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THE RIEMANN-HILBERT PROBLEM IN SMIRNOV CLASS WITH A VARIABLE EXPONENT AND AN ARBITRARY POWER WEIGHT

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Abstract. The paper considers the Riemann-Hilbert problem $\operatorname{Re}[(a(t)+ib(t))\phi^+(t)]=c(t)$ in the weighted Smirnov class $E^{p(t)}(D;\omega)$ with a variable exponent. The domain D is bounded by a piecewise smooth curve possessing one angular point A, at which the size of interior with respect to D angle is equal to $\nu\pi$, $0<\nu\leq 2$ and the weight ω is an arbitrary power function of type $\omega(z)=(z-A)^\beta$, $\beta\in\mathbb{R}$.

Depending on values of numbers ν, β, $p(A)$ and functions $a(t)$, $b(t)$, there arise different situations for solvability of the problem. All possible cases are investigated. The necessary and sufficient conditions of solvability are pointed out and all solutions (when they exist) are constructed.

რეზიუმე. ნაშრომში რიმან-ჰილბერტის ამოცანა $\operatorname{Re}[(a(t)+ib(t))\phi^+(t)]=c(t)$ გამოკვლეულია სმირნოვის $E^{p(t)}(D;\omega)$ კლასში ცვლადი $p(t)$ მაჩვენებლისა და ხარისხოვანი $\omega(z)=(z-A)^\beta$ წონის შემთხვევაში, სადაც A არის D არის საზღვრის კუთხური წერტილი, რომელშიც კუთხის სიდიდე (D არის მიმართ) არის $\pi\nu$, $0<\nu\leq 2$, β კი ნებისმიერი ნამდვილი რიცხვია.

იმისდამიხედვით თუ როგორია მოცემული $a(t)$, $b(t)$ ფუნქციები და თანაფარდობანი ν, β, $p(A)$ რიცხვებს შორის. გვაქვს ამოცანის ამოხსნადობის სხვადასხვა სურათი. განხილულია ყველა შესაძლო შემთხვევა. გამოვლენილია ამოხსნადობის აუცილებელი და საკმარისი პირობები. დათვლილია წრფივად დამოუკიდებელი ამონახსნების რაოდენობა, აგებულია ამოხსნები როცა ისინი არსებობს.

1^0. **Introduction.** Let D be a simply connected domain whose boundary Γ is a closed simple rectifiable curve. The real functions $a(t)$, $b(t)$ and

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$c(t)$ are prescribed on Γ and the domain D involves an $A(D)$ class of analytic functions $\phi(z)$ with boundary values $\phi^+(t)$, $t \in \Gamma$. We are required to define those functions ϕ from $A(D)$ for which

$$\operatorname{Re}[(a(t)+ib(t))\phi^+(t)] = c(t). \tag{1}$$

This problem is the Riemann-Hilbert problem. It has been thoroughly studied in various assumptions regarding Γ, a, b, c and $A(D)$ (see, e.g., [1–4], et al.).

In the recent years, a large number of works in the analysis and its applications are focused upon the boundary value problems in which unknown functions are Lebesgue integrable with a variable exponent.

In this connection, in [5-6] one can find generalizations of weighted V.I. Smirnov classes $E^p(D;\omega)$, analytic in the domain D, when p is a function of definite smoothness. Along with the above, V. Kokilashvili and S. Samko have proved that the Cauchy singular integral operator

$$S:\varphi \to S_\Gamma\varphi,\ \ (S_\Gamma\varphi)(t) = \frac{1}{\pi i}\int\limits_{\Gamma} \frac{\varphi(\tau)}{\tau - t}d\tau,\ \ t \in \Gamma, \quad \varphi \in L^{p(\cdot)}(\Gamma,\omega) \tag{2}$$

for wide classes of curved functions $p(t)$ and power weight functions

$$\omega(t) = \prod_{k=1}^{n}(t-t_k), \quad t_k \in \Gamma,$$
$$-\frac{1}{p(t_k)} < \alpha_k < \frac{1}{q(t_k)},\ \ q(t) = p(t)[p(t)-1]^{-1} \tag{3}$$

is bounded in the Lebesgue space with a variable exponent $L^{p(t)}(\Gamma,\omega)$ (see [7-8], et al.). It is stated in [9] that if $p(t)$ is Log is Hölder continuous on Γ and $\min p(t) > 1$, then S_Γ is continuous in $L^{p(\cdot)}(\Gamma)$, if and only if Γ is the Carleson curve (i.e., $\Gamma \in R$).

This fact made it possible to investigate in classes $E^{p(\cdot)}(D;\omega)$ various boundary value problems of the theory of analytic and harmonic functions (since problem (1) for $a(t)=1$ and $b(t)=0$ can be considered as the Dirichlet problem in the class $e^{p(\cdot)}(D;\omega) = \{u : u \in \operatorname{Re}\phi, \quad \phi \in E^{p(\cdot)}(D;\omega)\}$).

In particular, problem (1) has been investigated when Γ is a piecewice smooth curve free from an outer peak, and ω is given with condition (3) (see [10-15]). A part of the obtained results can be found in the book [16].

In the present work, we consider this problem in the class $E^{p(\cdot)}(D;\omega)$, when $p \in \widetilde{\mathcal{P}}(\Gamma)$ (for definition of $\widetilde{\mathcal{P}}(\Gamma)$, see below),

$$\Gamma \in C^1(A;\nu), \quad 0<\nu\le 2, \quad \omega(t) = (t-A)^\beta, \quad \beta \in \mathbb{R}. \tag{4}$$

Since in the weight $\omega(t) = (t-A)^\beta$ we take β_k as an arbitrary real number, there may arise the cases in which condition (3) violates, entailing considerable changes in investigation and in a picture of solvability.

In close formulations, problem (1) for constant p has been considered in [17–18]. In [18], problem (1) was treated under the assumptions (4) and $p = \text{const} > 1$. Following partially [18], using essentially the needed for investigation properties of functions from $E^{p(\cdot)}(D;\omega)$ and supplementing them with new facts, we have managed to state the results, analogous to those from [18], but somewhat different, since the statements contain values of the function $p(t)$ at the point A.

The Riemann-Hilbert problem is frequently considered in a class $K^{p(\cdot)}(D;\omega)$ of functions, representable by the Cauchy type integral with density from $L^{p(\cdot)}(\Gamma)$.

