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ON THE STRONG LOGARITHMIC SUMMABILITY OF THE DOUBLE FOURIER-WALSH-PALEY SERIES

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Abstract. The following theorem is proved: Suppose that $E \subset [0,1)^2$ is any Lebesgue measurable set, $\mu_2 E > 0$, and $\phi(u)$ is a nonnegative, continuous and nondecreasing function on $[0,\infty)$ such that $u\phi(u)$ is a convex function on $[0,\infty)$ and $\phi(u) = o(\ln u)$, $u \to \infty$. Then there exists a function $g \in L_1([0,1)^2)$ such that

$$\int_{[0,1)^2} \mid g(x,y) \mid \phi(\mid g(x,y) \mid) dxdy < \infty$$

and the sequence of the strong logarithmic means $Q_N^{(w)}(g;x,y)$ by squares of the Fourier series of g with respect to the double Walsh-Paley system, that is the sequence

$$Q_N^{(w)}(g;x,y) = \frac{1}{\ln N} \sum_{k=1}^{N} \frac{\mid S_{k,k}^{(w)}(g;x,y) - g(x,y) \mid}{k},$$

is not bounded in measure on E.

რეზიუმე. დამტკიცებულია შემდეგი თეორემა: დავუშვათ რომ $E \subset [0,1)^2$ ნებისმიერი ლებეგის აზრით ზომადი სიმრავლეა, $\mu_2 E > 0$, და $\phi(u)$ არის არაუარყოფითი, უწყვეტი და არაკლებადი ფუნქცია $[0,\infty)$-ზე და $\phi(u) = o(\ln u)$, $u \to \infty$. მაშინ არსებობს ფუნქცია $g \in L_1([0,1)^2)$ ისეთი რომ

$$\int_{[0,1)^2} \mid g(x,y) \mid \phi(\mid g(x,y) \mid) dxdy < \infty$$

და g ფუნქციის ორმაგი უოლშ-პელის სისტემის მიმართ ფურიეს მწკრივის ძლიერი ლოგარითმული $Q_N^{(w)}(g;x,y)$ საშუალოების მიმდევრობა კვადრატების გასწვრივ, ე.ი. მიმდევრობა

$$Q_N^{(w)}(g;x,y) = \frac{1}{\ln N} \sum_{k=1}^{N} \frac{\mid S_{k,k}^{(w)}(g;x,y) - g(x,y) \mid}{k},$$

არ არის ზომით შემოსაზღვრული E-ზე.

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1. Introduction

Let μ_d, $d = 1, 2, \dots$, denote the Lebesgue measure in the Euclidean space R^d.

Set

$$I = [0, 1). \tag{1}$$

If $f \in L\log^+ L(I^2)$, F. Móricz, F. Schipp and W. R. Wade [1] have shown that as $n_1, n_2 \to \infty$, the $(C, 1)$ means $\sigma^{(w)}_{n_1,n_2}(f; x, y)$ of the double Fourier-Walsh-Paley series of f converge to $f(x, y)$ for a.e. $(x, y) \in I^2$. Gát [2] proved that this result is best possible in the following sense. Given a positive measurable function $\delta(t)$ defined on $[0, \infty)$ with $\lim\limits_{t\to\infty} \delta(t) = 0$, there is $h \in L\delta(L)\log^+ L(I^2)$, i.e. $\int\limits_{I^2} |h|\delta(|h|)\log^+ |h| < \infty$, with $\limsup\limits_{n_1,n_2\to\infty} \sigma^{(w)}_{2^{n_1},2^{n_2}}(h; x, y) = \infty$ for a.e. $(x, y) \in I^2$.

Let $S^{(w)}_{m,n}(f; x, y)$ denote the rectagular partial sum of the Fourier series of $f \in L_1(I^2)$ with respect to the double Walsh-Paley system $(m, n = 1, 2, \dots)$.

Goginava [3] proved that given a function $f \in L_1(I^2)$, then sequence of Marcinkiewicz $(C, 1)$ means of it's double Fourier-Walsh-Paley series is convergent a.e. to $f(x, y)$, that is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (S^{(w)}_{n,n}(f; x, y) - f(x, y)) = 0$$

holds for almost all $(x, y) \in I^2$.

