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SOME FUNDAMENTAL INEQUALITIES FOR TRIGONOMETRIC POLYNOMIALS AND IMBEDDINGS OF GRAND BESOV SPACES

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Abstract. We establish Bernstein-Zygmund and Nikolsky type inequalities for trigonometric polynomials in the framework of grand Lebesgue spaces. It is revealed an influence of second parameter θ from the definition of grand Lebesgue spaces on the derived estimates.

Then we introduce grand Besov spaces and prove imbedding theorems for different metrics and different dimensions.

რეზიუმე. ნაშრომში დამტკიცებულია ბერნშტეინ-ზიგმუნდისა და ნიკოლსკის ტიპის უტოლობები ტრიგონომეტრიული პოლინომებისათვის გრანდ ლებეგის სივრცეებში. მიღებულ უტოლობებში გამოკვეთილია გრანდ ლებეგის სივრცეთა მეორე θ პარამეტრის გავლენა. შემოღებულია გრანდ ბესოვის სივრცეები და ამ სივრცეებში დამტკიცებულია ჩართვის თეორემები განსხვავებული მეტრიკებისა და განზომილებებისათვის.

1. Introduction

In this paper we prove Bernstein and Nikolsky type inequalities for trigonometric polynomials in grand Lebesgue spaces. Then on the base of certain subspace of grand Lebesgue space we introduce the periodic Besov type space and prove imbedding theorems of different metrics and different dimensions.

Let $\mathbb{T}^d = (-\pi, \pi)^d$ and $1 < p < \infty$, $\theta > 0$. The grand Lebesgue space $L^{p),\theta}(\mathbb{T}^d)$ of 2π-periodic functions in each variable separately $f : \mathbb{T}^d \to \mathbb{R}^1$ is defined as a set of measurable functions for which

$$\|f\|_{p),\theta} = \sup_{0<\epsilon<p-1} \left(\varepsilon^\theta \int_{\mathbb{T}^d} |f(x)|^{p-\epsilon} dx \right)^{1/p-\epsilon} < \infty.$$

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The grand Lebesgue spaces were introduced by T. Iwaniec and C. Sbordone [1] for $\theta = 1$, and by L. Greco, T. Iwaniec and C. Sbordone [2] for $\theta > 0$. It is known that these spaces are non-reflexive and non-separable Banach function spaces. The following continuous imbeddings

$$L^p \hookrightarrow L^{p),\theta} \hookrightarrow L^{p-\epsilon}, \quad 0 < \epsilon < p-1,$$

hold.

The closure $[L^p]$ of L^p by the norm of $L^{p),\theta}$ does not coincide with the latter space. For example the function $|\sin x|^{-\frac{1}{p}} \in [L^p]$ but $|\sin x|^{-\frac{1}{p}} \notin L^{p),\theta}$.

We denote the above-mentioned closure by $\dot{L}^{p),\theta}$. As is known [3] $\dot{L}^{p),\theta}$ is a subspace of the space $L^{p),\theta}$ of functions satisfying the condition

$$\lim_{\epsilon \to 0} \int_{\mathbb{T}^d} |f(x)|^{p-\epsilon} dx = 0.$$

By $E_{\nu_1,\nu_2,\dots,\nu_n}(f)$ we denote the best approximation by trigonometric polynomials of $f \in \dot{L}^{p),\theta}(\mathbb{T}^d)$

$$E_{\nu_1,\nu_2,\dots,\nu_n}(f) = \inf \|f - T\|_{p),\theta},$$

where the infimum is taken over all polynomials of degree not greater than ν_i with respect to the variable x_i $(i = 1, 2, \dots, n)$.

For $f \in \dot{L}^{p),\theta}(\mathbb{T}^d)$, we have

$$\lim_{\substack{\nu_i \to 0 \\ 1 \le i \le n}} E_{\nu_1,\nu_2,\dots,\nu_n}(f) = 0.$$

In the sequel we assume that $x = (x_1, \dots, x_d)$ is an element of $\mathbb{T}^d$.

