega} \frac{\zeta(x)k(x,y)}{K(x)} \Phi\left(|\mathbf{f}(y) - A_k\mathbf{f}(x)|\right)\Delta x)\Delta y \leq$$

$$\leq \int_{\Lambda} \eta(y)\Phi(\mathbf{f}(y))\Delta y. \tag{3.9}$$

Observe that inequality (3.9) gives a refinement of inequality (1.5) obtained by Donchev et al. [10].

Remark 3.6. The case $n = 1$ in Theorem 3.4 coincides with Theorem 1.4 obtained by Oguntuase and Persson in [23].

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