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REGULAR ELEMENTS OF THE SEMIGROUP $B_X(D)$ DEFINED BY SEMILATTICES OF THE CLASS $\Sigma_4(X,8)$ AND THEIR CALCULATION FORMULAS

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Abstract. The paper gives a full description of regular elements of the semigroup $B_X(D)$, which are defined by semilattices of the class $\Sigma_4(X,8)$. Formulas are derived, by means of which the number of regular elements of the semigroup is calculated when X is a finite set.

რეზიუმე. სტატიაში სრულადაა აღწერილი $\Sigma_4(X,8)$ კლასის გაერთიანებათა D ნახევარმესერებით განსაზღვრული $B_X(D)$ ნახევარჯგუფების რეგულარული ელემენტები. სასრული X სიმრავლის შემთხვევაში მოყვანილია ამ ნახევარჯგუფის რეგულარული ელემენტების რაოდენობის დათვლის ფორმულები.

Let X be an arbitrary nonempty set, D be an X-semilattice of unions, i.e. a nonempty set of subsets of the set X that is closed with respect to the set-theoretic operations of unification of elements from D, f be an arbitrary mapping from X into D. To each such a mapping f there correspondes a binary relation α_f on the set X that satisfies the condition

$$\alpha_f = \bigcup_{x\in X} \big(\{x\} \times f(x)\big).$$

The set of all such α_f is denoted by $B_X(D)$. It is easy to prove that $B_X(D)$ is a semigroup with respect to the operation of multiplication of binary relations, which is called a complete semigroup of binary relations defined by an X-semilattice of unions D.

We denote $\varnothing$ by an empty binary relation or empty subset of the set X. The condition $(x,y) \in \alpha$ will be written in the form $x\alpha y$. Further let $x,y \in X$, $Y \subseteq X$, $\alpha \in B_X(D)$, $T \in D$, $\varnothing \neq D' \subseteq D$, $\breve{D} = \cup D$ and $t \in \breve{D}$.

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Then by symbols we denote the following sets:

$$y\alpha = \{x \in X \mid y\,\alpha\,x\}, \ \ Y\,\alpha = \bigcup_{y \in Y} y\,\alpha,$$
$$2^X = \{Y \mid \ Y \subseteq X\}, \ \ X^* = 2^X \setminus \{\varnothing\},$$
$$V(D,\alpha) = \{Y\alpha \mid Y \in D\},$$
$$D'_T = \{T' \in D' \mid T \subseteq T'\}, \ \ \ddot{D}'_T = \{T' \in D' \mid T' \subseteq T\},$$
$$D'_t = \{Z' \in D' \mid t \in Z'\}, \quad l(D',T) = \cup(D' \setminus D'_T).$$

By symbol $\Lambda(D,D')$ we mean an exact lower bound of the set D' in the semilattice D.

Let X and $\Sigma_4(X,8)$ be respectively an arbitrary nonempty set and the class of X-semilattices of unions, where each element is isomorphic to some X-semilattice of unions $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\}$ that satisfies the conditions

$$\begin{gathered} Z_7 \subset Z_4 \subset Z_1 \subset \breve{D}, \ \ Z_7 \subset Z_6 \subset Z_1 \subset \breve{D}, \ \ Z_7 \subset Z_6 \subset Z_3 \subset \breve{D}, \\ Z_1 \setminus Z_2 \neq \varnothing, \ \ Z_2 \setminus Z_1 \neq \varnothing, \ \ Z_1 \setminus Z_3 \neq \varnothing, \ \ Z_3 \setminus Z_1 \neq \varnothing, \\ Z_2 \setminus Z_3 \neq \varnothing, \ \ Z_3 \setminus Z_2 \neq \varnothing, \ \ Z_4 \setminus Z_5 \neq \varnothing, \ \ Z_5 \setminus Z_4 \neq \varnothing, \\ Z_4 \setminus Z_6 \neq \varnothing, \ \ Z_6 \setminus Z_4 \neq \varnothing, \ \ Z_5 \setminus Z_6 \neq \varnothing, \ \ Z_6 \setminus Z_5 \neq \varnothing, \\ Z_1 \cup Z_2 = Z_1 \cup Z_3 = \breve{D}, \\ Z_2 \cup Z_3 = Z_4 \cup Z_2 = Z_4 \cup Z_3 = Z_5 \cup Z_2 = Z_5 \cup Z_3 = \breve{D}, \\ Z_4 \cup Z_5 = Z_4 \cup Z_6 = Z_5 \cup Z_6 = Z_1. \end{gathered} \tag{1}$$

An X-semilattice that satisfies conditions (1) is shown in Figure 1.