2. Bernstein and Nikolsky Type Inequalities in Grand Lebesgue Spaces

In this section first of all we prove the Bernstein and Nikolsky type inequalities. In the approximation theory the following inequalities for trigonometric polynomials are well-known:

$$\left\| \frac{\partial T_{\nu_1,\nu_2,\dots,\nu_d}}{\partial x_i} \right\|_{L^p} \le \nu_i \left\| T_{\nu_1,\nu_2,\dots,\nu_d} \right\|_{L^p}, \tag{1}$$

$$\left\| T_{\nu_1,\nu_2,\dots,\nu_d} \right\|_{L^q} \le 2^d \left(\prod_{k=1}^{d} \nu_k \right)^{\frac{1}{p}-\frac{1}{q}} \left\| T_{\nu_1,\nu_2,\dots,\nu_d} \right\|_{L^p}, \quad 1 \le p \le q \le \infty \tag{2}$$

and

$$\left\|\partial T_{\nu_1,\nu_2,\ldots,\nu_d}\right\|_{L^p(\mathbb{T}^m)} \leq$$
$$\leq 2^{d-m}\left(\prod_{i=m+1}^{d}\nu_i\right)^{\frac{1}{d}}\left\|T_{\nu_1,\nu_2,\ldots,\nu_d}\right\|_{L^p(\mathbb{T}^d)}, \quad 1 \leq m \leq d. \tag{3}$$

The first inequality is the Bernstein-Zygmund inequality and two others were proved by S. M. Nikolsky (see e. g. [4], Chapter IV).

Theorem 2.1 (Bernstein-Zygmund inequality)**.** *Let* $1 < p < \infty$ *and* $\theta > 0$. *Then for arbitrary trigonometric polynomial* $T_{\nu_1,\nu_2,\ldots,\nu_d}$ *the following inequality holds*

$$\left\|\frac{\partial T_{\nu_1,\nu_2,\ldots,\nu_d}}{\partial x_i}\right\|_{L^{p),\theta}(\mathbb{T}^d)} \leq \nu_i\left\|T_{\nu_1,\nu_2,\ldots,\nu_d}\right\|_{L^{p),\theta}(\mathbb{T}^d)}.$$

Proof. is a direct consequence of (1). Indeed,

$$\left\|\frac{\partial T_{\nu_1,\nu_2,\ldots,\nu_d}}{\partial x_i}\right\|_{L^{p),\theta}} =$$
$$= \sup_{0<\epsilon<p-1}\left(\varepsilon^\theta\int\limits_{\mathbb{T}^d}\left|\frac{\partial T_{\nu_1,\nu_2,\ldots,\nu_d}}{\partial x_i}\right|^{p-\epsilon}dx\right)^{\frac{1}{p-\epsilon}}, \quad x = (x_1,\ldots,x_d).$$

Applying (1) we obtain

$$\left\|\frac{\partial T_{\nu_1,\nu_2,\ldots,\nu_d}}{\partial x_i}\right\|_{L^{p),\theta}} \leq \nu_i \sup_{0<\epsilon<p-1}\left(\epsilon^\theta\int\limits_{\mathbb{T}^d}|T_{\nu_1,\nu_2,\ldots,\nu_d}(x)|^{p-\epsilon}dx\right)^{\frac{1}{p-\epsilon}} =$$
$$= \nu_i\|T_{\nu_1,\nu_2,\ldots,\nu_d}\|_{L^{p),\theta}}. \qquad \square$$

Theorem 2.2 (Nikolsky type inequality)**.** *Let* $1 < p < \infty$ *and* $\theta > 0$. *Then there is a constant* c_p *such that for arbitrary polynomial* T_ν *of order* $\nu = \nu_1, \nu_2, \ldots, \nu_d$ *we have*

$$\|T_\nu(x_1,\ldots,x_m,x_{m+1},\ldots,x_d)\|_{L^{p),\theta}(\mathbb{T}^m)} \leq$$
$$\leq c_{p,d,m}2^{d-m}\left(\prod_{j=m+1}^{d}\nu_j\right)^{\frac{1}{p}}\|T_\nu(x_1,\ldots,x_d)\|_{L^{p),\theta}(\mathbb{T}^d)}.$$

Proof. First observe that the norm $\|f\|_{L^{p),\theta}(\mathbb{T}^m)}$ is equivalent to

$$\sup_{0<\varepsilon<p-1}\varepsilon^{\frac{\theta}{p}}\left(\int\limits_{\mathbb{T}^m}|f(x)|^{p-\epsilon}dx_1\cdots dx_m\right)^{\frac{1}{p-\epsilon}}$$

since $\epsilon^{\frac{1}{p-\epsilon}} \sim \epsilon^{\frac{1}{p}}$ as $\varepsilon \to 0$.

Further we take σ, $0 < \sigma < p-1$, which will be chosen later.

