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SOME ASPECTS OF A NOVEL MATRIX SPECTRAL FACTORIZATION ALGORITHM

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Abstract. A new method of matrix spectral factorization has been recently published in [5]. In the present paper we consider some computational aspects of this algorithm.

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1. Introduction

Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle, and

$$S(t) = \begin{pmatrix} s_{11}(t) & s_{12}(t) & \cdots & s_{1m}(t) \\ s_{21}(t) & s_{22}(t) & \cdots & s_{2m}(t) \\ \vdots & \vdots & \vdots & \vdots \\ s_{m1}(t) & s_{m2}(t) & \cdots & s_{mm}(t) \end{pmatrix}, \quad t \in \mathbb{T}, \tag{1}$$

be a matrix function with integrable entries defined on $\mathbb{T}$, $s_{ij} \in L(\mathbb{T})$. Matrix Spectral Factorization Theorem (see, e.g., [8],[4]) asserts that if (1) is positive definite a.e. on $\mathbb{T}$ and the Paley-Wiener condition

$$\log \det S \in L(\mathbb{T}) \tag{2}$$

holds, then (1) can be factorized as

$$S(t) = S_+(t) S_+^*(t), \tag{3}$$

where $S_+(z)$, $|z| < 1$, is an analytic matrix function with entries from the Hardy space H_2 and boundary values $S_+(t)$ and $S_+^*(t)$ is its Hermitian conjugate. The spectral factor S_+ is unique up to a right constant unitary multiplier under the additional restriction that the analytic function $z \to \det S_+(z)$, $|z| < 1$, is outer.

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There are many contexts in which the matrix spectral factorization plays an important role, for example, Wiener filtering, linear quadratic control design, H^∞ robust control etc. Consequently, a number of different methods have been developed in order to actually compute S_+ for a given matrix function S (see survey papers [6], [7]). Recently, a new algorithm of matrix spectral factorization has been developed in [5]. In the present paper we make some theoretical and practical remarks on computational aspects of the proposed algorithm.

Let $L^+ \subset L$ be the class of functions from $L = L(\mathbb{T})$ whose Fourier coefficients with negative indices are equal to zero, and $L^- := \{f : \overline{f} \in L^+\}$. The superscript "+" (resp. "−") of a function f^+ (resp. f^-) emphasizes that this function belongs to L^+ (resp. L^-). $\mathcal{P}_N^+ := \big\{\sum_{n=0}^N c_n t^n : c_n \in \mathbb{C}\big\}$ denotes the set of polynomials of order at most N and $\mathcal{P}_N^- := \big\{\sum_{n=0}^N c_n t^{-n} : c_n \in \mathbb{C}\big\}$. We say that a matrix function belongs to some class if its entries belong to this class.

The central role in the proposed new method of matrix spectral factorization is played by a constructive proof of the following

Theorem 1. [5] *Let F be a (Laurent) polynomial matrix of the form*

$$F(t) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \zeta_1^-(t) & \zeta_2^-(t) & \zeta_3^-(t) & \cdots & \zeta_{m-1}^-(t) & f^+(t) \end{pmatrix}, \tag{4}$$

where

$$\zeta_j^- \in \mathcal{P}_N^-,\ j = 1, 2, \ldots, m-1, \ \text{ and } f^+ \in \mathcal{P}_N^+,\ f^+(0) \neq 0. \tag{5}$$

Then there exists a unitary matrix function U of the form

$$U(t) =$$

$$= \begin{pmatrix} u_{11}^+(t) & u_{12}^+(t) & \cdots & u_{1,m-1}^+(t) & u_{1m}^+(t) \\ u_{21}^+(t) & u_{22}^+(t) & \cdots & u_{2,m-1}^+(t) & u_{2m}^+(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{m-1,1}^+(t) & u_{m-1,2}^+(t) & \cdots & u_{m-1,m-1}^+(t) & u_{m-1,m}^+(t) \\ \overline{u_{m1}^+(t)} & \overline{u_{m2}^+(t)} & \cdots & \overline{u_{m,m-1}^+(t)} & \overline{u_{mm}^+(t)} \end{pmatrix}, \quad t \in \mathbb{T}, \tag{6}$$

$$u_{ij}^+ \in \mathcal{P}_N^+ \tag{7}$$

with

$$\det U(t) \equiv 1 \tag{8}$$

such that

$$FU \in \mathcal{P}_N^+. \tag{9}$$

In the following section, under an additional restriction on f^+ that $f^+(t) \neq 0$ for $|t| < 1$, we give a short (non-constructive) proof of this theorem based on Polynomial Matrix Spectral factorization Theorem a simple proof of which can be found in recently published paper [1].