It follows from a general result of Rodin ([4], p. 764) that, the following theorem is true

Theorem 1. *If $f \in LLn^+L(I^2)$ then*

$$\lim_{M,N\to\infty} \frac{1}{MN} \sum_{k=1}^{M} \sum_{l=1}^{N} | S^{(w)}_{k,l}(f; x, y) - f(x, y) |= 0$$

almost everywhere on I^2.

The strong logarithmic means $Q^{(w)}_N(f; x, y)$ by squares of the Fourier series of a function $f \in L_1(I^2)$ with respect to the double Walsh-Paley system are defined by $(N = 2, 3 \dots)$

$$Q^{(w)}_N(f; x, y) = \frac{1}{\ln N} \sum_{k=1}^{N} \frac{| S^{(w)}_{k,k}(f; x, y) - f(x, y) |}{k}. \tag{2}$$

It is natural to study the class of all those functions f that satisfy the following condition

$$\lim_{N\to\infty} Q^{(w)}_N(f; x, y) = 0$$

almost everywhere on I^2. In this paper we prove that this class cannot be wider than $LLn^+L(I^2)$. More precisely, we prove that the following theorem is true

Theorem 2. *Suppose that $E \subset I^2$ is any Lebesgue measurable set, $\mu_2 E > 0$, and $\phi(u)$ is a nonnegative, continuous and nondecreasing function on $[0,\infty)$ such that $u\phi(u)$ is a convex function on $[0,\infty)$ and*

$$\phi(u) = o(\ln u), \quad u \to \infty. \tag{3}$$

Then there exists a function $g \in L_1(I^2)$ such that

$$\int_{I^2} | g(x,y) | \phi(| g(x,y) |) dxdy < \infty$$

and the sequence $\{Q_N^{(w)}(g;x,y) : N = 1,2,\dots\}$ is not bounded in measure on E.

The corresponding results for the double trigonometric Fourier series were studied in papers [5]–[11].

2. Some Definitions and Auxiliary Propositions

Let Z_+ denote the set of all positive integers. For a finite set A Let $| A |$ denote the number of elements in A.

The Walsh-Paley system $\{w_m(x), \quad m = 0,1,2,\dots\}$ is defined on I in the following way (see, for example [12], p.1). Given a non-negative integer m it is possible to write the binary expansion of m uniquely as

$$m = \sum_{i=0}^{\infty} \alpha_i(m) 2^i, \tag{4}$$

where $\alpha_i(m) = 0$ or $\alpha_i(m) = 1$. Then

$$w_m(x) = \prod_{i=0}^{\infty} r_i^{\alpha_i(m)}(x), \tag{5}$$

where $\{r_i(x)\}$ is the Rademacher system.

Let $S_{m,m}^{(w)}(f;x,y)$ denote the square partial sum of the Fourier series of $f \in L_1(I^2)$ with respect to the double Walsh-Paley system $(m = 1,2,\dots)$:

$$S_{m,m}^{(w)}(f;x,y) = \sum_{i=0}^{m-1}\sum_{j=0}^{m-1} \int_0^1 \int_0^1 f(s,t) w_i(s) w_j(t) ds dt w_i(x) w_j(y). \tag{6}$$

We define Dirichlet kernels of the Walsh-Paley system by $D_0^{(w)}(x) = 0$ and

$$D_m^{(w)}(x) = \sum_{l=0}^{m-1} w_l(x), \quad x \in [0,1), \ m = 1,2,\dots. \tag{7}$$

The following is true (see [13], p.272)

Lemma 1. *Let $n \in Z_+$ and*

$$m = \sum_{i=0}^{n-1} \alpha_i(m) 2^i \tag{8}$$

be the binary expansion of $m \in Z_+$.