By Hölder's inequality we have

$$I := \|T_\nu(x_1,\dots,x_m,x_{m+1},\dots,x_d)\|_{L^{p),\theta}(\mathbb{T}^m)} \le$$

$$\le \sup_{\sigma\le\epsilon<p-1} \epsilon^{\frac{\theta}{p}} \bigg(\int\limits_{\mathbb{T}^m} |T_\nu(x)|^{p-\epsilon} dx_1,\dots,dx_m \bigg)^{\frac{1}{p-\epsilon}} +$$

$$+ \sup_{0<\epsilon<\sigma} \epsilon^{\frac{\theta}{p}} \bigg(\int\limits_{\mathbb{T}^m} |T_\nu(x)|^{p-\epsilon} dx_1,\dots,dx_m \bigg)^{\frac{1}{p-\epsilon}} \le$$

$$\le c\bigg(\sup_{0<\epsilon<p-1} \epsilon^{\frac{\theta}{p}} \bigg) \cdot \sigma^{-\frac{\theta}{p}} \sigma^{\frac{\theta}{p}} \bigg(\int\limits_{\mathbb{T}^m} |T_\nu(x)|^{p-\sigma} dx_1,\dots,dx_m \bigg)^{\frac{1}{p-\sigma}} +$$

$$+ \sup_{0<\epsilon<\sigma} \epsilon^{\frac{\theta}{p}} \bigg(\int\limits_{\mathbb{T}^m} |T_\nu(x)|^{p-\epsilon} dx_1,\dots,dx_m \bigg)^{\frac{1}{p-\epsilon}} \le$$

$$\le c_1 \max\Big((p-1)^{\frac{\theta}{p}} \sigma^{-\frac{\theta}{p}}, 1 \Big) \sup_{0<\epsilon\le\sigma} \epsilon^{\frac{\theta}{p}} \bigg(\int\limits_{\mathbb{T}^m} |T_\nu(x)|^{p-\epsilon} dx_1 \cdots dx_m \bigg)^{\frac{1}{p-\epsilon}}.$$

Then applying inequality (2) we get

$$I \le c_2 \bigg(\prod_{j=m+1}^{d} \nu_j \bigg)^{\frac{1}{p}} \|T_\nu\|_{L^{p),\theta}(\mathbb{T}^d)}$$

with a constant c_2 independent of T_ν.

Here we used the fact that

$$\bigg(\prod_{j=m+1}^{d} \nu_j \bigg)^{\frac{1}{p-\epsilon}} \le 2 \bigg(\prod_{j=m+1}^{d} \nu_j \bigg)^{\frac{1}{p}}$$

for $0<\epsilon\le\sigma$, where σ is sufficiently small.

Therefore

$$\|T_\nu(x_1,\dots,x_m,x_{m+1},\dots,x_d)\|_{L^{p),\theta}(\mathbb{T}^m)} \le$$

$$\le c_3 \bigg(\prod_{j=m+1}^{d} \nu_j \bigg)^{\frac{1}{p}} \|T_\nu(x_1,\dots,x_d)\|_{L^{p),\theta}(\mathbb{T}^d)},$$

where the constant c_3 depends only on p, m and d. □

In the sequel we prove Nikolsky type inequality for different metrics. Let $\varphi(x)$ be a continuous function on $[0,p-1]$, $\varphi(0)=0$, $\varphi(x)>0$ for $x>0$. Let $L^{p),\varphi(\cdot)}$ be a set of measurable functions for which

$$\|f\|_{L^{p),\varphi(\cdot)}} = \sup_{0<\epsilon<p-1} \bigg(\varphi(\epsilon) \int\limits_{\mathbb{T}^d} |f(x)|^{p-\epsilon} dx \bigg)^{\frac{1}{p-\epsilon}}$$

It is clear that when $\varphi(x) = x^\theta$, $\theta > 0$, then $L^{p),\varphi(\cdot)} = L^{p),\theta}$.

Introduce the notation:

$$\frac{1}{p} - \frac{1}{q} = A, \quad 1 < p \le q < \infty.$$

Let

$$\varphi(x) := \left[\frac{x-q}{1-A(x-q)} + p\right]^{1-(x-q)A}$$

Using the L′Hôpital rule we see that

$$\lim_{x\to 0}\left[\frac{x-q}{1-A(x-q)} + p\right] x^{-1} = \left(\frac{p}{q}\right)^2.$$

Consequently,

$$\varphi(x) \sim x^{\frac{q}{p}} \quad \text{as} \quad x \to 0.$$

Let

$$\psi(x) := \varphi(x^\theta), \quad \theta > 0.$$

Hence

$$\psi(x) \sim x^{\theta\frac{q}{p}} \quad \text{as} \quad x \to 0$$

The following theorem is true:

Theorem 2.3 (Nikolsky type inequality for different metrics)**.** *Let* $1 < p < q < \infty$ *and* $\theta > 0$. *Then*

$$\|T_\nu\|_{L^{q,\theta\frac{q}{p}}(\mathbb{T}^d)} \le c\left(\prod_{j=1}^d \nu_j\right)^{\frac{1}{p}-\frac{1}{q}} \|T_\nu\|_{L^{p),\theta}(\mathbb{T}^d)} \tag{4}$$

with a constant independent of T_ν.