A basic observation of the proof of Theorem 1 proposed in [5] is that the columns $U_1, U_2, \dots, U_m$ of matrix function (6),

$$U_k(t) = (u_{1k}^+(t), u_{2k}^+(t), \dots, \overline{u_{mk}^+(t)})^T, \ \ k = 1, \dots, m, \ t \in \mathbb{T},$$

are linearly independent solutions of the following system

$$\begin{cases} \zeta_1^- x_m^+ - f^+ \overline{x_1^+} \in L^+ \\ \zeta_2^- x_m^+ - f^+ \overline{x_2^+} \in L^+ \\ \cdot \qquad \cdot \qquad \cdot \\ \zeta_{m-1}^- x_m^+ - f^+ \overline{x_{m-1}^+} \in L^+ \\ \zeta_1^- x_1^+ + \zeta_2^- x_2^+ + \dots + \zeta_{m-1}^- x_{m-1}^+ + f^+ \overline{x_m^+} \in L^+ \end{cases} \tag{10}$$

($X(t) = (x_1(t), x_2(t), \dots, \overline{x_m(t)})$, $t \in \mathbb{T}$, is unknown vector function here). Solving the system (10) (i.e. finding the coefficients of $x_k^+, k = 1, 2, \dots, m$, for given coefficients of $\zeta_k^-, k = 1, 2, \dots, m-1$, and f^+, which can be reduced to solving $N \times N$ system of linear algebraic equations), it provides an algorithm for construction of the unitary matrix function (6). This procedure, which drastically reduces computational burden of the matrix spectral factorization algorithm, is described in [5] and this is a key component of the new factorization method.

The importance of system (10) in finding the corresponding unitary matrix (6) is further emphasized by the following

Theorem 2. (see [2, Lemma 5]) *Let F be a matrix function* (4) *satisfying* (5)*, and let U be the corresponding matrix function* (6) *satisfying* (7)*,* (8) *and such that* (9) *holds. Then each column of* (6) *satisfies the system* (10)*.*

In section 3 we provide a simple transparent proof of this theorem.

In actual computation of matrix function (6) according to the algorithm described in [5], we first construct m independent solutions of (10), viz., $V_1, V_2, \dots, V_m$,

$$V_k(t) = (v_{1k}^+(t), v_{2k}^+(t), \dots, \overline{v_{mk}^+(t)})^T, \ \ k = 1, 2, \dots, m, \ \ t \in \mathbb{T}, \tag{11}$$

satisfying additional requirement

$$\begin{aligned} &\big(c_0\{\zeta_1^- v_{mk}^+ - f^+ \overline{v_{1k}^+}\}, \dots, c_0\{\zeta_{m-1}^- v_{mk}^+ - f^+ \overline{v_{m-1,k}^+}\}, \\ &c_0\{\zeta_1^- v_{1k}^+ + \zeta_2^- v_{2k}^+ + \dots + f^+ \overline{v_{mk}^+}\}\big) = e_k = (\delta_{1k}, \delta_{2k}, \cdots, \delta_{mk}) \end{aligned} \tag{12}$$

(here, of course, $c_0\{g\}$ denotes the constant term in the Fourier series of g). It is then proved in [5] that

$$V^*(t)V(t) = C, \tag{13}$$

where $V(t) = [V_1(t), V_2(t), \dots, V_m(t)]$, $t \in \mathbb{T}$, is the $m \times m$ matrix function and C is a (constant, independent of t) nonsingular matrix. Consequently, the matrix $V(1)$ is nonsingular and the matrix function (6) in Theorem 1 is computed by the formula

$$U(t) = V(t)V^{-1}(1), \quad t \in \mathbb{T}$$

(see [5, formula (51)]). However, for practical computations of U according to these steps, it is important not only to show that $V(1)$ is non-singular but also that it is well conditioned, as we need to compute $V^{-1}(1)$. In section 4 we prove the following