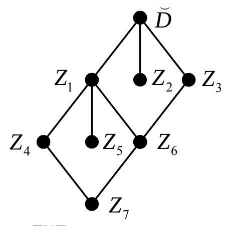

Figure 1

We call an element α taken from the semigroup $B_X(D)$ a regular element of the semigroup $B_X(D)$ if in $B_X(D)$ there exists an element β such that $\alpha \circ \beta \circ \alpha = \alpha$ (see [2] and [12]).

Let us give some definitions.

Definition 1. We say that a complete X-semilattice of unions D is an XI-semilattice of unions if it satisfies the following two conditions:

(a) $\Lambda(D, D_t) \in D$ for any $t \in \breve{D}$;

(b) $Z = \bigcup_{t \in Z} \wedge(D, D_t)$ for any nonempty element Z of D.

The following lemma is well know (see [11, Lemma 2.2]).

Lemma 1. *Let* $D = \{Z_7, Z_6, Z_5, Z_{4,} Z_3, Z_{2,} Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. *Then subsets of the following form*

(1) $\{Z_7\}$, $\{Z_6\}$, $\{Z_5\}$, $\{Z_4\}$, $\{Z_3\}$, $\{Z_2\}$, $\{Z_1\}$, $\{\breve{D}\}$ (*see Diagram* 1 *in Figure* 2)*;*

(2) $\{Z_7, Z_6\}$, $\{Z_7, Z_4\}$, $\{Z_7, Z_3\}$, $\{Z_7, Z_1\}$, $\{Z_7, \breve{D}\}$, $\{Z_6, Z_3\}$, $\{Z_6, Z_1\}$, $\{Z_6, \breve{D}\}$, $\{Z_5, Z_1\}$, $\{Z_5, \breve{D}\}$, $\{Z_4, Z_1\}$, $\{Z_4, \breve{D}\}$, $\{Z_3, \breve{D}\}$, $\{Z_2, \breve{D}\}$, $\{Z_1, \breve{D}\}$ (*see Diagram* 2 *in Figure* 2)*;*

(3) $\{Z_7, Z_6, \breve{D}\}$, $\{Z_7, Z_4, \breve{D}\}$, $\{Z_7, Z_3, \breve{D}\}$, $\{Z_7, Z_1, \breve{D}\}$, $\{Z_5, Z_1, \breve{D}\}$, $\{Z_7, Z_6, Z_3\}$, $\{Z_7, Z_6, Z_1\}$, $\{Z_7, Z_4, Z_1\}$, $\{Z_6, Z_3, \breve{D}\}$, $\{Z_6, Z_1, \breve{D}\}$, $\{Z_4, Z_1, \breve{D}\}$ (*see Diagram* 3 *in Figure* 2)*;*

(4) $\{Z_7, Z_6, Z_3, \breve{D}\}$, $\{Z_7, Z_6, Z_1, \breve{D}\}$, $\{Z_7, Z_4, Z_1, \breve{D}\}$ (*see Diagram* 4 *in Figure* 2)*;*

(5) $\{Z_7, Z_6, Z_4, Z_1\}$, $\{Z_7, Z_4, Z_3, \breve{D}\}$, $\{Z_6, Z_3, Z_1, \breve{D}\}$, $\{Z_7, Z_3, Z_1, \breve{D}\}$ (*see Diagram* 5 *in Figure* 2)*;*

(6) $\{Z_7, Z_6, Z_4, Z_1, \breve{D}\}$ (*see Diagram* 6 *in Figure* 2)*;*

(7) $\{Z_7, Z_6, Z_3, Z_1, \breve{D}\}$ (*see Diagram* 7 *in Figure* 2)*;*

(8) $\{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$ (*see Diagram* 8 *in Figure* 2)*,*

exhaust all XI*-subsemilattices of the semilattice* D*.*

According to the proof of Lemma 1, the diagrams of XI-semilattices will have the form of one of the diagrams given in Figure 2.

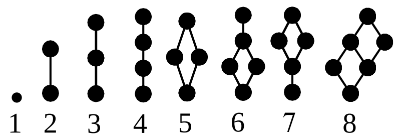

FIGURE 2

Now assume that $D \in \Sigma_4(X, 8)$. We introduce the following notation:

(1) $Q_1 = \{T\}$, where $T \in D$;

(2) $Q_2 = \{T, T'\}$, where $T, T' \in D$ and $T \subset T'$;

(3) $Q_3 = \{T, T', T''\}$, where $T, T', T'' \in D$ and $T \subset T' \subset T''$;

(4) $Q_4 = \{Z_7, T, T', \breve{D}\}$, where $T, T' \in D$ and $Z_7 \subset T \subset T' \subset \breve{D}$;

(5) $Q_5 = \{T, T', Z, T' \cup Z\}$, where $T, T', Z \in D$, $T \subset T'$, $T \subset Z$, $T' \setminus Z \neq \varnothing$, and $Z \setminus T' \neq \varnothing$;

(6) $Q_6 = \{Z_7, Z_6, Z_4, Z_1, \breve{D}\}$;

(7) $Q_7 = \{Z_7, Z_6, Z_3, Z_1, \breve{D}\}$;

(8) $Q_8 = \{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$.