Proof. Let us take some σ, $0 < \sigma < p-1$, which will be chosen later. We have

$$\|T_\nu\|_{L^{q),\psi}} = \sup_{0<\epsilon<q-1}\left(\psi(\epsilon)\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\epsilon}dx\right)^{\frac{1}{q-\epsilon}} =$$
$$= \max(B_1, B_2),$$

where

$$B_1 := \sup_{0<\epsilon\le\sigma} \psi^{\frac{1}{q-\epsilon}}(\epsilon)\left(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\epsilon}dx\right)^{\frac{1}{q-\epsilon}},$$

$$B_2 := \sup_{0<\epsilon\le q-1} \psi^{\frac{1}{q-\epsilon}}(\epsilon)\left(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\epsilon}dx\right)^{\frac{1}{q-\epsilon}}.$$

We have

$$B_2 = \sup_{\sigma<\epsilon\leq q-1} \psi^{\frac{1}{q-\epsilon}}(\epsilon)\psi^{-\frac{1}{q-\sigma}}(\sigma)\psi^{\frac{1}{q-\sigma}}(\sigma)\bigg(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\epsilon}dx\bigg)^{\frac{1}{q-\epsilon}}. \quad (5)$$

By Hölder's inequality we obtain

$$\bigg(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\epsilon}dx\bigg)^{\frac{1}{q-\epsilon}} \leq \bigg(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\sigma}dx\bigg)^{\frac{1}{q-\sigma}} \cdot |\mathbb{T}^d|^{\frac{\epsilon-\sigma}{(q-\epsilon)(q-\sigma)}} \leq$$

$$\leq c_q\bigg(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\sigma}dx\bigg)^{\frac{1}{q-\sigma}}.$$

Thus by (5)

$$B_2 \leq c_q \cdot (q-1)^{\theta\frac{q}{p}}\sigma^{-\theta\frac{q}{p(q-\sigma)}} \sup_{0<\epsilon\leq\sigma} \psi^{\frac{1}{q-\epsilon}}(\epsilon)\bigg(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\epsilon}dx\bigg)^{\frac{1}{q-\epsilon}}.$$

Consequently,

$$\|T_\nu\|_{L^{q),\psi(\cdot)}} \leq$$

$$\leq \max\Big(c_q(q-1)^{\theta\frac{q}{p}}\sigma^{-\theta\frac{q}{p(q-\sigma)}}, 1\Big) \sup_{0<\epsilon\leq\sigma} \psi^{\frac{1}{q-\epsilon}}(\epsilon)\bigg(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\epsilon}dx\bigg)^{\frac{1}{q-\epsilon}}. \quad (6)$$

Now for a given ε, $0<\varepsilon\leq q-1$ we choose η so that

$$\frac{1}{p-\eta} - \frac{1}{q-\epsilon} = \frac{1}{p} - \frac{1}{q}.$$

It is obvious that $\varepsilon\to 0$ is equivalent to $\eta\to 0$. If $0<\epsilon<\sigma$ for some small σ, then $0<\eta\leq\sigma_0<p-1$ for some small σ_0.

Let us prove that

$$[\psi(\epsilon)]^{\frac{1}{q-\epsilon}} \approx \eta^{\frac{\theta}{p-\eta}} \quad \text{as} \quad \varepsilon\to 0. \quad (7)$$

Since by definition $\psi(x)=\varphi(x^\theta)$, it is enough to show that

$$[\varphi(\epsilon)]^{\frac{1}{q-\epsilon}} \approx \eta^{\frac{1}{p-\eta}}$$

Indeed, if this is correct, then we get

$$\psi(\epsilon) = \varphi(\epsilon^\theta) \approx \epsilon^{\theta\frac{q}{p}} \approx (\varphi(\epsilon))^\theta = \eta^{\frac{\theta(q-\epsilon)}{p-\eta}}.$$

It remains to show (7). We have

$$\eta = p - \frac{q-\epsilon}{A(q-\epsilon)+1}.$$

Then

$$\eta^{\frac{1}{p-\eta}} = \left[p - \frac{q-\epsilon}{A(q-\epsilon)+1}\right]^{\frac{A(q-\epsilon)+1}{q-\epsilon}} =$$

$$= \left[\frac{\epsilon - q}{1 - A(\epsilon - q)}\right]^{\frac{A(q-\epsilon)+1}{q-\epsilon}} = [\varphi(\epsilon)]^{\frac{1}{q-\epsilon}}.$$