Theorem 3. *Let functions ζ_j^-, $j = 1, 2, \dots, m-1$, and f^+ satisfy* (5) *and let vector functions $V_1, V_2, \dots, V_m$ of the form* (11) *be solutions of the system* (10) *satisfying* (12)*. Then the matrix*

$$V = V(1) = [V_1(1), V_2(1), \dots, V_m(1)]$$

is well conditioned in a sense that

$$\sigma_{\max}(V)/\sigma_{\min}(V) < M,$$

where M can be explicitly written in terms of ζ_i^- and f^+.

Since in practical applications one deals mostly with real numbers, for simplicity of presentation, we will assume in the above theorem that the coefficients of ζ_i^- and f^+ are real.

2. Proof of Theorem 1

For a polynomial matrix

$$P(t) = \sum_{n=0}^{N} P_n t^n,$$

let

$$P^*(t) = \sum_{n=0}^{N} P_n^* t^{-n}.$$

Note that $P^*(t)$ coincides with the usual Hermitian conjugate whenever $|t| = 1$. Polynomial Matrix Spectral Factorization Theorem asserts that if

$$S(t) = \sum_{n=-N}^{N} S_n t^n,$$

where $S_n \in \mathbb{C}^{m\times m}$ (the class of $m \times m$ matrices), and $S(t)$ is positive definite for almost every $t \in \mathbb{T}$, then it admits the factorization (3), where

$S_+(t)$ is invertible inside $\mathbb{T}$. Such factorization is unique up to a constant unitary matrix with determinant 1 if we specify $\det S_+$ as a scalar spectral factor of $\det S$. Consequently, let P_+ be a spectral factor of FF^* with the determinant

$$\det P_+(t) = f^+(t) = \det F(t). \tag{14}$$

The relation

$$F(t)F^*(t) = P_+(t)P_+^*(t)$$

implies that

$$U(t) = F^{-1}(t)P_+(t) \tag{15}$$

is a unitary matrix function for $t \in \mathbb{T}$, while (14) yields $\det U(t) = 1$. Consequently (8) and (9) hold, and we need to prove that U has the form (6), (7). Since the inverse of matrix function (4) has the form

$$F^{-1} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ -\zeta_1^-/f^+ & -\zeta_2^-/f^+ & \cdots & -\zeta_{m-1}^-/f^+ & 1/f^+ \end{pmatrix} \tag{16}$$

and $P_+ \in \mathcal{P}_N^+$, it easily follows that the first $m-1$ rows of (15) belong to $\mathcal{P}_N^+$. Since $U^{-1}(t) = U^*(t)$ for $t \in \mathbb{T}$ and (8) holds, we have that the cofactor of $u_{mj}(t)$ is $\overline{u_{mj}(t)}$. On the other hand, this cofactor belongs to L^+ and thus $u_{mj} = \overline{u_{mj}^+}$. In addition, by virtue of (9),

$$\zeta_1^- u_{ij}^+ + \cdots + \zeta_{m-1}^- u_{(m-1),j}^+ + f^+\overline{u_{mj}^+} \in \mathcal{P}_N^+,$$

which implies that u_{mj}^+ is of order at most N since (5) holds. Therefore, it is obtained that U has the form (6), (7).

3. Proof of Theorem 2

The last condition in (10) is clear because of (9), and we need only to show that

$$\zeta_i^- u_{mj}^+ - f^+\overline{u_{ij}^+} \in L^+, \quad 1 \le i \le m-1,\ 1 \le j \le m. \tag{17}$$