In Section 3^0 we indicate the conditions under fulfilment of which $E^{p(\cdot)}(D;\omega) = K^{p(\cdot)}(D;\omega)$.

2^0. **Statement of the Problem.** First of all, we adduce the assumptions relative to the given elements of the problem (Items 2.1–2.3) and then to unknown functions (Items 2.4–2.6). In Item 2.6 we formulate the Riemann-Hilbert problem in a form this problem will be considered in the present work.

2.1. $\Gamma \in C^1(A,\nu)$ denotes that Γ is a closed, simple, piecewise smooth curve with only one angular point A at which angle size (with respect to a finite domain bounded by Γ) equals $\pi\nu$, $0 < \nu \leq 2$.

2.2. Classes of Exponents. By $\mathcal{P}(\Gamma)$ we denote a set of measurable on Γ real functions $p(t)$ for which: 1) there exists the constant $B(p)$ such that for any $t_1, t_2 \in \Gamma$ we have $|p(t_1) - p(t_2)| < \frac{B(p)}{|\ln|t_1-t_2||}$; 2) $\min_{t\in\Gamma} p(t) = \underline{p} > 1$.

By $\mathcal{P}_{1+\varepsilon}(\Gamma)$ we denote a subset of functions from $\mathcal{P}(\Gamma)$ for which condition 1) is replaced by the condition

$$|p(t_1) - p(t_2)| < \frac{B(p)}{|\ln|t_1 - t_2||^{1+\varepsilon}}, \quad \varepsilon > 0.$$

Assume

$$\widetilde{P}(\Gamma) = \bigcup_{\varepsilon>0} \mathcal{P}_{1+\varepsilon}(\Gamma).$$

2.3. Let $z = z(w)$-conformal mapping the circle $U = \{w : |w| < 1\}$ onto D and $z = z(w)$ its inverse function and $z(\tau)$ its extension on $\gamma = \{\tau : |\tau| = 1\}$.

If $\Gamma \in C^1(A,\nu)$, $0 < \nu \leq 2$, $p \in P(\gamma)$ then $l \in \widetilde{\mathcal{P}}(\Gamma)$, where $l(\tau) = p(z(\tau))$ ([16], p. 185).

2.4. The Spaces $L^{p(\cdot)}(\Gamma;\omega)$. Let f be a measurable on Γ function and $\omega(t)$ be a weight function (i.e., almost everywhere finite and different from zero measurable on Γ). Suppose

$$\|f\|_{L^{p(\cdot)}(\Gamma;\omega)} = \inf\{\lambda > 0, \int_0^l \Big|\frac{f(t(s))\omega(t(s))}{\lambda}\Big|^{p(t(s))} ds \leq 1\},$$

where $t = t(s)$, $0 \leq s \leq l$ is the equation of Γ with respect to the arc abscissa s.

Let

$$L^{p(\cdot)}(\Gamma;\omega) = \{f : \|f\|_{L^{p(\cdot)}(\Gamma;\omega)} < \infty\}, \quad L^{p(\cdot)}(\Gamma) := L^{p(\cdot)}(\Gamma;1).$$

2.5. The Class $\boldsymbol{E^{p(\cdot)}(D;\omega)}$. Let D be the simply connected domain bounded by the curve Γ; $p : \Gamma \to (1,\infty)$, and ω is the wighted function in D.

We say that the analytic in D function ϕ belongs to the class $E^{p(\cdot)}(D;\omega)$ if

$$\sup_{0<r<1} \int_0^{2\pi} |\phi(z(re^{i\theta}))\omega(z(re^{i\theta}))|^{p(z(e^{i\theta}))} d\theta < \infty. \tag{5}$$

where $z = z(re^{i\theta})$ is the function, mapping conformally the circle $\cup = \{w : w = re^{i\theta},\ 0 \leq r < 1,\ 0 \leq \theta < 2\pi\}$ onto D.

2.6. Classes of Functions Representable by the Cauchy Type Integral.

$$K^{p(\cdot)}(D;\omega) = \{\phi : \phi(z) = \frac{1}{2\pi i\omega(z)} \int_\Gamma \frac{f(t)dt}{t-z} \equiv$$

$$\equiv \frac{1}{\omega(z)} K_\Gamma f(z),\ f \in L^{p(\cdot)}(\Gamma),\ z \in D\}, \tag{6}$$

$$K^{p(\cdot)}(\Gamma,\omega) = \{\phi : \phi(z) = (K_\Gamma(f))(z),\ f \in L^{p(\cdot)}(\Gamma;\omega),\ z \overline{\in} \Gamma\} \tag{7}$$

$$\widetilde{K}^{p(\cdot)}(\Gamma;\omega) = \{\phi : \phi(z) = \phi_0(z) + q(z),$$

$$\phi_0 \in K^{p(\cdot)}(\Gamma;\omega),\ q(z) \quad \text{is the polynomial.}\} \tag{8}$$

Sometimes it is necessary to indicate an order of the polynomial $q(z)$. If in the representation $\phi(z)$ of (8) an order of $q(z)$ equals n, then we will write $\phi \in \mathcal{K}_n^{p(\cdot)}(\Gamma;\omega)$. If, however, $q(z) \equiv 0$, and ϕ_0 has zero of order n, we write $\phi \in \widetilde{K}_{-n}^{p(\cdot)}(\Gamma;\omega)$. In the notations for $\omega \equiv 1$, we do not indicate weight and write, respectively, $E^{p(\cdot)}(D)$, $K^{p(\cdot)}(\Gamma)$.

Denote by $\widetilde{K}_D^{p(\cdot)}(\Gamma;\omega)$ a set of functions which are restriction to on D of the functions from $\widetilde{K}^{p(\cdot)}(\Gamma;\omega)$.

In [6] (see also [16], p.168, Corollary 7), the following statement is proved.