Consequently, by (6), (2) and (7) we obtain

$$\|T_\nu\|_{L^{q),\psi(\cdot)}} \le c_{p,q} \sup_{0<\epsilon\le\sigma} \psi^{\frac{1}{q-\epsilon}}(\epsilon)\left(\int_{\mathbb{T}^d} |T_\nu(x)|^{q-\epsilon}dx\right)^{\frac{1}{q-\epsilon}} \le$$

$$\le c'_{p,q} \sup_{0<\eta\le\sigma_0} \eta^{\frac{\theta}{p-\eta}} \left(\prod_{j=1}^d \nu_j\right)^{\frac{1}{p-\eta}-\frac{1}{q-\epsilon}} \left(\int_{\mathbb{T}^d} |T_\nu(x)|^{p-\eta}dx\right)^{\frac{1}{p-\eta}} =$$

$$= c''_{p,q}\left(\prod_{j=1}^d \nu_j\right)^{\frac{1}{p}-\frac{1}{q}} \|T_\nu\|_{L^{p),\theta}}.$$

Therefore we have (4). □

3. Grand Besov Spaces. Imbedding for Different Matrices and Different Dimensions

Let $1 < p < \infty$ and $\theta > 0$. Suppose that $r > 0$ and $s > 0$. The grand Besov space is defined as

$$\dot{B}^{p),\theta}_{r,s}(\mathbb{T}^d) = \left\{f \in \dot{L}^{p),\theta}(\mathbb{T}^d) : \ \|f\|_{\dot{B}^{p),\theta}_{r,s}} < \infty\right\},$$

where

$$\|f\|_{\dot{B}^{p),\theta}_{r,s}} = \|f\|_{L^{p),\theta}} + \left(\sum_{k=0}^{\infty} 2^{rsk} E_k^s(f)_{L^{p),\theta}}\right)^{1/s}, \tag{8}$$

$$E_k(f)_{L^{p),\theta}} = E_{2^k,\dots,2^k}(f)_{L^{p),\theta}}.$$

For the Besov spaces see e. g. [5], [6].

The space $\dot{B}^{p),\theta}_{r,s}$ is a Banach function space.

In the sequel by $T_{\nu_1,\dots,\nu_d}$ we denote the best approximation polynomial for a given $f \in \dot{L}^{p),\theta}$:

$$E_{\nu_1,\dots,\nu_d}(f)_{L^{p),\theta}} = \|f - T_{\nu_1,\dots,\nu_d}\|_{L^{p),\theta}}.$$

Since

$$\lim_{\substack{\nu_i\to\infty \\ 1\le\nu_i\le d}} E_{\nu_1,\dots,\nu_d}(f) = 0$$

for arbitrary $f \in \dot{L}^{p),\theta}(\mathbb{T}^d)$, we have

$$f(x) = T_{1,\dots,1}(x) + \sum_{k=1}^{\infty} \big(T_{2^k,\dots,2^k}(x) - T_{2^{k-1},\dots,2^{k-1}}(x)\big) \quad x = (x_1, \dots, x_d),$$

by the norm of $L^{p),\theta}(\mathbb{T}^d)$. It is easy to see that

$$\|f - T_{1,\dots,1}\|_{L^{p),\theta}} = E_0(f)_{p),\theta} \le \|f\|_{L^{p),\theta}}.$$

Therefore,

$$\|T_{1,\dots,1}\|_{L^{p),\theta}} \le 2\|f\|_{L^{p),\theta}}. \tag{9}$$

The derivative

$$\frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}}, \quad \lambda = \sum_{i=1}^{n} \lambda_i$$

is assumed to be the generalized Sobolev derivative.

Theorem 3.1. *Let* $1 < p \le q < \infty$, $\theta > 0$, $1 \le m \le d$, $s > 0$, $s_1 \ge s$. *Let*

$$\varkappa = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\frac{m}{r} - \frac{1}{p}\frac{d-m}{r} - \frac{\lambda}{r} > 0. \tag{10}$$

Then for $f(x_1, \dots, x_d) \in \dot{B}^{p),\theta}_{r,s}(\mathbb{T}^d)$, *we have*

$$h(x_1, x_2, \dots, x_m) = \frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} f(x_1, x_2, \dots, x_m, x^0_{m+1}, \dots, x^0_n)$$

belongs to the space $\dot{B}^{q),\theta\frac{q}{p}}_{\rho,s_1}(\mathbb{T}^m)$ *for arbitrary* $x^0_{m+1}, \dots, x^0_n$, *where* $\rho = \varkappa r$. *Moreover, the following inequality*

$$\|h\|_{\dot{B}^{q),\theta\frac{q}{p}}_{r,s_1}(\mathbb{T}^m)} \le c\|f\|_{\dot{B}^{p),\theta}_{r,s}(\mathbb{T}^d)} \tag{11}$$

holds with a constant c *independent of* f.