Let

$$\Phi_+ := FU \in \mathcal{P}_N^+. \tag{18}$$

It follows from (4) and (8) that $\det \Phi_+(t) = f^+(t)$, $t \in \mathbb{T}$, and consequently

$$f^+ \cdot \Phi_+^{-1} = \operatorname{cof}(\Phi_+)^T \in L^+.$$

According to (18),

$$U^* F^{-1} = \Phi_+^{-1}.$$

Thus
$$U^* f^+ F^{-1} = f^+ \Phi_+^{-1} \in L^+.$$
On the other hand,
$$-U^* f^+ F^{-1} =$$
$$= \begin{pmatrix} \overline{u_{11}^+} & \cdots & \overline{u_{m-1,1}^+} & u_{m1}^+ \\ \overline{u_{12}^+} & \cdots & \overline{u_{m-1,2}^+} & u_{m2}^+ \\ \vdots & \vdots & \vdots & \vdots \\ \overline{u_{1m}^+} & \cdots & \overline{u_{m-1,m}^+} & u_{mm}^+ \end{pmatrix} \begin{pmatrix} -f^+ & 0 & \cdots & 0 & 0 \\ 0 & -f^+ & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -f^+ & 0 \\ \zeta_1^- & \zeta_2^- & \cdots & \zeta_{m-1}^- & -1 \end{pmatrix} \in L^+$$
and the functions in (17) appear as entries in the first $m-1$ columns of the above product matrix. Thus (17) holds.

4. Proof of Theorem 3

From the geometric interpretation of determinant, it is easy to derive the following

Lemma 1. *Let $y_i \in \mathbb{R}^m$, $i = 1, 2, \dots, m$. If*
$$\inf_{1 \le j \le m,\, \alpha_1, \alpha_2, \dots, \alpha_m \in \mathbb{R}} \|y_j - \sum_{i \ne j} \alpha_i y_i\| \ge c > 0, \tag{19}$$
then
$$\big|\det[y_1, y_2, \dots, y_m]\big| \ge c^m. \tag{20}$$

We proceed with the proof of Theorem 3 introducing notation used in [5]. Let $f^+(t) = \sum_{n=0}^N d_n t^n$ and $\zeta_i^-(t) = \sum_{n=0}^N \gamma_{in} t^{-n}$, $i = 1, 2, \dots, m-1$. Since $f^+(0) \ne 0$, $\frac{1}{f^+}$ can be represented as a power series in the neighborhood of 0:
$$\frac{1}{f^+(z)} = \sum_{n=0}^{\infty} b_n z^n.$$
Let also
$$\left[\frac{\zeta_i^-}{f^+}\right]_- = \left[\sum_{k=0}^N \gamma_{in} t^{-n} \cdot \sum_{n=0}^N b_n t^n\right]_- = \sum_{n=0}^N \eta_{in} t^{-n}, \tag{21}$$
where $[\cdot]_-$ denotes the projection operator, i.e., $\left[\sum_{k=-N}^N c_k t^n\right]_- = \sum_{n=-N}^0 c_n t^n$. Then
$$\|(\eta_{i0}, \eta_{i1}, \cdots, \eta_{iN})\|^2 = \frac{1}{2\pi} \left\| \sum_{n=0}^N \eta_{in} t^{-n} \right\|_{L^2(\mathbb{T})}^2 \le \frac{M_0^2}{2\pi} \|\zeta_i^-(t)\|_{L^2(\mathbb{T})}^2, \tag{22}$$

where

$$M_0 = \sup_{t\in\mathbb{T}} \Big| \sum_{n=0}^{N} b_n t^n \Big|.$$

Let

$$D = \begin{pmatrix} d_0 & d_1 & d_2 & \cdots & d_{N-1} & d_N \\ 0 & d_0 & d_1 & \cdots & d_{N-2} & d_{N-1} \\ 0 & 0 & d_0 & \cdots & d_{N-3} & d_{N-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & d_0 \end{pmatrix}.$$

Then

$$D^{-1} = \begin{pmatrix} b_0 & b_1 & b_2 & \cdots & b_{N-1} & b_N \\ 0 & b_0 & b_1 & \cdots & b_{N-2} & b_{N-1} \\ 0 & 0 & b_0 & \cdots & b_{N-3} & b_{N-2} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 0 & b_0 \end{pmatrix},$$

Let also

$$\Gamma_i = \begin{pmatrix} \gamma_{i0} & \gamma_{i1} & \gamma_{i2} & \cdots & \gamma_{i,N-1} & \gamma_{iN} \\ \gamma_{i1} & \gamma_{i2} & \gamma_{i3} & \cdots & \gamma_{iN} & 0 \\ \gamma_{i2} & \gamma_{i3} & \gamma_{i4} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \gamma_{iN} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \quad i=1,2,\ldots,m-1, \quad \Gamma_m = D.$$