If Γ is the Carleson curve, $\frac{1}{\omega} \in E^\delta(D)$, $\delta > 0$ and the function $\omega^+(t)$ belongs to $W^{p(\cdot)}(\Gamma)$, (i.e., the operator

$$S_{\Gamma,\omega^+} : \varphi \to S_{\Gamma,\omega^+}, \quad (S_{\omega^+}\varphi)(t) = \frac{\omega^+(t)}{\pi i} \int_\Gamma \frac{\varphi(\tau)d\tau}{\omega^+(\tau)(\tau - t)}, \quad t \in \Gamma,$$

is continuous in $L^{p(\cdot)}(\Gamma)$)*, then*

$$K^{p(\cdot)}(D;\omega) = K_D^{p(\cdot)}(\Gamma;\omega). \tag{9}$$

Let now the conditions of this statement be fulfilled, and $p \in \mathcal{P}(\Gamma)$.

If $\phi \in K^{p(\cdot)}(D;\omega)$, then $\phi(z) = \frac{1}{\omega(z)}(K_\Gamma f)(z)$ where $(K_\Gamma f)(z) = \frac{1}{2\pi i} \times \int_\Gamma \frac{f(t)dt}{t-z}$, $f \in L^{p(\cdot)}(\Gamma) \subset L^{\underline{p}}(\Gamma)$. Since Γ is the Carleson curve, therefore $(K_\Gamma f)(z) \in E^{\underline{p}(\cdot)}(D)$ (see [19], p.29). Consequently, $\phi \in E^\eta(D)$ for some $\eta > 0$ (since $\frac{1}{\omega} \in E^\delta(D)$ and $(K_\Gamma f) \in E^p(D)$).

Further, according to the Sokhotskii-Plemelj formula, $(K_\Gamma f)^+(t) = \frac{1}{2}f(t) + \frac{1}{2}(S_\Gamma f)(t)$. By virtue of our assumptions (i.e., that $p \in P(\Gamma)$, Γ is the Carleson curve and $\phi^+ \in L^{p(\cdot)}(\Gamma)$), we can conclude that $\eta > 0$. (This follows from the theorem on the boundedness in $L^{p(\cdot)}(\Gamma)$ of the Cauchy singular operator (see [9] and [16], pp. 41-58).

Thus, $\phi(z) \in E^\eta(D)$ and $\phi^+ \in L^{p(\cdot)}(\Gamma)$. Let us make use of the following result from [6] (see also [16], p.85).

If $\phi \in E^{p(\cdot)}(D)$ *and* $\phi^+ \in L^{p_1(\cdot)}(\Gamma)$*, where* $p > 0$*,* $p_1 \in \mathcal{P}(\Gamma)$ *and* Γ *is the piecewise smooth curve, then* $\phi \in E^{\widetilde{p}(\cdot)}(D)$*, where* $\widetilde{p}(t) = \max(p(t), p_1(t))$*.*

In the sequel, this theorem proven by Smirnov for constant p will be called Smirnov's generalized theorem ([20], see also [21], p.203).

In our case, $\max(\eta, p(t(s))) = \underline{p} > 1$, hence $\phi^+ \in L^{\underline{p}}(\Gamma) \subset L^1(\Gamma)$, and thus $\phi \in E^{\underline{p}}(D) \subset E^1(D)$.

Due to this fact, ϕ is representable by the Cauchy integral $\phi(z) = (K_\Gamma \phi^+)(z)$ (see, e.g., [21], pp. 205–6).

Since $\phi^+ \in L^{p(\cdot)}(\Gamma)$, we find that ϕ is represented by the Cauchy integral with density from $L^{p(\cdot)}(\Gamma)$.

On the other hand, $\omega\phi = K_\Gamma f \in E^{\underline{p}}(D)$ and $(\phi\omega)^+ \in L^{p(\cdot)}(\Gamma)$.

By Smirnov's generalized theorem, we can conclude that $\omega\phi \in E^{p(\cdot)}(D)$.

Thus the following theorem is proved.

Theorem 1. *If* $\phi \in L^{p(\cdot)}(D;\omega)$*,* $\frac{1}{\omega} \in E^\delta(D)$ *for some* $\delta > 0$*,* $\omega^+ \in W^{p(\cdot)}(\Gamma)$*,* Γ *is the Carleson curve, and* $p \in \mathcal{P}(\Gamma)$*, then*

$$K^{p(\cdot)}(D;\omega) = K_D^{p(\cdot)}(\Gamma;\omega) = E^{p(\cdot)}(D;\omega). \tag{10}$$

2.7. Statement of the Riemann-Hilbert Problem.

We are now able to formulate the Riemann-Hilbert problem in a form we intend to solve it.

Let D be the bounded simply connected domain with the boundary Γ, and

$$\Gamma \in C^1(A,\nu),\ \ 0 < \nu \le 2, \quad p \in \mathcal{P}(\Gamma), \quad \omega = (t-A)^\beta, \quad A \in \Gamma, \quad \beta \in \mathbb{R}.$$

Next, let there be given on Γ the piecewise Hölder real functions $a(t)$, $b(t)$ and $c(t)$ such that

$$\inf(a^2(t)-b^2(t))>0,\quad c(t)\in L^{p(\cdot)}(\Gamma;\omega).$$

We are required to define the function $\phi\in E^{p(\cdot)}(D;\omega)$ such that for almost all t on Γ, we have

$$\operatorname{Re}[(a(t)+ib(t))\phi^+(t)]=c(t).$$

3^0. **The Function $\widetilde{X}(W)$.**

Let $z:z(w)$ be the conformal mapping of the circle U onto D, and $w=w(z)$ be its inverse mapping.

Let

$$A(\tau)=a(z(\tau)),\quad B(\tau)=b(z(\tau)),\quad C(\tau)=c(z(\tau)). \tag{11}$$

Since $a(\tau)$, $b(\tau)$ are piecewise Hölder functions on Γ and $\Gamma\in C^1(A,\nu)$, it is easy to prove that the functions $A(\tau)$ and $B(\tau)$ on the circumference $\gamma=\{r:|\tau|=1\}$ are the same; thus the function $\mathcal{G}_1(\tau)=[A(\tau-iB(\tau))][A(\tau)+iB(\tau)]^{-1}$ is likewise the same.

Let $b_1,b_2,\dots,b_n$, be all points of $\mathcal{G}_1(\tau)$ discontinuity. Since $|\mathcal{G}_1(t)|=1$ everywhere on γ, except the points $b_1,b_2,\dots,b_n$, therefore $|\mathcal{G}_1(b_j\pm)|=1$. Thus, if $\mathcal{G}_1(b_j-)[\mathcal{G}_1(b_j+)]^{-1}=\exp 2\pi i u_j$, then u_j are the real numbers.