Let k be an integer such that $1 \leq k \leq n$ and let x be a real number that satisfies the inequality

$$\frac{1}{2^k} \leq x < \frac{1}{2^{k-1}}. \tag{9}$$

Then

a) *if $\alpha_{k-1}(m) = 0$ we have*

$$D_m^{(w)}(x) = w_m(x) \sum_{i=0}^{k-1} \alpha_i(m) 2^i \tag{10}$$

and

b) *if $\alpha_{k-1}(m) = 1$ we have*

$$D_m^{(w)}(x) = -w_m(x)[1 + \sum_{i=0}^{k-1} (1 - \alpha_i(m)) 2^i]. \tag{11}$$

Let for a number $h \in I$, I_h denote the interval $[0, 1-h)$.

If F is a Lebesgue measurable set in R^2, with $0 < \mu_2 F < \infty$, then let $L^0(F)$ denote the set of all Lebesgue measurable functions on F that are finite a.e. on F.

A set Q of Lebesgue measurable functions on F is called bounded in measure on F if for any $\epsilon > 0$ there is a constant $R > 0$ such that $\mu_2\{(x,y) \in F : \mid f(x,y) \mid \geq R\} \leq \epsilon$ for any function $f \in Q$.

A sequence $\{f_n(x,y),\ n = 1, 2, \dots\}$ of Lebesgue measurable functions on F is called bounded in measure on F if for any $\epsilon > 0$ there is a constant $R_1 > 0$ such that $\mu_2\{(x,y) \in F : \mid f_n(x,y) \mid \geq R_1\} \leq \epsilon$ for any $n = 1, 2, \dots$.

An operator $T : L_1(I^2) \to L^0(I^2)$ is called superlinear ([14], p.131) if for any $f_0 \in L_1(I^2)$ there is a linear operator $G_{f_0} : L_1(I^2) \to L^0(I^2)$ such that

$$G_{f_0}(f_0)(x,y) = T(f_0)(x,y) \tag{12}$$

and

$$\mid G_{f_0}(f)(x,y) \mid \leq \mid T(f)(x,y) \mid \quad \text{for any } \ f \in L_1(I^2) \tag{13}$$

and for almost all points (x,y) in I^2.

A superlinear operator $T : L_1(I^2) \to L^0(I^2)$ is said to be bounded in measure on I^2 if the set of functions

$$Q = \{T(f) : \ \parallel f \parallel_{L_1} \leq 1\}$$

is bounded in measure on I^2.

For each pair of numbers $(\theta,\eta)\in I_h^2$ and a number $h\in I$ introduce the following function of two variables (x,y) defined on I^2 by

$$\delta_{\theta,\eta,h}(x,y)=\begin{cases} h^{-2}, & \text{if} \quad (x,y)\in[\theta,\theta+h]\times[\eta,\eta+h]; \\ 0, & \text{otherwise on} \quad I^2. \end{cases} \tag{14}$$

The Kernel for a superlinear operator $T:L_1(I^2)\to L^0(I^2)$ is defined by

$$K(x,y,\theta,\eta)=\lim_{h\to\infty} T(\delta_{\theta,\eta,h}(.,.))(x,y)\,, \quad (x,y,\theta,\eta)\in I^4, \tag{15}$$

provided the limit exists for a.e. $(x,y,\theta,\eta)\in I^4$.

In [15] we have proved the following

Theorem 3. *Suppose that $E\subset I^2$ is any Lebesgue measurable set, $\mu_2 E>0$, and $\phi(u)$ is a nonnegative, continuous and nondecreasing function on $[0,\infty)$ such that $u\phi(u)$ is a convex function on $[0,\infty)$.*

Let $\{T_n : L_1(I^2)\to L^0(I^2),\ n=1,2,\dots\}$ be a sequence of superlinear operators that are bounded in measure on I^2 and let $K_n(x,y,\theta,\eta)$,

$$\| K_n(x,y,\theta,\eta)\|_\infty<\infty, \tag{16}$$

be the kernel for T_n, $n=1,2,\dots$.

