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NEW ESTIMATIONS OF THE REMAINDER IN THREE-POINT AND FOUR-POINT QUADRATURE FORMULAE VIA THE CHEBYSHEV FUNCTIONAL

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Abstract. We derive some new bounds for general weighted three-point and four-point quadrature formulae by using recently obtained inequality for the Chebyshev functional. As special cases, we provide some new estimates for the error in Gauss-Chebyshev quadrature rules.

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1. Introduction

The well known Chebyshev functional [4] is defined by

$$T(f,g)=\frac{1}{b-a}\int_a^b f(s)\,g(s)\,\mathrm{d}s-\frac{1}{b-a}\int_a^b f(s)\,\mathrm{d}s\cdot\frac{1}{b-a}\int_a^b g(s)\,\mathrm{d}s.$$

where $f,g:[a,b]\to\mathbf{R}$ are two real functions such that $f,g,f\cdot g\in L^1[a,b]$. In paper [2] P. Cerone and S. S. Dragomir proved the following result:

Lemma 1. *If $h:[a,b]\to\mathbf{R}$ is an absolutely continuous function with*

$$(\cdot-a)(b-\cdot)(h')^2\in L^1[a,b],$$

then the following inequality holds

$$T(h,h)\le\frac{1}{2(b-a)}\int_a^b (s-a)(b-s)\left[h'(s)\right]^2\mathrm{d}s. \tag{1.1}$$

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The constant $1/2$ *is the best possible.*

Many researchers have investigated the Chebyshev functional and inequalities related to the Chebyshev functional (see [4], [5], [6] and the references cited therein). In this note we will give some new bounds for three-point and four-point quadrature formulae using Lemma 1 and general weighted three-point and four-point quadrature formulae recently published in [7] and [8]. We will use the above results to get the error estimates for Simpson's, dual Simpson's and Maclaurin's three-point formula and for three-point Gauss-Chebyshev formulae of the first kind and of the second kind. Also, the corresponding error estimates for Simpson's 3/8 formula and Lobatto four-point formula will be derived. More about quadrature formulae and error estimations (from the point of view of inequality theory) can be found in monographs [1] and [3]. The usual convention $f^{(0)} = f$, $0! = 1$ and $\sum_{i=0}^{-1} \cdot = 0$ will be used.

2. Three-point quadrature formulae

Here and hereafter the nonnegative normalized weighted function $w : [a,b] \to [0,\infty)$ is integrable function satisfying $\int_a^b w(s)\,ds = 1$, and $W(s) = \int_a^s w(u)\,du$ for $s \in [a,b]$, $W(s) = 0$ for $s < a$ and $W(s) = 1$ for $s > b$. J. Pečarić and M. Ribičić Penava [7] proved the following general weighted three-point quadrature formula:

Theorem 1. *Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $f : I \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 1$. Let $w : [a,b] \to [0,\infty)$ be some nonnegative normalized weighted function and $A : [a, \frac{a+b}{2}) \to R^+$. Then for each $x \in [a, \frac{a+b}{2})$ the following identity holds*

$$\int_a^b w(s)f(s)\mathrm{d}s = Q_n(f) + \frac{1}{(n-1)!}\int_a^b F_n^w(x,s)f^{(n)}(s)\mathrm{d}s, \tag{2.1}$$

where

$$\begin{aligned} Q_n(f) =& A(x)\left[\sum_{i=0}^{n-1}\frac{f^{(i)}(x)}{i!}\int_a^b w(s)(s-x)^i\mathrm{d}s + \right.\\ &\left. + \sum_{i=0}^{n-1}\frac{f^{(i)}(a+b-x)}{i!}\int_a^b w(s)(s-a-b+x)^i\mathrm{d}s\right] + \\ &+ \left(1-2A(x)\right)\sum_{i=0}^{n-1}\frac{f^{(i)}\left(\frac{a+b}{2}\right)}{i!}\int_a^b w(s)\left(s-\frac{a+b}{2}\right)^i\mathrm{d}s \end{aligned} \tag{2.2}$$

and the function $F_n^w(x,s)$ *satisfies the conditions*

$$F_n^w(x,s) =$$
$$= \begin{cases} -\int\limits_a^s w(u)(u-s)^{n-1}\,du, & a \le s \le x, \\ (A(x)-1)\int\limits_a^s w(u)(u-s)^{n-1}\,du + \\ \quad +A(x)\int\limits_s^b w(u)(u-s)^{n-1}\,du, & x < s \le \frac{a+b}{2}, \\ -A(x)\int\limits_a^s w(u)(u-s)^{n-1}\,du - \\ \quad -(A(x)-1)\int\limits_s^b w(u)(u-s)^{n-1}\,du, \\ & \frac{a+b}{2} < s \le a+b-x, \\ \int\limits_s^b w(u)(u-s)^{n-1}\,du, & a+b-x < s \le b. \end{cases} \tag{2.3}$$