Proof. For the simplicity let us assume that $x^0_{m+1} = x^0_{m+2} = \dots = x^0_d = 0$. As it was mentioned above

$$f(x) = \sum_{k=0}^{\infty} t_k(x),$$

where $t_0(x) = T_{1,\dots,1}(x)$ and $t_k(x) = T_{2^k,\dots,2^k}(x) - T_{2^{k-1},\dots,2^{k-1}}(x)$.

Then

$$h(x_1, \dots, x_m) = \sum_{k=0}^{\infty} \frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} t_k(x_1, \dots, x_m, 0, \dots, 0). \tag{12}$$

According to the definition of the grand Besov space it is enough to show that

$$\|h\|_{L^{q),\frac{\theta}{q}p}(\mathbb{T}^m)} \le c\Bigg(\|f\|_{L^{p),\theta}} + \bigg(\sum_{k=0}^{\infty} 2^{rks} E^s_k(f)_{L^{p),\theta}(\mathbb{T}^d)}\bigg)^{1/s}\Bigg) \tag{13}$$

and

$$\left(\sum_{k=0}^{\infty} 2^{r\varkappa s_1 k} E_k^{s_1}(h)_{L^{q),\theta\frac{q}{p}}(\mathbb{T}^m)}\right)^{1/s_1} \leq c\left(\sum_{k=0}^{\infty} 2^{rks} E_k^s(f)_{L^{p),\theta}(\mathbb{T}^d)}\right)^{1/s}. \quad (14)$$

Applying the Nikolsky and Bernstein type inequalities for $k = 1, 2, \ldots$ we obtain

$$\left\|\frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} t_k(x_1, \ldots, x_m, 0, \ldots, 0)\right\|_{L^{q),\theta\frac{q}{p}}(\mathbb{T}^m)} \leq$$

$$\leq c 2^{mk\left(\frac{1}{p}-\frac{1}{q}\right)} \left\|\frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} t_k(x_1, \ldots, x_m, 0, \ldots, 0)\right\|_{L^{p),\theta}(\mathbb{T}^m)} \leq$$

$$\leq c 2^{mk\left(\frac{1}{p}-\frac{1}{q}\right)} \cdot 2^{(d-m)\frac{1}{p}} \left\|\frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} t_k(x_1, \ldots, x_m, 0, \ldots, 0)\right\|_{L^{p),\theta}(\mathbb{T}^d)} \leq$$

$$\leq c 2^{mk\left(\frac{1}{p}-\frac{1}{q}\right)} \cdot 2^{(d-m)\frac{1}{p}} \cdot \prod_{k=1}^{d} 2^{k\lambda_i} \|t_k\|_{L^{p),\theta}(\mathbb{T}^d)} =$$

$$= c 2^{k(1-\varkappa)r} \|t_k\|_{L^{p),\theta}(\mathbb{T}^d)} \leq c 2^{k(1-\varkappa)r} E_{k-1}(f)_{L^{p),\theta}(\mathbb{T}^d)}. \quad (15)$$

By Hölder's inequality

$$\sum_{k=0}^{\infty} 2^{-rk\varkappa} \cdot 2^{kr} E_k(f)_{L^{p),\theta}} \leq \left(\sum_{k=0}^{\infty} 2^{-k\varkappa r s'}\right)^{1/s'} \left(\sum_{k=0}^{\infty} 2^{rks} E_k^s(f)_{L^{p),\theta}}\right)^{1/s} \leq$$

$$\leq c\left(\sum_{k=0}^{\infty} 2^{rks} E_k^s(f)_{L^{p),\theta}}\right)^{1/s}, \quad (16)$$

where $s' = \frac{s}{s-1}$.