Assume

$$r_j(w)=\begin{cases}(w-b_j)^{u_j}, & |w|<1,\\ \overline{\Big(\dfrac{1}{\overline{w}}-b_j\Big)^u_j}, & |w|>1.\end{cases}\qquad r(w)=\prod_{j=1}^n r_j(w),\quad R_j(\tau)=\frac{r_j^+(\tau)}{r_j^-(\tau)}.$$

Then $R_j(\tau)=\exp(-2\pi u_j)$, and hence, the function $\mathcal{G}(r)=\mathcal{G}_1(r)\prod_{j=1}^n R_j(\tau)$ belongs to the Hölder class on γ and is everywhere different from zero. Let $\widetilde{\mathcal{G}}(\tau)=\mathcal{G}(\tau)(\tau-w_0)^{-\varkappa}$, where $w_0\in U$ and

$$\varkappa=\operatorname{ind}\widetilde{\mathcal{G}}(\tau)=\frac{1}{2\pi}[\arg\widetilde{\mathcal{G}}(\tau)]_\gamma. \tag{12}$$

Assume

$$\widetilde{X}(w)=\begin{cases}\exp\Big(\dfrac{1}{2\pi i}\int_\gamma\dfrac{\ln\widetilde{\mathcal{G}}(\tau)d\tau}{\tau-w}\Big), & |w|<1,\\ (w-w_0)^{-\varkappa}\exp\Big(\dfrac{1}{2\pi i}\int_\gamma\dfrac{\ln\widetilde{\mathcal{G}}(\tau)d\tau}{\tau-w}\Big), & |w|>1.\end{cases} \tag{13}$$

In Section 4^0, using the function $\widetilde{X}(w)$, we will construct a canonical function for the Riemann problem to which we reduce problem (1).

4^0. **The Basic Theorem.**

Theorem 2. *Let D be the simply connected domain bounded by the curve Γ and let conditions* (4) *be fulfilled.*

Assume

$$k^* = \nu\beta + \frac{\nu - 1}{p(A)} + \varkappa, \quad k = [k^*], \quad W(A) = a, \quad a \in \gamma,$$

($i.e.$, *k is the integer of k^*), and let Z be a set of integers.*

Then:

I. *If $k^* \notin Z$, $k \geq 0$ then problem (1) is solvable, and its general solution is given by the equality*

$$\phi(z) = \phi_0(z) + \phi_C(z),$$

in which

$$\phi_0(z) = Q_k(w(z))[(w(z) - a)^k]^{-1}, \tag{14}$$

where $Q_k(w) = \sum_{j=0}^{k} a_j w^j$ is an arbitrary polynomial with coefficients satisfying the condition

$$c_k \overline{a}_j = a_{k-j}, \quad c_k = (-1)^{k+1} a^k, \tag{15}$$

and

$$\phi_C(z) = \frac{1}{2\pi} \frac{1}{[w(z) - a]} \int\limits_{\Gamma} \frac{c(t)(w(t) - a)}{w(t) - w(z)} \Big(1 + \frac{w(z)}{w(t)}\Big)^{k+1} dt. \tag{16}$$

II. *For $k^* \notin Z$, $k \leq -1$*

(a) *if $k = -1$, then the problem is uniquely solvable, and a solution is given by the equality $\phi(z) = \phi_C(z)$ (see (16));*

(b) *if $k \leq -2$, then for the problem to be solvable, it is necessary and sufficient that*

$$\int\limits_{\Gamma} C(t)(w(t) - a)^k w^j(t) w'(t) dt = 0, \ \ j = \overline{0, -k-2}, \tag{17}$$

and if these conditions are fulfilled, we have a unique solution $\phi(z) = \phi_C(z)$.

III. *For $k^* \in Z$, $k \geq 0$ for the problem to be solvable, it is necessary and sufficient that*

$$\frac{1}{w(z) - a} \int\limits_{\Gamma} \frac{c(\tau)(w(\tau) - a)^k}{w(\tau) - w(t)} w'(\tau) d\tau \in L^{p(\cdot)}(\Gamma; \omega), \tag{18}$$

and if these conditions are fulfilled, then:

if either (a)

$$\frac{w(t)}{(w(t) - a)^k} \in L^{p(\cdot)}(\Gamma), \tag{19}$$

or (b)

$$\frac{w(t)}{(w(t) - a)^k} \notin L^p(\cdot)(\Gamma) \quad \text{and} \quad Q(a) = 0,$$

then a solution is unique and given by equalities (13)–(16)*.*

IV. *For $k^* \in Z$, $k \leq -1$.*

(a) *if* $k=-1$, *then for the problem to be solvable, it is necessary that condition (18) is fulfilled, and if this condition is fulfilled, the function* $\phi_C(z)$ *is a solution of the problem;*

(b) *if* $k \le -2$, *then for the problem to be solvable, it is necessary and sufficient that conditions* (17)–(18) *are fulfilled, and if they are fulfilled, then the function* $\phi_C(z)$ *is a solution.*

We divide the proof into several stages. At the first stage, problem (1) reduces to the Riemann problem for a circle, but with a supplementary condition (Item 5.1).

We solve the obtained problem first without a supplementary condition (Item 5.2) and then consider the cases $k^* \notin Z$ (Item 5.2.1) and $k^* \in Z$ (Item 5.2.2). Finally, from the obtained set of solutions we distinguish those satisfying the supplementary condition, and obtain thereby solutions of problem (1).

5.1. Reduction of Problem (1) to the Riemann Problem in a Circle U with a Supplementary Condition.

The function $\phi(z(\omega))$ satisfies the condition

$$\operatorname{Re}[(a(z(\tau))) + ib(z(\tau))\phi^+(z(\tau))] = c(z(\tau)). \tag{20}$$

From the definition it follows that if $\phi \in E^{p(\cdot)}(D;\omega)$, then the function $\phi(z(\omega))$ in the circle U belongs to the class $L^{p(\cdot)}(U;\omega(z(w))|z'(\omega)|^{\frac{1}{p(z(e^{i\theta}))}})$.