Using identity (2.1) and Lemma 1 we get some new bounds for the remainders in general weighted three-point formula. Let us recall the divided difference of function $f^{(n)}$ is defined as

$$\left[f^{(n)};a,b\right] = \frac{f^{(n)}(b)-f^{(n)}(a)}{b-a}.$$

Theorem 2. *Let I be an open interval in* $\mathbf{R}$, $[a,b]\subset I$, *and let* $w:[a,b]\to[0,\infty)$ *be some nonnegative normalized weighted function. Let* $f: I \to \mathbf{R}$ *be such that* $f^{(n)}$ *is absolutely continuous and* $A:(a,\frac{a+b}{2})\to R^{+}$. *Then for each* $x\in[a,\frac{a+b}{2})$ *we have*

$$\int\limits_a^b w(s)f(s)\mathrm{d}s = Q_n(f)+\frac{1}{(n-1)!}\int\limits_a^b F_n^w(x,s)\mathrm{d}s\left[f^{(n-1)};a,b\right]+ \\ + G_n^w(f,x) \tag{2.4}$$

and the remainder $G_n^w(f,x)$ *satisfies the estimation*

$$\left|G_n^w(f,x)\right| \le \frac{\sqrt{(b-a)}}{\sqrt{2}\,(n-1)!}\left[T\left(F_n^w\left(x,\cdot\right),F_n^w\left(x,\cdot\right)\right)\right]^{1/2}\times$$
$$\times\left[\int_a^b (s-a)(b-s)\left(f^{(n+1)}(s)\right)^2 \mathrm{d}s\right]^{1/2}, \tag{2.5}$$

where $F_n^w(x,\cdot)$ *is defined by* (2.3).

Proof. The identity (2.1) can be rewritten as

$$\int_a^b w(s)f(s)\mathrm{d}s =$$
$$= Q_n(f)+\frac{1}{(n-1)!\,(b-a)}\int_a^b F_n^w(x,s)\mathrm{d}s\int_a^b f^{(n)}(s)\mathrm{d}s + G_n^w(f,x).$$

Since

$$\int_a^b f^{(n)}(s)\,\mathrm{d}s = f^{(n-1)}(b)-f^{(n-1)}(a).$$

then

$$G_n^w(f,x)=\frac{1}{(n-1)!}\int_a^b F_n^w(x,s)f^{(n)}(s)\mathrm{d}s -$$
$$-\frac{1}{(n-1)!}\cdot\frac{f^{(n-1)}(b)-f^{(n-1)}(a)}{b-a}\int_a^b F_n^w(x,s)\mathrm{d}s. \tag{2.6}$$

Now, by using Cauchy-Schwartz inequality for double integrals and applying Lemma 1 with $f^{(n)}$ in place of h, we obtain

$$\left|\frac{1}{b-a}\int_a^b F_n^w(x,s)f^{(n)}(s)\mathrm{d}s-\frac{1}{b-a}\int_a^b F_n^w(x,s)\mathrm{d}s\cdot\frac{1}{b-a}\int_a^b f^{(n)}(s)\mathrm{d}s\right| \le$$
$$\le\left[T\big(F_n^w(x,\cdot),F_n^w(x,\cdot)\big)\right]^{1/2}\cdot\left[T\left(f^{(n)},f^{(n)}\right)\right]^{1/2} <$$
$$<\frac{1}{\sqrt{2\,(b-a)}}\left[T\big(F_n^w(x,\cdot),F_n^w(x,\cdot)\big)\right]^{1/2}\times$$
$$\times\left[\int_a^b (s-a)(b-s)\left(f^{(n+1)}(s)\right)^2\mathrm{d}s\right]^{1/2}. \tag{2.7}$$

Finally, after multiplying (2.7) by $\frac{b-a}{(n-1)!}$ and combining this with (2.6) we get the estimation (2.5). $\square$

Now, we apply the previous results to obtain some error estimates for Gauss-Chebyshev quadrature rules (see [9]). For $w(s) = \frac{1}{\pi\sqrt{1-s^2}}, s \in (-1,1)$ we get some new bounds for Gauss-Chebyshev three-point formulae of the first kind (Corollaries 1, 2, 3). Further, for $w(s) = \frac{2}{\pi}\sqrt{1-s^2}, s \in [-1,1]$ we derive some new bounds for Gauss-Chebyshev three-point formulae of the second kind (Corollaries 4, 5, 6).