On the other hand, by Bernstein and Nikolsky type inequalities, and (9)

$$\left\|\frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} t_0(x_1, \ldots, x_m, 0, \ldots, 0)\right\|_{L^{q),\frac{\theta q}{p}}(\mathbb{T}^m)} \leq$$

$$\leq c\|t_0(x_1, \ldots, x_m, 0, \ldots, 0)\|_{L^{p),\theta}(\mathbb{T}^m)} = c\|T_{1,\ldots,1}(x_1, \ldots, x_d)\|_{L^{p),\theta}(\mathbb{T}^d)} \leq$$

$$\leq c\|f\|_{L^{p),\theta}(\mathbb{T}^d)}. \quad (17)$$

Therefore by (15) and (17) we conclude that

$$\|h\|_{L^{q),\theta\frac{q}{p}}(\mathbb{T}^m)} \leq \left\|\frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} t_0(x_1, \ldots, x_m, 0, \ldots, 0)\right\|_{L^{q),\theta\frac{q}{p}}(\mathbb{T}^m)} +$$

$$+ \sum_{k=1}^{\infty} \left\|\frac{\partial^\lambda}{\partial x_1^{\lambda_1} \cdots \partial x_d^{\lambda_d}} t_k(x_1, \ldots, x_m, 0, \ldots, 0)\right\|_{L^{q),\theta\frac{q}{p}}(\mathbb{T}^m)} \leq$$

$$\leq c\left(\|f\|_{L^{p),\theta}(\mathbb{T}^d)} + \sum_{k=1}^{\infty} 2^{rk(1-\varkappa)} E_{k-1}(f)_{L^{p),\theta}(\mathbb{T}^d)}\right) \leq$$

$$\le c\bigg(\|f\|_{L^{p),\theta}(\mathbb{T}^d)} + \bigg(\sum_{k=1}^{\infty} 2^{rks} E_k^s(f)_{L^{p),\theta}(\mathbb{T}^d)}\bigg)^{1/s}\bigg).$$

Thus (13) is proved. Then we have

$$\bigg(\sum_{k=0}^{\infty} 2^{r\varkappa s_1 k} E_k^s(h)_{L^{q),\theta\frac{q}{p}}(\mathbb{T}^m)}\bigg)^{s/s_1} \le \sum_{k=0}^{\infty} 2^{r\varkappa sk} E_k^s(h)_{L^{q),\theta\frac{q}{p}}(\mathbb{T}^m)} \le$$
$$\le \sum_{k=0}^{\infty} 2^{r\varkappa sk}\bigg(\sum_{\mu=k+1}^{\infty}\bigg\|\frac{\partial^\lambda t_\mu(x_1,\dots,x_m,0,\dots,0)}{\partial x_1^{\lambda_1}\dots\partial x_d^{\lambda_d}}\bigg\|_{L^{q),\frac{\theta q}{p}}(\mathbb{T}^m)}\bigg)^s.$$

Let $0<\delta<\varkappa$. Then using (15) and Hölder's inequality we get the following chain of inequalities:

$$\bigg(\sum_{k=0}^{\infty} 2^{r\varkappa s_1 k} E_k^{s_1}(h)_{L^{q),\theta\frac{q}{p}}(\mathbb{T}^m)}\bigg)^{\frac{s}{s_1}} \le c\sum_{k=0}^{\infty} 2^{r\varkappa sk}\bigg(\sum_{\mu=k}^{\infty} 2^{\mu r(1-\varkappa)} E_\mu(f)_{L^{p),\theta}(\mathbb{T}^d)}\bigg)^s =$$
$$= c\sum_{k=0}^{\infty} 2^{r\varkappa sk}\bigg(\sum_{\mu=k}^{\infty} 2^{-r\mu(\varkappa-\delta)}\cdot 2^{r\mu(1-\delta)} E_\mu(f)_{L^{p),\theta}(\mathbb{T}^d)}\bigg)^s \le$$
$$\le c\sum_{k=0}^{\infty} 2^{r\varkappa sk}\cdot 2^{-rk(\varkappa-\delta)s}\bigg(\sum_{\mu=k}^{\infty} 2^{r\mu(1-\delta)s} E_\mu^s(f)_{L^{p),\theta}(\mathbb{T}^d)}\bigg) =$$
$$= c\sum_{k=0}^{\infty}\sum_{k=0}^{\mu} 2^{r\varkappa sk}\cdot 2^{-rk(\varkappa-\delta)s}\cdot 2^{r\mu(1-\delta)s} E_\mu^s(f)_{L^{p),\theta}(\mathbb{T}^d)} =$$
$$= c\sum_{\mu=0}^{\infty} 2^{r\mu(1-\delta)s} E_\mu^s(f)_{L^{p),\theta}(\mathbb{T}^d)}\sum_{\mu=0}^{\mu} 2^{r\varkappa sk} 2^{-rk(\varkappa-\delta)s} \le$$
$$\le c\sum_{\mu=0}^{\infty} 2^{r\mu(1-\delta)s} 2^{r\mu\delta s} E_\mu^s(f)_{L^{p),\theta}(\mathbb{T}^d)} = c\sum_{\mu=0}^{\infty} 2^{r\mu s} E_\mu^s(f)_{L^{p),\theta}(\mathbb{T}^d)}.$$

Thus (14) and, consequently, the theorem is proved. □

In one dimensional case N. K. Bari established Bernstein-Zygmund and Nikolsky type inequalities for arbitrary intervals $[a,b]\subset(-\pi,\pi)$. In the sequel we present the similar results for the spaces $L^{p),\theta}$, $1<p<\infty$, $\theta>0$.