Then direct computations show that

$$\Theta_i := D^{-1}\,\Gamma_i = \begin{pmatrix} \eta_{i0} & \eta_{i1} & \eta_{i2} & \cdots & \eta_{i,k-1} & \eta_{iN} \\ \eta_{i1} & \eta_{i2} & \eta_{i3} & \cdots & \eta_{iN} & 0 \\ \eta_{i2} & \eta_{i3} & \eta_{i4} & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \eta_{iN} & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{23}$$

$i = 1, 2, \ldots, m-1$, (note that Θ_i is a symmetric matrix) and for any vectors $x = (x_0, x_1, \ldots, x_N)^T$ and $y = (y_0, y_1, \ldots, y_N)^T$ one has $y = \Theta_i\, x$ if and only if

$$\left[\sum_{n=0}^{N} \eta_{in} t^{-n} \sum_{n=0}^{N} x_n t^n \right]_- = \sum_{n=0}^{N} y_n t^{-n}.$$

Consequently,

$$\|(y_0, y_1, \cdots, y_N)\| \le M_i \|(x_0, x_1, \cdots, x_N)\|, \tag{24}$$

where

$$M_i = \sup_{t\in\mathbb{T}} \Big| \sum_{n=0}^{N} \eta_{in} t^{-n} \Big|.$$

Now we estimate the maximal singular value $\sigma_{\max}(V)$ of $V(1)$. Let matrix function V, described in Introduction, be

$$V(t)=\begin{pmatrix} v^+_{11}(t) & v^+_{12}(t) & \cdots & v^+_{1,m-1}(t) & v^+_{1m}(t) \\ v^+_{21}(t) & v^+_{22}(t) & \cdots & v^+_{2,m-1}(t) & v^+_{2m}(t) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ v^+_{m-1,1}(t) & v^+_{m-1,2}(t) & \cdots & v^+_{m-1,m-1}(t) & v^+_{m-1,m}(t) \\ \overline{v^+_{m1}(t)} & \overline{v^+_{m2}(t)} & \cdots & \overline{v^+_{m,m-1}(t)} & \overline{v^+_{mm}(t)} \end{pmatrix}, \quad t\in\mathbb{T},$$

where

$$v^+_{ij}(t)=\sum_{n=0}^{N}\alpha^{ij}_n t^n.$$

For any matrix A, we have

$$\sum_i\sum_j A^2_{ij}=\sum_k\sigma^2_k(A),$$

where A_{ij} are entries of A and $\sigma_k(A)$ are singular values of A. Thus,

$$\sigma^2_{\max}(V)\le\sum_{i=1}^m\sum_{j=1}^m V^2_{ij}.$$

By virtue of (13), the singular values of $V(t)$ do not depend on $t\in\mathbb{T}$. Consequently, $\sigma^2_{\max}(V)\le\sum_{i=1}^m\sum_{j=1}^m(v^+_{ij}(t))^2$ for each $t\in\mathbb{T}$. Integrating over $\mathbb{T}$:

$$\sigma^2_{\max}(V)\le\frac{1}{2\pi}\sum_{i=1}^m\sum_{j=1}^m\int\limits_{\mathbb{T}}(v^+_{ij}(t))^2\,dt=\sum_{i=1}^m\sum_{j=1}^m\sum_{n=0}^N(\alpha^{ij}_n)^2. \tag{25}$$

On the other hand, as it is proved in [5], $X=(\alpha^{mj}_0,\alpha^{mj}_1,\dots,\alpha^{mj}_N)^T$ is a solution of the system

$$\Delta X=\Theta_j D^{-1}\mathbb{1},\quad j=1,2,\dots,m, \tag{26}$$

where $\Delta=\Theta_1^2+\Theta_2^2+\cdots+\Theta^2_{m-1}+I_m$ and $\mathbb{1}=(1,0,\dots,0)^T$, and

$$(\alpha^{ij}_0,\alpha^{ij}_1,\dots,\alpha^{ij}_N)^T=\Theta_i(\alpha^{mj}_0,\alpha^{mj}_1,\dots,\alpha^{mj}_N)^T-\delta_{ij}D^{-1}\mathbb{1} \tag{27}$$