Suppose that for each integer $n>n_0$ there exist: positive numbers h_n, ξ_n, and a Lebesgue measurable set E_n, $E_n\subset E$, $\mu_2 E_n\geq\gamma_1>0$, such that:

i) *For each set $F\subset E_n$, with $\mu_2 F\geq\frac{\gamma_1}{6}$, there exists a positive number $\lambda_n(F)$ with the property*

$$\mu_4\{(x,y,\theta,\eta)\in F\times I^2 : | K_n(x,y,\theta,\eta)|\geq C_1\lambda_n(F)\}\geq\frac{\xi_n}{\lambda_n(F)}>0. \tag{17}$$

ii)

$$\lim_{n\to\infty}\xi_n=\infty, \tag{18}$$

iii)

$$\phi(h_n^{-2})=o(\xi_n)\ \ (n\to\infty), \tag{19}$$

iv)

$$\mu_4\{(x,y,\theta,\eta)\in E\times \\ \times I_{t_n}^2 : | T_n(\delta_{\theta,\eta,h_n})(x,y)-K_n(x,y,\theta,\eta)|>1\}\leq\frac{\xi_n}{20\Lambda_n} \tag{20}$$

and

v)

$$h_n\leq t_n, \tag{21}$$

where

$$\Lambda_n=sup_{\{F:\ F\subset E_n,\ \mu_2 F\geq\frac{\gamma_1}{6}\}}\lambda_n(F), \tag{22}$$

$$t_n=\frac{\xi_n}{50\Lambda_n} \tag{23}$$

and C_1, γ_1 and n_0 are positive constants , independent of n and (x,y).

Then there exists a function $g \in L_1(I^2)$ such that

$$\int_{I^2} | g(x,y) | \phi(| g(x,y) |)dxdy < \infty$$

and the sequence of functions $\{T_n(g),\, n = 1,2,\dots\}$ is not bounded in measure on E.

3. Proof of Theorem 4

Set $(N = 3,4,\dots)$

$$B_N = \bigcup_{k=[\frac{N}{4}]+1}^{[\frac{N}{3}]} \Big[\frac{1}{2^k}, \frac{1}{2^{k-1}}\Big) \times \Big[\frac{1}{2^{N-k}}, \frac{1}{2^{N-k-1}}\Big). \tag{24}$$

It is clear that for $N \ge 13$

$$\mu_2 B_N = \sum_{k=[\frac{N}{4}]+1}^{[\frac{N}{3}]} \frac{1}{2^k}\frac{1}{2^{N-k}} \ge \frac{1}{24}\frac{N}{2^N}. \tag{25}$$

Now we prove

Lemma 2. *Let the set B_N be defined by* (24). *Then for all integers $N \ge N_1$ and $(x,y) \in B_N$ the following inequality holds*

$$\frac{1}{N}\sum_{m=1}^{2^N} \frac{| D_m^{(w)}(x) D_m^{(w)}(y) |}{m} \ge C_2 2^N, \tag{26}$$

where C_2 and N_1 are positive constants.

Proof. We choose a positive integer q such that

$$C = \frac{1}{4}\Big(\frac{1}{3} - 64\frac{1}{2^q}\Big)2^{-2q} > 0. \tag{27}$$

We keep q fixed.

Let $(x,y) \in B_N$. Then (see (24)) there exists an integer $k = k(x,y)$ such that

$$\Big[\frac{N}{4}\Big] + 1 \le k \le \Big[\frac{N}{3}\Big], \tag{28}$$

$$\frac{1}{2^k} \le x < \frac{1}{2^{k-1}} \tag{29}$$

and

$$\frac{1}{2^{N-k}} \le y < \frac{1}{2^{N-k-1}}. \tag{30}$$

Then according to the Abel's transformation

$$\sum_{m=1}^{2^N-1} \frac{1}{m} \mid D_m^{(w)}(x) D_m^{(w)}(y) \mid =$$

$$= \sum_{j=1}^{2^N-1} \Big(\frac{1}{j} - \frac{1}{j+1}\Big) \sum_{m=1}^{j} \mid D_m^{(w)}(x) D_m^{(w)}(y) \mid + \frac{1}{2^N} \sum_{m=1}^{2^N} \mid D_m^{(w)}(x) D_m^{(w)}(y) \mid \geq$$

$$\geq \sum_{j=1}^{2^N-1} \Big(\frac{1}{j} - \frac{1}{j+1}\Big) \sum_{m=1}^{j} \mid D_m^{(w)}(x) D_m^{(w)}(y) \mid \geq$$