Corollary 1. *Let I be an open interval in* $\mathbf{R}$, $[-1,1] \subset I$, *and let* $f : I \to \mathbf{R}$ *be such that* f' *is absolutely continuous. Then the following inequality holds*

$$\left| \int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}} \mathrm{d}s - \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right) \right] \right| <$$

$$< C_1\left(-\frac{\sqrt{3}}{2}\right) \cdot \left[\int_{-1}^{1} \left(1-s^2\right) \left(f''(s)\right)^2 \mathrm{d}s \right]^{1/2},$$

where $C_1\left(-\frac{\sqrt{3}}{2}\right) = \left(\frac{2\pi-6}{3}\right)^{1/2}$.

Proof. This is a special case of Theorem 2 for $n = 1$, $a = -1$, $b = 1$, $x = -\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$ and $w(s) = \frac{1}{\pi\sqrt{1-s^2}}$, $s \in (-1,1)$. $\square$

Corollary 2. *Let I be an open interval in* $\mathbf{R}$, $[-1,1] \subset I$, *and let* $f : I \to \mathbf{R}$ *be such that* f'' *is absolutely continuous. Then the following inequality holds*

$$\left| \int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}} \mathrm{d}s - \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right) \right] - \right.$$

$$\left. - \frac{\pi\sqrt{3}}{6}\left[f'\left(-\frac{\sqrt{3}}{2}\right) - f'\left(\frac{\sqrt{3}}{2}\right) \right] - \frac{\pi}{2}\left[f'; -1, 1\right] \right| <$$

$$< C_2\left(-\frac{\sqrt{3}}{2}\right) \cdot \left[\int_{-1}^{1} \left(1-s^2\right) \left(f'''(s)\right)^2 \mathrm{d}s \right]^{1/2},$$

where $C_2\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{12\sqrt{3}}(256 + 16\pi - 27\pi^2)^{1/2}$.

Proof. Applying Theorem 2 with $n = 2$, $a = -1$, $b = 1$, $x = -\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$ and $w(s) = \frac{1}{\pi\sqrt{1-s^2}}$, $s \in (-1,1)$ we get above inequality. $\square$

Corollary 3. *Let I be an open interval in* $\mathbf{R}$, $[-1,1] \subset I$, *and let $f : I \to \mathbf{R}$ be such that f''' is absolutely continuous. Then the following inequality holds*

$$\begin{aligned}\Bigg| \int_{-1}^{1} \frac{f(s)}{\sqrt{1-s^2}} \mathrm{d}s - \frac{\pi}{3}\left[f\left(-\frac{\sqrt{3}}{2}\right) + f(0) + f\left(\frac{\sqrt{3}}{2}\right)\right] - \\ - \frac{\pi\sqrt{3}}{6}\left[f'\left(-\frac{\sqrt{3}}{2}\right) - f'\left(\frac{\sqrt{3}}{2}\right)\right] - \\ - \frac{\pi}{12}\left[\frac{5}{2}f''\left(-\frac{\sqrt{3}}{2}\right) + f''(0) + \frac{5}{2}f''\left(\frac{\sqrt{3}}{2}\right)\right]\Bigg| < \\ < C_3\left(-\frac{\sqrt{3}}{2}\right) \cdot \left[\int_{-1}^{1} \left(1-s^2\right)\left(f^{(4)}(s)\right)^2 \mathrm{d}s\right]^{1/2},\end{aligned}$$

where $C_3\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{120\sqrt{30}}(-32768 + 24655\pi)^{1/2}$.

Proof. Applying Theorem 2 with $n = 3$, $a = -1$, $b = 1$, $x = -\frac{\sqrt{3}}{2}$, $A\left(-\frac{\sqrt{3}}{2}\right) = \frac{1}{3}$ and $w(s) = \frac{1}{\pi\sqrt{1-s^2}}$, $s \in (-1,1)$ we get above inequality. $\square$

Corollary 4. *Let I be an open interval in* $\mathbf{R}$, $[-1,1] \subset I$, *and let $f : I \to \mathbf{R}$ be such that f' is absolutely continuous. Then the following inequality holds*

$$\begin{aligned}\Bigg| \int_{-1}^{1} \sqrt{1-s^2} f(s) \,\mathrm{d}s - \frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right) + 2f(0) + f\left(\frac{\sqrt{2}}{2}\right)\right]\Bigg| < \\ < C_1\left(-\frac{\sqrt{2}}{2}\right) \cdot \left[\int_{-1}^{1} \left(1-s^2\right)\left(f''(s)\right)^2 \mathrm{d}s\right]^{1/2},\end{aligned}$$

where $C_1\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{24\sqrt{10}}\left(-2048 + 60(8+5\sqrt{2})\pi - 45\sqrt{2}\pi^2\right)^{1/2}$.