(see [5, formulas (31) and (30)]), while

$$\Theta_i D^{-1}\mathbb{1}=b_0(\eta_{i0},\eta_{i1},\dots,\eta_{iN})^T,\quad i=1,2,\dots,m-1, \tag{28}$$

due to (23). Since Δ is positive definite with eigenvalues greater than or equal to 1, Δ^{-1} has all eigenvalues not exceeding than 1, and it follows from

(26), (28) and (22) that

$$\|(\alpha_0^{mj}, \alpha_1^{mj}, \dots, \alpha_N^{mj})\| \le |b_0| \|(\eta_{j0}, \eta_{j1}, \dots, \eta_{jn})\| \le$$
$$\le \frac{|b_0| M_0}{\sqrt{2\pi}} \|\zeta_j^-(t)\|_{L^2(\mathbb{T})}, \tag{29}$$

$j = 1, 2, \dots, m-1$, while

$$\|(\alpha_0^{mm}, \alpha_1^{mm}, \dots, \alpha_N^{mm})\| \le |b_0|. \tag{30}$$

By virtue of (27), (24) and (29) or (30), we have

$$\|(\alpha_0^{ij}, \alpha_1^{ij}, \dots, \alpha_N^{ij})\| \le$$
$$\le \frac{M_i |b_0| M_0}{\sqrt{2\pi}} \|\zeta_j^-(t)\|_{L^2(\mathbb{T})} + \delta_{ij}|b_0|, \quad 1 \le i, j < m, \tag{31}$$

and

$$\|(\alpha_0^{im}, \alpha_1^{im}, \dots, \alpha_N^{im})\| \le M_i |b_0|, \quad 1 \le i \le m-1. \tag{32}$$

It follows now from (29), (30), (31) and (32) that

$$\sum_{i=1}^m \sum_{j=1}^m \sum_{n=0}^N (\alpha_n^{ij})^2 = \sum_{i=1}^m \sum_{j=1}^m \|(\alpha_0^{ij}, \alpha_1^{ij}, \dots, \alpha_N^{ij})\|^2 \le$$
$$\le b_0^2 \left(\frac{M_0^2}{\pi} \sum_{j=1}^{m-1} \|\zeta_j^-(t)\|^2_{L^2(\mathbb{T})} + 1 \right) \left(1 + \sum_{i=1}^{m-1} M_i^2 \right) \tag{33}$$

and (25) implies that

$$\delta_{\max}(V) \le C_0, \tag{34}$$

where C_0 is the square root of (33).

Next we give a lower estimate for $\delta_{\min}(V)$. Namely, we prove that

$$\det(V) \ge c^{-m} \tag{35}$$

for a suitably chosen constant c. It will then follow from (34) and (35) that $C_0^{m-1}\delta_{\min}(V) \ge c^{-m}$, and consequently

$$\frac{\delta_{\max}(V)}{\delta_{\min}(V)} \le (C_0 c)^m.$$

For a vector function $\mathbf{x}(t) = (x_1(t), \dots, x_m(t)) \in L_\infty^m(\mathbb{T})$, let

$$\|\mathbf{x}(t)\|_{L_2(\mathbb{T})} = \left(\int_{\mathbb{T}} \sum_{i=1}^m (x_i(t))^2 \, dt \right)^{\frac{1}{2}} = \left(\int_{\mathbb{T}} \|\mathbf{x}(t)\|^2 \, dt \right)^{\frac{1}{2}}. \tag{36}$$

Direct computations show that the map $\mathcal{L} : L_2^m(\mathbb{T}) \longrightarrow \mathbb{R}^m$ defined by

$$\mathcal{L}(x_1, x_2, \ldots, x_m) =$$
$$= \frac{1}{2\pi}\Bigg(\int\limits_{\mathbb{T}} \big(\zeta_1^-(t)x_m(t) - f^+(t)\overline{x_1(t)}\big)dt, \ldots,$$
$$\int\limits_{\mathbb{T}} \big(\zeta_{m-1}^-(t)x_m(t) - f^+(t)\overline{x_{m-1}(t)}\big)\,dt,$$
$$\int\limits_{\mathbb{T}} \big(\zeta_1^-(t)x_1(t) + \cdots + \zeta_{m-1}^-(t)x_{m-1}(t) + f^+(t)\overline{x_m(t)}\big)\,dt\Bigg)$$

is bounded:

$$\|\mathcal{L}(\mathbf{x})\| \le C_1\|\mathbf{x}\|_{L_2(\mathbb{T})}\,. \tag{37}$$