$$\geq \frac{1}{2} \sum_{j=1}^{2^N-1} \frac{1}{j^2} \sum_{m=1}^{j} \mid D_m^{(w)}(x) D_m^{(w)}(y) \mid \geq$$

$$\geq \frac{1}{2} \sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2} \frac{1}{j^2} \sum_{m=1}^{j} \mid D_m^{(w)}(x) D_m^{(w)}(y) \mid \geq$$

$$\geq \frac{1}{2} \sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2} \frac{1}{j^2} \sum_{m=[\frac{j}{2}]+1}^{j} \mid D_m^{(w)}(x) D_m^{(w)}(y) \mid \geq$$

$$\geq \frac{1}{2} \sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2} \frac{1}{j^2} \sum_{m \in P_0} \mid D_m^{(w)}(x) D_m^{(w)}(y) \mid \tag{31}$$

where (see (28), (4))

$$P_0 = P_0(j, N, k, q) = \Big(\Big[\Big[\frac{j}{2}\Big] + 1, j\Big] \setminus (P \cup Q)\Big) \cap Z_+, \tag{32}$$

$$P = \{m \in Z_+ : \ 1 \leq m \leq j, \ \alpha_{k-1}(m) = \alpha_{k-2}(m) \cdots = \alpha_{k-q}(m) = 1\} \cup$$
$$\cup \{m \in Z_+ : \ 1 \leq m \leq j, \ \alpha_{k-1}(m) = \alpha_{k-2}(m) \cdots = \alpha_{k-q}(m) = 0\} \tag{33}$$

and

$$Q = \{m \in Z_+ : \ 1 \leq m \leq j, \ \alpha_{N-k-1}(m) =$$
$$= \alpha_{N-k-2}(m) \cdots = \alpha_{N-k-q}(m) = 1\} \cup$$
$$\cup \{m \in Z_+ : \ 1 \leq m \leq j, \ \alpha_{N-k-1}(m) =$$
$$= \alpha_{N-k-2}(m) \cdots = \alpha_{N-k-q}(m) = 0\}. \tag{34}$$

Let j be an integer such that

$$2^{[\frac{3N}{4}]+1} + 1 \leq j \leq 2^N - 2. \tag{35}$$

It is obvious that then (see (28), (27))

$$[\log_2 j] > N - k - 1 > N - k - q > k - 1 > k - q > 0, \tag{36}$$

for all $N > N_2$, where N_2 is a certain positive constant.

It is clear that the number of elements of the set $P \cup Q$ satisfies the following inequality (see (33), (34))

$$| P \cup Q | \leq 2^{[\log_2 j]+6-q} \leq 64 \frac{j}{2^q}.$$

Therefore (see (32))

$$| P_0 | \geq \frac{j}{2} - 1 - 64 \frac{j}{2^q}. \tag{37}$$

Let $m \in P_0$. Then (see (32), (4), (36))

$$m = \alpha_{[\log_2 j]}(m) 2^{[\log_2 j]} + \alpha_{[\log_2 j]-1}(m) 2^{[\log_2 j]-1} + \dots + \alpha_{[\frac{3N}{4}]}(m) 2^{[\frac{3N}{4}]} + \dots$$
$$+\alpha_{N-k-1}(m) 2^{N-k-1} + \dots + \alpha_{N-k-q}(m) 2^{N-k-q} + \dots$$
$$+\alpha_{k-1}(m) 2^{k-1} + \dots + \alpha_{k-q}(m) 2^{k-q} + \dots + \alpha_0(m) \geq 2^{[\frac{3N}{4}]}. \tag{38}$$

Now we will prove that

$$| D_m^{(w)}(x) | \geq 2^{k-q}. \tag{39}$$

Indeed, we consider two cases:

Case 1. $\alpha_{k-1}(m) = 1$. Then (see (33), (32)) there exists an integer i_0 such that

$$k - 1 > i_0 \geq k - q$$

and

$$\alpha_{i_0}(m) = 0.$$

Thus (see (38), (11), (29))

$$| D_m^{(w)}(x) | = (1 - \alpha_{k-1}(m)) 2^{k-1} +$$
$$+ \dots + (1 - \alpha_{k-q}(m)) 2^{k-q} \geq (1 - \alpha_{i_0}(m)) 2^{i_0} \geq 2^{k-q}. \tag{40}$$