Assume

$$\omega_1(w) = \omega(z(w))|z'(w)|^{\frac{1}{p(z(e^{i\theta}))}}, \quad w = re^{i\theta}. \tag{21}$$

Since $\Gamma \in C^1(A;\nu)$, therefore

$$z(w) - z(a) = (w-a)^{\nu} \exp\left(\frac{1}{\pi}\int_{\gamma} \frac{\varphi_1(\tau)d\tau}{\tau - w}\right),$$

$$z'(w) = (w-a)^{\nu-1} \exp\left(\frac{1}{\pi}\int_{\gamma} \frac{\varphi_2(\tau)d\tau}{\tau - w}\right)$$

where φ_1, φ_2, are the real continuous on γ functions (see [19], pp. 144-153).

Taking into account these equalities, we obtain

$$\begin{aligned} \omega_1(w) \sim (w-a)^{\nu\beta}(w-a)^{\frac{\nu-1}{p(z(e^{i\theta}))}} \times \\ \times \exp\frac{1}{\pi}\int_{\gamma}\frac{\varphi(\tau)d\tau}{\tau - w}\, \varphi = \varphi_1 + \varphi_2, \end{aligned} \tag{22}$$

(the writing $\psi(w) \sim g(w)$ here denotes that $0 < m \le \left|\frac{\psi(w)}{g(w)}\right| \le M < \infty$).

Let $\phi_1(w) = \phi(z(w))\omega_1(w)$. We are unable to consider problem (20) as the Riemann-Hilbert problem of the class $E^{l(\cdot)}(U)$, $l(\tau) = p(z(\tau))$ with respect to $\phi_1(w)$. The reason is that $[z'(re^{i\theta})]^{\frac{1}{p(z(e^{i\theta}))}}$ is not analytic, but equivalent to the analytic function. Towards this end, we apply the following result (see [16], p.218),

Lemma 1. *If $l \in \widetilde{\mathcal{P}}(\gamma)$, then*

$$\omega_1(w) \sim \rho(w) = (w-a)^{\nu\beta}(w-a)^{\frac{\nu-1}{l(a)}} \exp\left(\frac{1}{\pi}\int\limits_{\gamma} \frac{\psi(\zeta)d\zeta}{\zeta - w}\right), \tag{23}$$

where $\psi(\zeta) = \frac{\varphi(\zeta)}{l(\zeta)}$, $l(\zeta) = p(z(\zeta))$, $\varphi(\zeta) = \varphi_1(\zeta) + \varphi_2(\zeta)$.

Corollary. *Under the assumptions of Lemma* 1, $\phi(z(w)) \in E^{l(\cdot)}(U,\rho)$.

Following [1] (Ch. II), we can now reduce problem (1) to the Riemann problem in a circle, but with a supplementary condition.

Condition (1) takes the form

$$\mathrm{Re}[(A(\tau) + iB(\tau))\phi^+(z(\tau))] = C(\tau), \tag{24}$$

that is,

$$\mathrm{Re}\Big[\frac{A(\tau) + iB(\tau)}{\rho(\tau)}\rho(\tau)\phi^+(z(\tau))\Big] = C(\tau). \tag{25}$$

Let

$$\Psi(w) = \rho(w)\phi(z(w)), \quad |w| < 1,$$

Then from (25), we successively obtain

$$\frac{A(\tau) + iB(\tau)}{\rho(\tau)}\Psi^+(\tau) + \frac{A(\tau) - iB(\tau)}{\overline{\rho(\tau)}}\overline{\Psi^+(\tau)} = 2C(\tau),$$

$$\Psi^+(\tau) = \frac{A(\tau) - iB(\tau)}{A(\tau) + iB(\tau)}\frac{\rho(\tau)}{\overline{\rho(\tau)}}\overline{\Psi^+(\tau)} = \frac{2C(\tau)\rho(\tau)}{A(\tau) + iB(\tau)}. \tag{26}$$

The function $\overline{\Psi^+(\tau)} = \overline{\phi(z(\tau))\rho(\tau)}$ is the boundary value of the analytic in the domain $U^- = C\backslash\overline{U}$ function $\overline{\Psi\Big(\frac{1}{\overline{w}}\Big)\rho\Big(\frac{1}{\overline{w}}\Big)}$.

Suppose

$$\Omega(w) = \begin{cases} \Psi(w), & |w| < 1, \\ \overline{\Psi\Big(\frac{1}{\overline{w}}\Big)}, & |w| > 1. \end{cases} \tag{27}$$

Then (26) takes the form

$$\Omega^+(\tau) = \mathcal{G}_1(\tau)\Omega^-(\tau) + C_1(\tau), \tag{28}$$

where

$$\mathcal{G}_1(\tau) = \frac{A(\tau) - iB(\tau)}{A(\tau) + iB(\tau)}\frac{\rho(\tau)}{\overline{\rho(\tau)}}, \quad C_1(\tau) = \frac{2C(\tau)\rho(\tau)}{A(\tau) + iB(\tau)}. \tag{29}$$

Since $\phi \in E^{p(\cdot)}(D,\omega)$, therefore $\Omega(w) \in E^{l(\cdot)}(U)$, and from the definition of Ω in the domain U^- it follows that there exists the polynomial $q(w)$ such that $(\Omega - q) \in E^{p(\cdot)}(U^-)$.

The function

$$\widetilde{\Omega}(w)=\begin{cases}\Omega(w), & |w|<1,\\ \Omega(w)-q(w), & |w|>1,\end{cases}$$

is representable by the Cauchy type integral with density $\widetilde{\Omega}^{+}-\widetilde{\Omega}^{-}=(\Omega^{+}-\Omega^{-}-q)\in L^{l(\cdot)}(\gamma)$,

$$\Omega(w)=\frac{1}{2\pi i}\int\limits_{\gamma}\frac{\Omega_1(\tau)d\tau}{\tau-w}+q(w),\ \ |w|\neq 1,\ \ \Omega_1(\tau)=\widetilde{\Omega}^{+}(\tau)-\widetilde{\Omega}^{-}(\tau)$$

and hence $\Omega(w)\in\widetilde{K}^{l(\cdot)}(\Gamma)$.