Indeed, let

$$C_1 = \sqrt{3m}\sup_{1\le i<m,\, t\in\mathbb{T}}(|\zeta_i^-(t)|, |f^+(t)|).$$

Then

$$2\pi\|\mathcal{L}\big(x_1, x_2, \ldots, x_m\big)\|^2 \le$$
$$\le \sum_{i=1}^{m-1}\int\limits_{\mathbb{T}} \big(\zeta_i^-(t)\overline{x_m(t)} - f^+(t)x_i(t)\big)^2\,dt + \int\limits_{\mathbb{T}}\Bigg(\sum_{i=1}^{m-1}\zeta_i^-(t)x_i(t) +$$
$$+ f^+(t)\overline{x_m(t)}\Bigg)^2 dt \le \sum_{i=1}^{m-1} 2\int\limits_{\mathbb{T}} \big(|\zeta_i^-(t)|^2|x_m(t)|^2 + |f^+(t)|^2|x_i(t)|^2\big)\,dt +$$
$$+ \int\limits_{\mathbb{T}} m\Bigg(\sum_{i=1}^{m-1}|\zeta_i^-(t)|^2|x_i(t)|^2 + |f^+(t)|^2|x_m(t)|^2\Bigg)\,dt =$$
$$= \sum_{i=1}^{m-1}\int\limits_{\mathbb{T}} \big(m|\zeta_i^-(t)|^2 + 2|f^+(t)|^2\big)|x_i(t)|^2\,dt +$$
$$+ \int\limits_{\mathbb{T}}\Bigg(2\sum_{i=1}^{m-1}|\zeta_i^-(t)|^2 + m|f^+(t)|^2\Bigg)|x_m(t)|^2\,dt \le C_1^2\|\mathbf{x}\|^2_{L_2(\mathbb{T})}$$

and (37) follows.

Since $\langle V_i(t), V_j(t)\rangle$, the scalar products of the columns of $V(t)$, are independent of t due to (13), we have

$$\Bigg\|\sum_{i=1}^{n}\alpha_i V_i(t)\Bigg\| = \text{const}, \tag{38}$$

i.e., the left hand side of (38) is independent of t.

Since $\mathcal{L}(V_k) = e_k$ according to (12), we have

$$\Big\|\mathcal{L}\Big(V_j(t) - \sum_{i\neq j}\alpha_i V_i(t)\Big)\Big\| = \Big\|\mathcal{L}(V_j(t)) - \sum_{i\neq j}\alpha_i \mathcal{L}(V_i(t))\Big\| =$$
$$= \|(-\alpha_1, \dots, -\alpha_{j-1}, 1, -\alpha_{j+1}, \dots, -\alpha_m)\| \geq 1$$

and, due to (37),

$$\Big\|V_j(t) - \sum_{i\neq j}\alpha_i V_i(t)\Big\|_{L_2(\mathbb{T})} \geq C_1^{-1}. \tag{39}$$

However, by virtue of (36) and (38),

$$\Big\|V_j(t) - \sum_{i\neq j}\alpha_i V_i(t)\Big\|^2_{L_2(\mathbb{T})} = \int_{\mathbb{T}} \Big\|V_j(t) - \sum_{i\neq j}\alpha_i V_i(t)\Big\|^2 dt =$$
$$= 2\pi \Big\|V_j(1) - \sum_{i\neq j}\alpha_i V_i(1)\Big\|^2 \tag{40}$$

and (39) and (40) imply that

$$\Big\|V_j(1) - \sum_{i\neq j}\alpha_i V_i(1)\Big\| \geq \frac{1}{\sqrt{2\pi}C_1}.$$

It remains now to apply Lemma 1 to get (35).

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