Case 2. $\alpha_{k-1}(m) = 0$. Then (see (33), (32)) there exists an integer j_0 such that

$$k - 1 > j_0 \geq k - q$$

and

$$\alpha_{j_0}(m) = 1.$$

Thus (see (8), (9), (10), (29))

$$| D_m^{(w)}(x) | = \alpha_{k-1}(m) 2^{k-1} + \dots + \alpha_{k-q}(m) 2^{k-q} \geq \alpha_{j_0}(m) 2^{j_0} \geq 2^{k-q}.$$

The inequality (39) (see (40)) is proved.

Similarly we can prove that (see (30), (32), (34))

$$| D_m^{(w)}(y) | \geq 2^{N-k-q}. \tag{41}$$

Now we have (see (41), (39), (35)–(37), (27))

$$\frac{1}{2}\sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2}\frac{1}{j^2}\sum_{m\in P_0}\mid D_m^{(w)}(x)D_m^{(w)}(y)\mid\geq$$

$$\geq\frac{1}{2}\sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2}\frac{1}{j^2}\sum_{m\in P_0}2^{k-q}2^{N-k-q}\geq$$

$$\geq\frac{1}{2}\sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2}\frac{1}{j^2}\mid P_0\mid 2^{N-2q}\geq$$

$$\geq\frac{1}{2}\sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2}\frac{1}{j^2}\Big(\frac{j}{2}-1-16\frac{j}{2^q}\Big)2^{N-2q}\geq$$

$$\geq\frac{1}{2}\sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2}\frac{1}{j^2}j\Big(\frac{1}{3}-16\frac{1}{2^q}\Big)2^{N-2q}\geq$$

$$\geq\frac{1}{2}\sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2}\frac{1}{j}\Big(\frac{1}{3}-64\frac{1}{2^q}\Big)2^{N-2q}\geq$$

$$\geq\frac{1}{4}\Big(\frac{1}{3}-64\frac{1}{2^q}\Big)2^{N-2q}\sum_{j=2^{[\frac{3N}{4}]+1}+1}^{2^N-2}\frac{1}{j}\geq C_2N2^N$$

for all $N\geq N_1$ (for certain positive constants N_1 and C_2).

Lemma 2 (see (31), (26)) is proved. □

Let $(x,y)\in I^2$. Consider the set (for the definition and properties of the operation $\dot{+}$ see [12], p. 10-13)

$$B_N\dot{+}(x,y)=\{(\theta,\eta)\in I^2:\ (\theta,\eta)=(\theta_1\dot{+}x,\eta_1\dot{+}y),\ (\theta_1,\eta_1)\in B_N\}.$$

It is clear that if $(\theta,\eta)\in B_N\dot{+}(x,y)$ then there exists a point $(\theta_1,\eta_1)\in B_N$ such that $(\theta,\eta)=(\theta_1\dot{+}x,\eta_1\dot{+}y)$ and, consequently, according to Lemma 2, for a.e. $(\theta,\eta)\in B_N\dot{+}(x,y)$ and for all integers $N\geq N_1$ the following inequality holds (see (26))

$$\frac{1}{N}\sum_{m=1}^{2^N}\frac{\mid D_m^{(w)}(\theta\dot{+}x)D_m^{(w)}(\eta\dot{+}y)\mid}{m}\geq$$

$$\geq\frac{1}{N}\sum_{m=1}^{2^N}\frac{\mid D_m^{(w)}(\theta_1)D_m^{(w)}(\eta_1)\mid}{m}\geq C_22^N. \tag{42}$$

Let $F \subset E$ be any set such that $\mu_2 F \geq \frac{\gamma_1}{6}$, where $\gamma_1 = \mu_2 E > 0$. Consider the set

$$\Omega_N = \Big\{ (x, y, \theta, \eta) \in F \times$$

$$\times I^2 : \ \frac{1}{N} \sum_{m=1}^{2^N} \frac{| D_m^{(w)}(\theta \dot{+} x) D_m^{(w)}(\eta \dot{+} y) |}{m} \geq C_2 2^N \Big\}. \tag{43}$$