Proof. This is a special case of Theorem 2 for $n = 1$, $a = -1$, $b = 1$, $x = -\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$ and $w(s) = \frac{2\sqrt{1-s^2}}{\pi}$, $s \in [-1,1]$. $\square$

Corollary 5. *Let I be an open interval in* $\mathbf{R}$, $[-1,1] \subset I$, *and let $f : I \to \mathbf{R}$ be such that f'' is absolutely continuous. Then the following* *inequality holds*

$$\Bigg|\int_{-1}^{1}\sqrt{1-s^2}f(s)\mathrm{d}s-\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]-$$

$$-\frac{\pi\sqrt{2}}{16}\left[f'\left(-\frac{\sqrt{2}}{2}\right)-f'\left(\frac{\sqrt{2}}{2}\right)\right]-\frac{\pi}{8}\left[f';-1,1\right]\Bigg|<$$

$$<C_2\left(-\frac{\sqrt{2}}{2}\right)\cdot\left[\int_{-1}^{1}\left(1-s^2\right)\left(f'''(s)\right)^2\mathrm{d}s\right]^{1/2},$$

where $C_2\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{240\sqrt{21}}\left(65536-105\pi\left(64-92\sqrt{2}+15\left(3+\sqrt{2}\right)\pi\right)\right)^{1/2}$.

Proof. This is a special case of Theorem 2 for $n=2$, $a=-1$, $b=1$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ and $w(s)=\frac{2}{\pi}\sqrt{1-s^2}$, $s\in[-1,1]$. □

Corollary 6. *Let I be an open interval in $\mathbf{R}$, $[-1,1]\subset I$, and let $f:I\to\mathbf{R}$ be such that f''' is absolutely continuous. Then the following inequality holds*

$$\Bigg|\int_{-1}^{1}\sqrt{1-s^2}f(s)\mathrm{d}s-\frac{\pi}{8}\left[f\left(-\frac{\sqrt{2}}{2}\right)+2f(0)+f\left(\frac{\sqrt{2}}{2}\right)\right]-$$

$$-\frac{\pi\sqrt{2}}{16}\left[f'\left(-\frac{\sqrt{2}}{2}\right)-f'\left(\frac{\sqrt{2}}{2}\right)\right]-$$

$$-\frac{\pi}{64}\left[3f''\left(-\frac{\sqrt{2}}{2}\right)+2f''(0)+3f''\left(\frac{\sqrt{2}}{2}\right)\right]\Bigg|<$$

$$<C_3\left(-\frac{\sqrt{2}}{2}\right)\cdot\left[\int_{-1}^{1}\left(1-s^2\right)\left(f^{(4)}(s)\right)^2\mathrm{d}s\right]^{1/2},$$

where $C_3\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{20160\sqrt{10}}(-16777216+2520\pi(1376+3887\sqrt{2})-1554525\sqrt{2}\pi^2)^{1/2}$.

Proof. Applying Theorem 2 with $n=3$, $a=-1$, $b=1$, $x=-\frac{\sqrt{2}}{2}$, $A\left(-\frac{\sqrt{2}}{2}\right)=\frac{1}{4}$ and $w(s)=\frac{2}{\pi}\sqrt{1-s^2}$, $s\in[-1,1]$ we get above inequality. □

In non-weighted case for a special choice of the function A, $A(x)=\frac{(b-a)^2}{6(a+b-2x)^2}$, $x\in[a,\frac{a+b}{2})$ and special choices of x ($x=a, x=\frac{3a+b}{4}, x=\frac{5a+b}{6}$) we obtain some new bounds for the well-known Simpson's, dual Simpson's and Maclaurin's formula, respectively. In the following corollaries we will use the Beta function and the incomplete Beta function of Euler type defined by

$$B(u,v)=\int_0^1 s^{u-1}(1-s)^{v-1}\,\mathrm{d}s,\quad B_r(u,v)=\int_0^r s^{u-1}(1-s)^{v-1}\,\mathrm{d}s,\quad u,v>0.$$

Corollary 7. *Let I be an open interval in $\mathbf{R}$, $[a,b]\subset I$, and let $f:I\to\mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous. Then the following identity holds*

$$\begin{aligned}\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s &= \frac{1}{6}\sum_{i=0}^{n-1}\left[f^{(i)}(a)+(-1)^i f^{(i)}(b)\right]\frac{(b-a)^i}{(i+1)!}+\\ &+\frac{2}{3}\sum_{i=0}^{n-1} f^{(i)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^i\right)(b-a)^i}{2^{i+1}(i+1)!}+\\ &+\frac{\left(2^{n-1}+1\right)(1+(-1)^n)(b-a)^n}{3\cdot 2^n(n+1)!}\left[f^{(n-1)};a,b\right]+G_n(f,a).\end{aligned}$$