Theorem 3.2. *Let $1<p<\infty$ and $\theta>0$. Then for arbitrary trigonometric polynomial $T_n(x)$ and arbitrary interval $(a,b)\subset(-\pi,\pi)$, the inequality*

$$\|T_n'\|_{L^{p),\theta}(a,b)} \le c(a,b)n^2\|T_n\|_{L^{p),\theta}(a,b)} \tag{18}$$

holds.

Theorem 3.3. *Let $1<p<\infty$ and $\theta>0$. Then for arbitrary $[a,b]\subset(-\pi,\pi)$, $(a',b')\subset(a,b)$ and arbitrary trigonometric polynomial T_n we have*

$$\|T_n'\|_{L^{p),\theta}(a',b')} \le c(a,b,a',b')n\|T_n\|_{L^{p),\theta}(a,b)}. \tag{19}$$

Theorem 3.4. *Let $1 < p < q < \infty$ and $\theta > 0$. Then for arbitrary interval $(a,b) \subset (-\pi,\pi)$ and arbitrary polynomial T_n we have the inequality*

$$\|T_n\|_{L^{q),\theta\frac{q}{p}}(a,b)} \le c(a,b,a',b',p,q) n^{2\left(\frac{1}{p}-\frac{1}{q}\right)} \|T_n\|_{L^{p),\theta}(a,b)} \tag{20}$$

holds.

Theorem 3.5. *Let $1 < p < q < \infty$ and $\theta > 0$. Then for arbitrary $[a,b] \subset (-\pi,\pi)$, $(a',b') \subset (a,b)$ and arbitrary polynomial T_n we have*

$$\|T_n\|_{L^{q),\theta\frac{q}{p}}(a',b')} \le c(a,b,a',b',p,q) n^{\left(\frac{1}{p}-\frac{1}{q}\right)} \|T_n\|_{L^{p),\theta}(a,b)}. \tag{21}$$

The proofs of Theorems 3.2–3.5 are based on the Bari's inequalities and are similar to the proofs of Theorems 2.1 and 2.3 therefore we omit them.

In the sequel by $E_n(f)_{L^{p),\theta}(a,b)}$ denote

$$E_n(f)_{L^{p),\theta}(a,b)} = \inf \|f - T_k\|_{L^{p),\theta}(a,b)}$$

where the infimum is taken over all trigonometric polynomials T_k with order not greater than n.

The definition of the space $\dot{B}^{p),\theta}_{r,s}(a,b)$ is similar to that of the space $\dot{B}^{p),\theta}_{r,s}(\mathbb{T})$.

Theorem 3.6. *Let $1 < p < q < \infty$, $\theta > 0$, $r > 0$, $s_1 \ge s > 0$. Suppose that*

$$\varkappa = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\frac{2}{r} - \frac{2\lambda}{r} > 0.$$

Then for arbitrary $[a,b] \subset (-\pi,\pi)$ the following continuous embedding

$$\dot{B}^{p),\theta}_{r,s}(a,b) \hookrightarrow \dot{B}^{q),\theta\frac{q}{p}}_{\rho,s_1}(a,b), \quad \rho = \varkappa r,$$

holds.

Theorem 3.7. *Let $1 < p < q < \infty$, $\theta > 0$, $r > 0$, $s_1 \ge s > 0$. Suppose that*

$$\varkappa = 1 - \left(\frac{1}{p} - \frac{1}{q}\right)\frac{1}{r} - \frac{\lambda}{r} > 0.$$

Then for arbitrary $[a',b'] \subset (a,b)$, $[a,b] \subset (-\pi,\pi)$ the following continuous embedding

$$\dot{B}^{p),\theta}_{r,s}(a,b) \hookrightarrow \dot{B}^{q),\theta\frac{q}{p}}_{\rho,s_1}(a',b'), \quad \rho = \varkappa r$$

holds.

The proofs are similar to that of Theorem 3.1. It is enough to apply Theorems 3.2–3.5 instead of Theorems 2.1 and 2.3.

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