It is easy to see from (25), (42), (43) that for a.e. $(x, y) \in I^2$

$$\int_{I^2} \chi_{\Omega_N}(x, y, \theta, \eta) d\theta d\eta \geq \mu_2 B_N \geq \frac{1}{24} \frac{N}{2^N}$$

and, consequently,

$$\mu_4 \Omega_N = \int_F \int_{I^2} \chi_{\Omega_N}(x, y, \theta, \eta) dx dy d\theta d\eta \geq \frac{\gamma_1}{6} \frac{1}{24} \frac{N}{2^N}. \tag{44}$$

We set in Theorem 3 for $N \geq N_1$, $N \in Z_+$, (see (43))

$$E_N = E, \tag{45}$$

$$h_N = \frac{1}{2^{9N}}, \tag{46}$$

$$\xi_N = \frac{\gamma_1}{6} \frac{1}{24} N, \tag{47}$$

$$C_1 = C_2, \tag{48}$$

and

$$\lambda_N(F) = 2^N. \tag{49}$$

Then (see (22), (23)) we have

$$\Lambda_N = 2^N \tag{50}$$

and

$$t_N = \frac{\frac{\gamma_1}{6} \frac{1}{24} N}{50 \cdot 2^N} \geq h_N. \tag{51}$$

In Theorem 3 we set also

$$T_N(f)(x, y) = \frac{1}{N} \sum_{m=1}^{2^N} \frac{| S_{m,m}^{(w)}(f; x, y) |}{m}, \tag{52}$$

that is clearly (see (6), (12), (13)) superlinear and bounded in measure. Then it is easy to see that the kernel (see (14), (15), (52), (6))

$$K_N^{(w)}(x, y, \theta, \eta) = \frac{1}{N} \sum_{m=1}^{2^N} \frac{| D_m^{(w)}(\theta \dot{+} x) D_m^{(w)}(\eta \dot{+} y) |}{m}. \tag{53}$$

Introduce the following set

$$P_N = \bigcup_{i=1}^{2^N} \bigcup_{j=1}^{2^N} \Big[\frac{i-1}{2^N}, \frac{i}{2^N} - \frac{1}{2^{8N}}\Big) \times \Big[\frac{j-1}{2^N}, \frac{j}{2^N} - \frac{1}{2^{8N}}\Big). \tag{54}$$

It is clear that

$$\mu_2 P_N \geq 1 - \frac{2}{2^{7N}}. \tag{55}$$

It is obvious that (see (14), (4)–(6)) we have for almost all $(x, y, \theta, \eta) \in I^2 \times P_N$ and for all $1 \leq m \leq 2^N$ and $N > 16$

$$S_{m,m}^{(w)}(\delta_{\theta,\eta,h_N}; x, y) = \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} h_N^{-2} \int_{\theta}^{\theta+h_N} \int_{\eta}^{\eta+h_N} w_i(s) w_j(t) ds dt w_i(x) w_j(y) =$$

$$= \sum_{i=0}^{m-1} \sum_{j=0}^{m-1} w_i(\theta) w_j(\eta) w_i(x) w_j(y) = D_m^{(w)}(\theta \dot{+} x) D_m^{(w)}(\eta \dot{+} y).$$

Set

$$\Theta_N = \Big\{(x, y, \theta, \eta) \in I \times I_{t_N} : \Big| \frac{1}{N} \sum_{m=1}^{2^N} \frac{| S_{m,m}^{(w)}(\delta_{\theta,\eta,h_n}; x, y) |}{m} -$$

$$- \frac{1}{N} \sum_{m=1}^{2^N} \frac{| D_m^{(w)}(\theta \dot{+} x) D_m^{(w)}(\eta \dot{+} y) |}{m} \Big| > 1 \Big\}.$$

It is obvious that (see (51)) $\Theta_N \subset I^2 \times (I^2 \setminus P_N)$ and, consequently, (see (55))

$$\mu_4 \Theta_N \leq \frac{2}{2^{7N}}. \tag{56}$$

Taking account of (1)–(3), (43)–(53), (16)–(23), (57), (56) we can conclude that according to Theorem 3 we have proved Theorem 2.

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