The remainder $G_n(f,a)$ satisfies the estimation

$$\begin{aligned}\left|G_n(f,a)\right| &\le \frac{\sqrt{b-a}}{\sqrt{2}\cdot n!}\left[T\big(F_n(a,\cdot),F_n(a,\cdot)\big)\right]^{1/2}\times\\ &\times\left[\int_a^b (s-a)(b-s)\left(f^{(n+1)}(s)\right)^2\mathrm{d}s\right]^{1/2},\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}&T\big(F_n(a,\cdot),F_n(a,\cdot)\big)=\\ &=\frac{(b-a)^{2n-2}}{9}\left[\frac{2^{2n-2}+3}{2^{2n-1}(2n+1)}+\frac{5(-1)^n B(n+1,n+1)}{2}-\right.\\ &\left.-\left(\frac{\left(2^{n-1}+1\right)(1+(-1)^n)}{2^n(n+1)}\right)^2\right].\end{aligned}$$

Proof. This is a special case of Theorem 2 for $w(s)=\frac{1}{b-a}$, $s\in[a,b]$, $x=a$ and $A(a)=\frac{1}{6}$. ☐

Remark 1. For $n = 1$ in Corollary 7 we have

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s - \frac{1}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right)\right| <$$
$$< \frac{\sqrt{b-a}}{6\sqrt{2}} \cdot \left[\int_a^b (s-a)(b-s)\left(f''(s)\right)^2 \mathrm{d}s\right]^{1/2}.$$

Corollary 8. *Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $f : I \to \mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous. Then the following identity holds*

$$\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s =$$
$$= \frac{2}{3}\sum_{i=0}^{n-1}\left[f^{(i)}\left(\frac{3a+b}{4}\right) + (-1)^i f^{(i)}\left(\frac{a+3b}{4}\right)\right]\frac{\left[3^{i+1} - (-1)^{i+1}\right](b-a)^i}{4^{i+1}(i+1)!} -$$
$$- \frac{1}{3}\sum_{i=0}^{n-1} f^{(i)}\left(\frac{a+b}{2}\right)\frac{\left(1 + (-1)^i\right)(b-a)^i}{2^{i+1}(i+1)!} +$$
$$+ \frac{\left(3^{n+1} - 2^n + 1\right)\left(1 + (-1)^n\right)(b-a)^n}{3 \cdot 2^{2n+1}(n+1)!}\left[f^{(n-1)}; a, b\right] + G_n\left(f, \frac{3a+b}{4}\right).$$

The remainder $G_n\left(f, \frac{3a+b}{4}\right)$ satisfies the bound

$$\left|G_n\left(f, \frac{3a+b}{4}\right)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}\cdot n!}\left[T\left(F_n\left(\frac{3a+b}{4}, \cdot\right), F_n\left(\frac{3a+b}{4}, \cdot\right)\right)\right]^{1/2} \times$$
$$\times \left[\int_a^b (s-a)(b-s)\left(f^{(n+1)}(s)\right)^2 \mathrm{d}s\right]^{1/2}, \tag{2.9}$$

where

$$T\left(F_n\left(\frac{3a+b}{4}, \cdot\right), F_n\left(\frac{3a+b}{4}, \cdot\right)\right) =$$
$$= \frac{4(b-a)^{2n-2}}{9}\left[\frac{3^{2n+1} - 3 \cdot 2^{2n-1} + 2}{2^{4n+1}(2n+1)} +\right.$$
$$+ (-1)^n\left(B_{\frac{3}{4}}(n+1, n+1) - B_{\frac{1}{4}}(n+1, n+1)\right) -$$
$$\left. - \left(\frac{\left(3^{n+1} - 2^n + 1\right)\left(1 + (-1)^n\right)}{2^{2n+2}(n+1)}\right)^2\right].$$

Proof. This is a special case of Theorem 2 for $w(s) = \frac{1}{b-a}$, $s \in [a,b]$, $x = \frac{3a+b}{4}$ and $A\left(\frac{3a+b}{4}\right) = \frac{2}{3}$. $\square$

Remark 2. Let us consider the special case $n = 1$ in Corollary 8. We have

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s - \frac{1}{3}\left(2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right)\right| <$$

$$< \frac{\sqrt{b-a}}{6}\cdot\left[\int_a^b (s-a)(b-s)\left(f''(s)\right)^2 \mathrm{d}s\right]^{1/2}.$$

Corollary 9. *Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $f : I \to \mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous. Then the following identity holds*

$$\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s =$$

$$= \frac{3}{8}\sum_{i=0}^{n-1}\left[f^{(i)}\left(\frac{5a+b}{6}\right) + (-1)^i f^{(i)}\left(\frac{a+5b}{6}\right)\right]\frac{\left[5^{i+1} - (-1)^{i+1}\right](b-a)^i}{6^{i+1}(i+1)!} +$$

$$+ \frac{1}{4}\sum_{i=0}^{n-1} f^{(i)}\left(\frac{a+b}{2}\right)\frac{\left(1+(-1)^i\right)(b-a)^i}{2^{i+1}(i+1)!} +$$

$$+ \frac{\left(5^{n+1} + 2\cdot 3^n + 1\right)\left(1+(-1)^n\right)(b-a)^n}{2^{n+4}\cdot 3^n (n+1)!}\left[f^{(n-1)}; a, b\right] + G_n\left(f, \frac{5a+b}{6}\right).$$