For the function F, analytical outside of γ, we put

$$F_*(w)=\overline{F}\Big(\frac{1}{\overline{w}}\Big),\quad |w|\neq 1.$$

We can now formulate the problem with respect to Ω: Find the function analytical outside of γ, satisfying the conditions

$$\begin{cases}\Omega\in\widetilde{K}^{l(\cdot)}(\gamma), \quad \Omega_*(w)=\Omega(w), \quad l(\tau)=p(z(\tau)), \quad |w|\neq 1,\\ \Omega^{+}(\tau)=\mathcal{G}_1(\tau)\Omega^{-}(\tau)+C_1(\tau), \quad \tau\in\gamma\end{cases}\tag{30}$$

Omitting for the time being the condition $\Omega_*(w)=\Omega(w)$, $w\neq 1$ we first solve the problem: Find the functions Ω for which

$$\begin{cases}\Omega\in\widetilde{K}^{l(\cdot)}(\gamma)\\ \Omega^{+}(\tau)=\mathcal{G}_1(\tau)\Omega^{-}(\tau)+C_1(\tau)\end{cases}\tag{31}$$

5.2. Solution of the Riemann Problem (31).

5.2.1. The Case $k^*\notin Z$. We write the function $\mathcal{G}_1(\tau)$ defined by equality (29) as follows:

$$\mathcal{G}_1(\tau)=\mathcal{G}(\tau)\frac{\rho(\tau)}{\overline{\rho(\tau)}},\quad \text{where}\quad \mathcal{G}(\tau)=\frac{A(\tau)-iB(\tau)}{A(\tau)+iB(\tau)}.\tag{32}$$

Let

$$X(w)=\begin{cases}\widetilde{X}(w)\rho(w)(w-a)^{-k}, & |w|<1,\\ \overline{\widetilde{X}\Big(\frac{1}{\overline{w}}\Big)\rho\Big(\frac{1}{\overline{w}}\Big)}(w-a)^{-k}, & |w|>1.\end{cases}\tag{33}$$

Lemma 2. *If $l(\tau)\in\widetilde{\mathcal{P}}(\gamma)$ and $k^*\notin Z$, then $X(w)$ is the factor-function for $\mathcal{G}_\tau$ in $K^{l(\cdot)}(\gamma)$.*

Proof. By the definition of the factor-function (see [4], p.110), we are to have: (1) $X\in\widetilde{K}_k^{l(\cdot)}(\gamma)$, $X^{-1}\in\widetilde{K}_{-k}^{l(\cdot)}(\gamma)$; (2) $X^{+}(\tau)[X^{-}(\tau)]^{-1}=\mathcal{G}(\tau)$; (3) $X^{+}(\tau)\in W^{l(\cdot)}(\gamma)$.

1) The function $\exp\Big(\frac{1}{\pi}\int_\gamma \frac{\ln\widetilde{\mathcal{G}}(\tau)d\tau}{\tau-w}\Big)$ is bounded in $C\backslash\gamma$ and $\rho(w)\sim (z(w)-z(a))^\beta z'(w)^{\frac{1}{l(a)}}$, therefore

$$(w-a)^k X(w)\in K_k^{l(\cdot)}(\gamma),\quad \frac{1}{X(w)}\in \widetilde{K}_{-k}^{l(\cdot)}(\gamma),\quad l'(\tau)=\frac{l(\tau)}{l(\tau)-1}. \tag{34}$$

2) $\displaystyle\frac{X^+(\tau)}{X^-(\tau)}=\frac{\widetilde{X}^+(\tau)}{\widetilde{X}^-(\tau)}\frac{[\rho(w)]^+}{\Big[\rho\Big(\frac{1}{\overline{w}}\Big)\Big]^-}=\mathcal{G}_1(\tau)\frac{\rho(\tau)}{\rho\Big(\frac{1}{\overline{\tau}}\Big)}=\mathcal{G}(\tau).$

3) We have

$$\begin{aligned}X(\tau)&\sim\rho(\tau)=(\tau-a)^{\nu\beta+\frac{\nu-1}{l(a)}}(\tau-a)^{-k}=(\tau-a)^{k^*-k-\frac{1}{l(a)}}=\\&=(\tau-a)^\alpha\mu(\tau),\quad \mu(\tau)=\exp\Big(\frac{1}{-\pi}\int\limits_\gamma\frac{\psi(\zeta)}{l(\zeta)}\frac{d\zeta}{\zeta-\tau}\Big),\end{aligned}\tag{35}$$

where $\alpha=k^*-k-\frac{1}{l(a)}$ Since $0<k^*-k<1$,

$$-\frac{1}{l(a)}<k^*-k-\frac{1}{l(a)}<1-\frac{1}{l(a)}=\frac{1}{l'(a)},$$

i.e.,

$$-\frac{1}{l(a)}<\alpha<\frac{1}{l'(a)}. \tag{36}$$

But when condition (36) is fulfilled, the function $(\tau-a)^k\mu(\tau)$ belongs to $W^{l(\cdot)}$ ([17], [7], p.111), whence it follows that

$$X^+\in W^{l(\cdot)}(\gamma). \tag{37}$$

Note that if $\nu>0$ and $p\in\widetilde{\mathcal{P}}(\Gamma)$, then $l(\tau)\in\widetilde{\mathcal{P}}(\gamma)$ (see [16], p. 85). $\square$

5.2.2. The Case $k^*\in Z$.

Lemma 3. *If $l\in\widetilde{\mathcal{P}}(\gamma)$, then there exists the number $\varepsilon>0$ such that*

$$X\in\mathcal{K}^{l(\cdot)-\varepsilon}(\gamma),\quad \frac{1}{X}\in\bigcap_{0<\delta<\varepsilon}K^{l(a)+\delta}(\gamma),\quad X^+\in W^{l(\cdot)-\varepsilon}(\gamma). \tag{38}$$

Moreover, an order of the function X at the point $z=\infty$ equals $-k$.

Proof. In this case $k^*=k$, and from (25), we get

$$X^+(t)=(t-a)^{-\frac{1}{l(a)}}\exp\int\limits_\gamma\frac{\varphi(\zeta)}{l(\zeta)}\frac{d\zeta}{\zeta-t}.$$

Let λ be an arbitrary number from the interval $(1,l(a))$, then we have $-\frac{1}{\lambda}<-\frac{1}{l(a)}<\frac{1}{\lambda'}$.