The remainder $G_n\left(f, \frac{5a+b}{6}\right)$ satisfies the bound

$$\left|G_n\left(f, \frac{5a+b}{6}\right)\right| \le \frac{\sqrt{b-a}}{\sqrt{2}\cdot n!}\left[T\left(F_n\left(\frac{5a+b}{6}, \cdot\right), F_n\left(\frac{5a+b}{6}, \cdot\right)\right)\right]^{1/2} \times$$

$$\times \left[\int_a^b (s-a)(b-s)\left(f^{(n+1)}(s)\right)^2 \mathrm{d}s\right]^{1/2}, \tag{2.10}$$

where

$$T\left(F_n\left(\frac{5a+b}{6}, \cdot\right), F_n\left(\frac{5a+b}{6}, \cdot\right)\right) =$$

$$= \frac{(b-a)^{2n-2}}{16}\left[\frac{3\cdot 5^{2n+1} + 16\cdot 3^{2n} + 13}{2^{2n+2}\cdot 3^{2n}(2n+1)} +\right.$$

$$+ \frac{15}{2}(-1)^n\left(B_{\frac{5}{6}}(n+1, n+1) - B_{\frac{1}{6}}(n+1, n+1)\right) -$$

$$\left. - \left(\frac{\left(5^{n+1} + 2\cdot 3^n + 1\right)\left(1+(-1)^n\right)}{3^n\cdot 2^{n+2}(n+1)}\right)^2\right].$$

Proof. This is a special case of Theorem 2 for $w(s) = \frac{1}{b-a}$, $s \in [a,b]$, $x = \frac{5a+b}{6}$ and $A\left(\frac{5a+b}{6}\right) = \frac{3}{8}$. $\square$

Remark 3. Let us consider the special case $n = 1$ in Corollary 3. We have

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s - \frac{1}{8}\left(3f\left(\frac{5a+b}{6}\right) + 2f\left(\frac{a+b}{2}\right) + 3f\left(\frac{a+5b}{6}\right)\right)\right| <$$

$$< \frac{\sqrt{b-a}}{8\sqrt{3}} \cdot \left[\int_a^b (s-a)(b-s)\left(f''(s)\right)^2 \mathrm{d}s\right]^{1/2}.$$

3. Four-point quadrature formulae

Using weighted Montgomery identity the following general weighted closed four-point quadrature formula was proved in [8]:

Theorem 3. *Let I be an open interval in $\mathbf{R}$, $[a,b] \subset I$, and let $f : I \to \mathbf{R}$ be such that $f^{(n-1)}$ is absolutely continuous for some $n \geq 1$. Let $w : [a,b] \to [0,\infty)$ be some nonnegative normalized weighted function and $A : \left(a, \frac{a+b}{2}\right] \to R^+$. Then for each $x \in \left(a, \frac{a+b}{2}\right]$ the following representation holds*

$$\int_a^b w(s)f(s)\mathrm{d}s = P_n(f) + \frac{1}{(n-1)!}\int_a^b S_n^w(x,s)f^{(n)}(s)\mathrm{d}s, \tag{3.1}$$

where

$$\begin{aligned} P_n(f) = A(x)&\left[\sum_{i=0}^{n-1}\frac{f^{(i)}(x)}{i!}\int_a^b w(s)(s-x)^i\mathrm{d}s + \right.\\ &\left.+\sum_{i=0}^{n-1}\frac{f^{(i)}(a+b-x)}{i!}\int_a^b w(s)(s-a-b+x)^i\mathrm{d}s\right] +\\ &+\left(\frac{1}{2}-A(x)\right)\left[\sum_{i=0}^{n-1}\frac{f^{(i)}(a)}{i!}\int_a^b w(s)(s-a)^i\,\mathrm{d}s + \right.\\ &\left.+\sum_{i=0}^{n-1}\frac{f^{(i)}(b)}{i!}\int_a^b w(s)(s-b)^i\,\mathrm{d}s\right] \end{aligned}$$

and the function $S_n^w(x,s)$ satisfies the conditions

$$S_n^w(x,s) =$$
$$= \begin{cases} -\left(\frac{1}{2}+A(x)\right)\int_a^s w(u)(u-s)^{n-1}\,du + \\ \quad +\left(\frac{1}{2}-A(x)\right)\int_s^b w(u)(u-s)^{n-1}\,du, \quad a\le s\le x, \\ -\frac{1}{2}\left[\int_a^s w(u)(u-s)^{n-1}\,du - \int_s^b w(u)(u-s)^{n-1}\,du\right], \\ \qquad\qquad x< s\le a+b-x, \\ -\left(\frac{1}{2}-A(x)\right)\int_a^s w(u)(u-s)^{n-1}\,du + \\ +\left(\frac{1}{2}+A(x)\right)\int_s^b w(u)(u-s)^{n-1}\,du, \quad a+b-x< s\le b. \end{cases} \tag{3.2}$$

Now, we obtain some new bound for the remainder in general weighted four-point formula. This will be done using identity (3.1) and Lemma 1.

Theorem 4. *Let I be an open interval in $\mathbf{R}$, $[a,b]\subset I$, and let $w:[a,b]\to[0,\infty)$ be some nonnegative normalized weighted function. Let $f:I\to\mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous and $A:\left(a,\frac{a+b}{2}\right]\to R^+$. Then for each $x\in\left(a,\frac{a+b}{2}\right]$ the following identity holds*

$$\int_a^b w(s)f(s)\mathrm{d}s = P_n(f) + \frac{1}{(n-1)!}\int_a^b S_n^w(x,s)\mathrm{d}s\left[f^{(n-1)};a,b\right] + \\ + R_n^w(f,x). \tag{3.3}$$

The remainder $R_n^w(f,x)$ satisfies the estimation

$$\left|R_n^w(f,x)\right| \le \frac{\sqrt{(b-a)}}{\sqrt{2}\,(n-1)!}\left[T\left(S_n^w(x,\cdot),S_n^w(x,\cdot)\right)\right]^{1/2}\times \\ \times\left[\int_a^b (s-a)(b-s)\left(f^{(n+1)}(s)\right)^2\mathrm{d}s\right]^{1/2}, \tag{3.4}$$

where $S_n^w(x,\cdot)$ is define by (3.2).

Proof. The proof is similar to the proof of Theorem 2. □

For $w(s)=\frac{1}{b-a}$, $s\in[a,b]$ and $A(x)=\frac{(b-a)^2}{12(x-a)(b-x)}$, $x\in\left(a,\frac{a+b}{2}\right]$ and special choices of variable x, $\left(x=\frac{2a+b}{3}\text{ and } x=-\frac{\sqrt{5}}{5}, \text{ for } [a,b]=[-1,1]\right)$, we get some new error estimates for the well-known Simpson's 3/8 formula and Lobatto four-point formula.

Corollary 10. *Let I be an open interval in $\mathbf{R}$, $[a,b]\subset I$, and let $f:I\to\mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous. Then the following identity holds*

$$\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s=$$
$$=\frac{1}{8}\sum_{i=0}^{n-1}\left[f^{(i)}\left(\frac{2a+b}{3}\right)+(-1)^i f^{(i)}\left(\frac{a+2b}{3}\right)\right]\frac{\left[2^{i+1}+(-1)^i\right](b-a)^i}{3^i(i+1)!}+$$
$$+\frac{1}{8}\sum_{i=0}^{n-1}\left[f^{(i)}(a)+(-1)^i f^{(i)}(b)\right]\frac{(b-a)^i}{(i+1)!}+$$
$$+\frac{\left(3^n+2^{n+1}+1\right)(1+(-1)^n)(b-a)^n}{8\cdot 3^n(n+1)!}\left[f^{(n-1)};a,b\right]+R_n\left(f,\frac{2a+b}{3}\right).$$

The remainder $R_n\left(f,\frac{2a+b}{3}\right)$ satisfies the bound

$$\left|R_n\left(f,\frac{2a+b}{3}\right)\right|\le\frac{\sqrt{b-a}}{\sqrt{2}\cdot n!}\left[T\left(S_n\left(\frac{2a+b}{3},\cdot\right),S_n\left(\frac{2a+b}{3},\cdot\right)\right)\right]^{1/2}\times$$
$$\times\left[\int_a^b(s-a)(b-s)\left(f^{(n+1)}(s)\right)^2\mathrm{d}s\right]^{1/2},\tag{3.5}$$

where

$$T\left(S_n\left(\frac{2a+b}{3},\cdot\right),S_n\left(\frac{2a+b}{3},\cdot\right)\right)=$$
$$=\frac{(b-a)^{2n-2}}{16}\left[\frac{3^{2n}+5\cdot 2^{2n+1}+11}{2\cdot 3^{2n}(2n+1)}+\right.$$
$$+(-1)^n\left(8\cdot B_{\frac{2}{3}}(n+1,n+1)-B_{\frac{1}{3}}(n+1,n+1)\right)-$$
$$\left.-\left(\frac{\left(3^n+2^{n+1}+1\right)(1+(-1)^n)}{2\cdot 3^n(n+1)}\right)^2\right].$$