Taking $\lambda = l(a)-\varepsilon$, from the inequality

$$-\frac{1}{l(a)-\varepsilon} < -\frac{1}{l(a)} < \frac{1}{l(a)-\varepsilon}$$

and from (35), we can conclude that $X^+ \in W^{l(a)-\varepsilon}(\gamma)$, $0<\varepsilon<l(a)$, and $X \in \bigcap\limits_{0<\varepsilon<l(a)} K^{l(\cdot)-\varepsilon}(\gamma)$.

Since $\dfrac{1}{X(w)} = (w-a)^{\frac{1}{l(a)}} \exp\Big(-\dfrac{1}{\pi}\int_\gamma \dfrac{\psi(\tau)}{l(\tau)}\dfrac{d\tau}{\tau-t}\Big)$, obviously, $\dfrac{1}{X} \in \bigcap\limits_{0<\delta<\varepsilon} \widetilde{K}^{l(a)-\varepsilon}(\gamma)$. □

5.2.3. Reduction of Problem (31) to the Problem of a Jump. According to Lemmas 2 and 3, $-\frac{1}{X} \in \widetilde{K}^{l(\cdot)}(\gamma)$, which imples that $\frac{\Omega}{X} \in \widetilde{\mathcal{K}}^1(\gamma)$, and hence, we can write (31) in the form

$$\begin{cases} \widetilde{\Omega}^+(\tau) = \widetilde{\Omega}^-(\tau) + \frac{C_1(\tau)}{X^+(\tau)} \\ \widetilde{\Omega} \in \widetilde{\mathcal{K}}^1(\gamma). \end{cases} \quad \text{where } \ \widetilde{\Omega} = \tfrac{\Omega}{X} \tag{39}$$

The homogeneous problem corresponding to problem (39) is

$$\begin{cases} \widetilde{\Omega}^+(\tau) = \widetilde{\Omega}^-(\tau), \\ \widetilde{\Omega} \in \widetilde{K}^1(\gamma). \end{cases} \tag{39_0}$$

Since $\Omega \in \widetilde{K}_0^{l(\cdot)}(\gamma)$, therefore:

(i) when $k \geq 0$, a solution of problem (39_0) is any polynomial $Q_k(w) = \sum_{j=0}^k a_j w^j$, and a particular solution of the inhomogeneous problem is

$$\widetilde{\Omega}_c(w) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{C_1(\tau)}{X^+(\tau)}\frac{d\tau}{\tau-w}.$$

Owing to this fact, a general solution of problem (39) is given by the equality

$$\widetilde{\Omega}(w) = \frac{1}{2\pi i}\int\limits_\Gamma \frac{C_1(\tau)}{X^+(\tau)}\frac{d\tau}{\tau-w} + Q_k(w).$$

(ii) (a) When $k=-1$, the problem (39) is unconditionally solvable and unique. $\widetilde{\Omega}_c(w)$ is its solution.

(b) If $k=-2$, then for problem (39) to be solvable, it is necessary and sufficient that

$$\int\limits_\gamma \frac{C_1(\zeta)}{\widetilde{X}^+(\zeta)}\zeta^j cl\zeta = 0, \quad j = \overline{0,-k-2} \tag{40}$$

and if these conditions are fulfilled, then $\widetilde{\Omega}_c(w)$ is again its solution.

5.2.4. Solution of Problem (31).

Let us consider all possible cases: I_0. $k^* \notin Z$, $k \geq 0$; II_0. $k^* \notin Z$, $k \leq -1$; III_0. $k^* \in Z$, $k \geq 0$ (a) $X_0^+ \in L^{l(\cdot)}(\gamma)$, (b) $X^+ \notin L^{l(\cdot)}(\gamma)$; IV_0. $k^* \in Z$, $k < 0$.

I_0. In this case, $X^+ \in W^{l(\cdot)}(\gamma)$ and

$$\Omega_c(w) = \frac{X(w)}{2\pi i} \int\limits_\gamma \frac{\mathcal{C}_1(\zeta)}{X^+(\zeta)} \frac{d\zeta}{\zeta - w} \in \mathcal{K}^{l(\cdot)}(\gamma).$$

Moreover, $X \in K^{l(\cdot)}(\gamma)$, and hence a solution of the homogeneous problem will be

$$\Omega_0(w) = X(w)Q_k(w), \quad Q_k(w) = \sum_{j=0}^{k} a_j w^j, \tag{41}$$

and a general solution of problem (39) will be

$$\Omega(w) = \Omega_0(w) + \Omega_c(w), \tag{42}$$

where

$$\Omega_c(w) = \frac{X(w)}{2\pi} \int\limits_\gamma \frac{C_1(\tau)}{X^+(\tau)} \frac{d\tau}{\tau - w}, \quad C_1(\tau) = \frac{2C(\tau)\rho(\tau)}{A(\tau) + iB(\tau)}. \tag{43}$$

II_0. By Lemma 2, $X^+ \in W^{l(\cdot)}(\gamma)$. Therefore, if $k = -1$, then $\Omega_c \in K_0^{l(\cdot)}(\gamma)$, and hence problem (31) has a unique solution $\Omega(w) = \Omega_c(w)$. If, however, $k \leq -2$, then according to statement (b) of item 5.2.3, for the problem to be solvable, it is necessary and sufficient that conditions (40) are fulfilled, and here we have a unique solution $\Omega(w) = \Omega_c(w)$.

III_0. $k^* \in Z$, $k \geq 0$. By Lemma 3, $X \in K^{l(\cdot)}(\gamma)$ (see (38)), therefore the condition $X \in K^{l(\cdot)}(\gamma)$ is equivalent to the condition $X^+ \in L^{l(\cdot)}(\gamma)$.

For $k^* \in Z$, there may occur both cases (a) $X^+ \in L^{l(\cdot)}(\gamma)$ and (b) $X^+ \notin L^{l(\cdot)}(\gamma)$.

For (a) $X \in \mathcal{K}_{-k}^{l(\cdot)}(\gamma)$, the function

$$\Omega_0(w) = X(w)Q_k(w) \tag{44}$$

is a solution of the homogeneous problem (39_0), and in case (b) it is not difficult to calculate that

$$X^+(\zeta) = (\zeta - a)^{-\frac{1}{l(a)}} \exp\left(\frac{1}{\pi} \int\limits_\gamma \frac{\varphi(\tau)}{l(\tau)} \frac{d\tau}{\tau - \zeta}\right).$$

Assume

$$Q_k(a) = 0. \tag{45}$$

Then (38) implies that $X \in K^{l(\cdot)-\varepsilon}(\gamma)$, and due to the fact that $Q_k(\zeta)$ is bounded, we can conclude that $XQ_k \in K^{l(\cdot)-\varepsilon}(\gamma)$. Taking into account

(45), we can apply the generalized Smirnov's theorem, and as a result we find that Ω_0 is a solution of problem (39_0).

As for the inhomogeneous problem, here: if $S_{\alpha,X^+}C_1 \notin L^{p(\cdot)}(\gamma)$, the problem is unsolvable , if, however, $S_{\alpha,X^+}C_1 \in L^{l(\cdot)}(\gamma)$, the problem is solvable, and a solution is given by equalities (42)–(44) for $X^+ \in L^{p(\cdot)}(\gamma)$ and by (42)–(44),(45) for $X^+ \notin L^{p(\cdot)}(\gamma)$.

IV_0. $k^* \in Z$, $k < 0$.

(a) If $k = -1$, then the one possible solution of the problem is the function $\Omega_c(w)$ if and only if it belongs to $K^{l(\cdot)}(\Gamma)$.

(b) If $k \leq -2$, then as it has been shown in item 5.2.3, for the problem to be solvable, it is necessary and sufficient that equalities (40) are fulfilled.

5.3. Proof of the Basic Theorem. (Solution of Problem (30)).

For the above constructed solutions of problem (31) to be likewise solutions of problem (30), it is necessary that the equality

$$(\Omega_0 + \Omega_c)_* = \Omega_0 + \Omega_c \tag{46}$$

is fulfilled.

Let us clear up in what cases equality (46) holds. First, we find out when the equality

$$(\Omega_0)_*(w) = \Omega_0(w). \tag{47}$$

is valid.

It follows from equality (44) that

$$\begin{aligned}\Omega_0(w) = \overline{X\Big(\frac{1}{\overline{w}}\Big)Q_k\Big(\frac{1}{\overline{w}}\Big)} = (-1)^{k+1}a^k X(w)\sum_{j=0}^{k}\overline{a}_j w^{k-j} = \\ = c_k X(w)\sum_{j=0}^{k}\overline{a}_j w^{k-j},\end{aligned} \tag{48}$$

where

$$c_k = (-1)^{k+1}a^k. \tag{49}$$

Analogously one can prove that (48) holds for $|w| > 1$, as well.

It is easy to see that (46) follows from (48), if and only if

$$c_k\overline{a}_j = a_{k-j}, \quad j = \overline{0, \Big[\frac{k+1}{2}\Big]}. \tag{50}$$

Next, if $\Omega_c(w)$ is a solution of problem (30), then $(\Omega_c)_*$ is likewise a solution (see [1], §41).

Consequently, the function

$$\widetilde{\Omega}_c(w) = \frac{1}{2}[\Omega_c(w) + (\Omega_c)_*(w)] \tag{51}$$

will be a solution of problem (30) of the required property, and a general solution will be

$$\Omega(w) = \Omega_0(w) + \widetilde{\Omega}_c(w),$$

where

$$\Omega_0(w) = X(w)\sum_{j=0}^{k} a_j w^j,$$

in which a_j are arbitrary numbers satisfying the conditions (49)–(50).

It follows from (46) that $\overline{X\left(\frac{1}{\overline{w}}\right)} = c_k X(w)$, $c_k = (-1)^{k+1}a^k$ and from (33) and (41) we have

$$\frac{C_1(\zeta)}{X^+(\zeta)} = 2C(\zeta)(\zeta - a)^k. \tag{52}$$

Therefore,

$$\begin{aligned}\Omega_c(w) &= \frac{1}{2}\frac{X(w)}{2\pi i}\left[\int_\gamma \frac{\mathcal{G}_1(zt)}{X^+(\zeta)}\frac{d\zeta}{\zeta - w} - c_k w^k \overline{\int_\gamma \frac{2C(z(\zeta))(\zeta-a)^k d\zeta}{\zeta - \frac{1}{\overline{w}}}}\right] = \\ &= \frac{X(w)}{2\pi i}\int_\gamma \frac{C(z(\zeta))(\zeta-a)^k)}{\zeta - w}\Big(1 + \frac{w^{k+1}}{\zeta^{k+1}}\Big)d\zeta. \end{aligned} \tag{53}$$

Here we have omitted a number of transformations; we omit also details dealt with derivation of the remaining statements of the theorem, as they do not differ at that stage from the case $P = \text{const}$ (see [18]).

We will dwell only on the consideration of case III, $k^* \in Z$, $k \geq 0$.

As it has been shown in item 5.1.4, solutions do exist if and only if

$$\frac{X^+(\tau)}{2\pi i}\int_\gamma \frac{C_1(\zeta)}{X^+(\zeta)}\frac{d\zeta}{\zeta - w} \in L^{l(\cdot)}(\gamma). \tag{54}$$

It follows from (54), (33) and (53) that this is equivalent to the condition

$$\frac{1}{(w(t)-a)^k}\int_\Gamma C(\tau)(w(\tau)-w(a))w(\tau)\frac{d\tau}{w(\tau)-w(t)} \in L^{p(\cdot)}(\Gamma). \tag{55}$$

Finally, some particular cases are worth mentioning.

(1) If Γ is a smooth curve, then $\nu = 1$, and hence $k^* = \beta + \varkappa$.

(2) If $\omega = 1$, then $\beta = 0$, and $k^* = \dfrac{\nu - 1}{l(a)} + \varkappa = \dfrac{\nu-1}{P(A)} + \varkappa$, $k = \left[\dfrac{\nu-1}{p(A)} + \varkappa\right]$.

(3) If we consider the Dirichlet problem in the class $e^{p(\cdot)}(D,\omega) = \operatorname{Re} E^{p(\cdot)}(D,\omega)$, then $a+ib=1$, and hence $\varkappa=0$, while $k^*=\nu\beta+\dfrac{\nu-1}{p(A)}$, $k=\Big[\nu\beta+\dfrac{\nu-1}{p(A)}\Big]$.

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