Proof. This is a special case of Theorem 4 for $w(s)=\frac{1}{b-a}$, $s\in[a,b]$, $x=\frac{2a+b}{3}$ and $A\left(\frac{2a+b}{3}\right)=\frac{3}{8}$. □

Remark 4. For $n=1$ in Corollary 10 we have

$$\left|\frac{1}{b-a}\int_a^b f(s)\mathrm{d}s-\frac{1}{8}\left(f\left(a\right)+3f\left(\frac{2a+b}{3}\right)+3f\left(\frac{a+2b}{3}\right)+f\left(b\right)\right)\right|<$$

$$<\frac{\sqrt{b-a}}{8\sqrt{3}}\cdot\left[\int_a^b\left(s-a\right)\left(b-s\right)\left(f''\left(s\right)\right)^2\mathrm{d}s\right]^{1/2}.$$

Corollary 11. *Let I be an open interval in $\mathbf{R}$, $[-1,1]\subset I$, and let $f:I\to\mathbf{R}$ be such that $f^{(n)}$ is absolutely continuous. Then the following identity holds*

$$\int_{-1}^{1} f(s)\mathrm{d}s=$$

$$=\frac{5}{6}\sum_{i=0}^{n-1}\left[f^{(i)}\left(-\frac{\sqrt{5}}{5}\right)+(-1)^i f^{(i)}\left(\frac{\sqrt{5}}{5}\right)\right]\frac{\left(5+\sqrt{5}\right)^{i+1}+(-1)^i\left(5-\sqrt{5}\right)^{i+1}}{2\cdot 5^{i+1}\left(i+1\right)!}+$$

$$+\frac{1}{6}\sum_{i=0}^{n-1}\left[f^{(i)}(-1)+(-1)^i f^{(i)}\left(1\right)\right]\frac{2^i}{(i+1)!}+$$

$$+\frac{1+(-1)^n}{12\cdot 5^n\left(n+1\right)!}\left[\left(5+\sqrt{5}\right)^{n+1}+\left(5-\sqrt{5}\right)^{n+1}+2\cdot 10^n\right]\left[f^{(n-1)};-1,1\right]+$$

$$+2\cdot R_n\left(f,-\frac{\sqrt{5}}{5}\right).$$

The remainder $R_n\left(f,-\frac{\sqrt{5}}{5}\right)$ satisfies the bound

$$\left|R_n\left(f,-\frac{\sqrt{5}}{5}\right)\right|\le\frac{1}{n!}\left[T\left(S_n\left(-\frac{\sqrt{5}}{5},\cdot\right),S_n\left(-\frac{\sqrt{5}}{5},\cdot\right)\right)\right]^{1/2}\times$$

$$\times\left[\int_{-1}^{1}\left(1-s^2\right)\left(f^{(n+1)}\left(s\right)\right)^2\mathrm{d}s\right]^{1/2},\tag{3.6}$$

where

$$T\left(S_n\left(-\frac{\sqrt{5}}{5},\cdot\right),S_n\left(-\frac{\sqrt{5}}{5},\cdot\right)\right)=$$

$$=\frac{2\cdot 10^{2n}+17\left(5-\sqrt{5}\right)^{2n+1}+7\left(5+\sqrt{5}\right)^{2n+1}}{576\left(2n+1\right)5^{2n}}+$$

$$+\frac{2^{2n}\left(-1\right)^n}{144}\left(18B_{\frac{5+\sqrt{5}}{10}}\left(n+1,n+1\right)-7B_{\frac{5-\sqrt{5}}{10}}\left(n+1,n+1\right)\right)-$$

$$-\left(\frac{1+(-1)^n}{48\cdot 5^n\,(n+1)}\right)^2\left(\left(5+\sqrt{5}\right)^{n+1}+\left(5-\sqrt{5}\right)^{n+1}+2\cdot 10^n\right)^2.$$

Proof. This is a special case of Theorem 4 for $a=-1$, $b=1$, $x=-\frac{\sqrt{5}}{5}$, $A\left(-\frac{\sqrt{5}}{5}\right)=\frac{5}{12}$ and $w\,(s)=\frac{1}{2}$, $s\in[-1,1]$. $\square$

Remark 5. For $n=1$ in Corollary 11 we have

$$\left|\int_{-1}^{1} f(s)\mathrm{d}s-\frac{1}{6}\left(f\,(-1)+5f\left(-\frac{\sqrt{5}}{5}\right)+5f\left(\frac{\sqrt{5}}{5}\right)+f\,(1)\right)\right|\le$$

$$\le\frac{\sqrt{13-5\sqrt{5}}}{6}\cdot\left[\int_{-1}^{1}\left(1-s^2\right)\left(f''\,(s)\right)^2\mathrm{d}s\right]^{1/2}.$$

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