Definition 2. The one-to-one mapping φ between the complete X-semilattices of unions D' and D'' is called a complete isomorphism if the condition

$$\varphi(\cup D_1) = \bigcup_{T' \in D_1} \varphi(T')$$

is fulfilled for each nonempty subset D_1 of the semilattice D' (see [4, Definition 6.3.2] or [5, Definition 6.3.2]).

Definition 3. Let α be some binary relation of the semigroup $B_X(D)$. We say that the complete isomorphism φ between the complete semilattices of unions Q and D' is a complete α-isomorphism if

(a) $Q = V(D, \alpha)$;

(b) $\varphi(\varnothing) = \varnothing$ for $\varnothing \in V(D, \alpha)$ and $\varphi(T)\alpha = T$ for any $T \in V(D, \alpha)$ (see [4, Definition 6.3.3] or [5, Definition 6.3.2]).

Definition 4. Let the symbol $\Sigma'_{\mathrm{XI}}(X, D)$ denote a set of all XI-subsemilattices of the semilattice D.

Let further, $D', D'' \in \Sigma'_{\mathrm{XI}}(X, D)$ and $\vartheta_{\mathrm{XI}} \subseteq \Sigma'_{\mathrm{XI}}(X, D) \times \Sigma'_{\mathrm{XI}}(X, D)$. It is assumed that $D' \vartheta_{\mathrm{XI}} D''$ if and only if there exist some complete isomorphism φ between the semilatices D' and D''. One can easily verify that the binary relation ϑ_{XI} is an equivalence relation on the set $\Sigma'_{\mathrm{XI}}(X, D)$.

Let the by symbol $Q_i \vartheta_{\mathrm{XI}}$ denote the ϑ_{XI}-class of equivalence ϑ_{XI} of the set $\Sigma'_{\mathrm{XI}}(X, D)$, whore every element is isomorphic to the X-semilattice Q_i.

Remark, that

$$\begin{aligned} Q_1 \vartheta_{\mathrm{XI}} &= \Big\{ \{Z_7\}, \{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\breve{D}\} \Big\}, \\ Q_2 \vartheta_{\mathrm{XI}} &= \Big\{ \{Z_7, Z_6\}, \{Z_7, Z_4\}, \{Z_7, Z_3\}, \{Z_7, Z_1\}, \{Z_7, \breve{D}\}, \\ &\quad \{Z_6, Z_3\}, \{Z_6, Z_1\}, \{Z_6, \breve{D}\}, \{Z_5, Z_1\}, \{Z_5, \breve{D}\}, \\ &\quad \{Z_4, Z_1\}, \{Z_4, \breve{D}\}, \{Z_3, \breve{D}\}, \{Z_2, \breve{D}\}, \{Z_1, \breve{D}\} \Big\}, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ Q_7 \vartheta_{\mathrm{XI}} &= \Big\{ \{Z_7, Z_6, Z_3, Z_1, \breve{D}\} \Big\}, \\ Q_8 \vartheta_{\mathrm{XI}} &= \Big\{ \{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\} \Big\}. \end{aligned}$$

Assume that $D' \in Q_i\theta_{\mathrm{XI}}$ and denote by the symbol $R(D')$ the set of all regular elements α of the semigroup $B_X(D)$, for which the semilattices D' and Q_i are mutually α-isomorphic and $V(D,\alpha) = D'$ and

$$R^*(Q_i) = \bigcup_{D' \in Q_i \vartheta_{\mathrm{XI}}} R(D')$$

(see [4, definition 6.3.5] or [5, definition 6.3.5]).

Let $\alpha \in B_X(D)$, $T \in V(D,\alpha)$ and $Y_T^\alpha = \{x \in X \mid x\alpha = T\}$.

Theorem 1. *Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X,8)$. Then a binary relation α from the semigroup $B_X(D)$ whose quasinormal representation has the form $\alpha = \bigcup_{T \in V(D,\alpha)} (Y_T^\alpha \times T)$ will be a regular element of this semigroup if and only if there exists a complete φ α-isomorphism from the semilattice $V(D,\alpha)$ to some subsemilattice D' of the semilattice D which satisfies one of the following conditions:*

(a) *$\alpha = X \times T$, for some element $T \in D$;*

(b) *$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T')$ for some $T, T' \in D$, $T \subset T'$, $Y_T^\alpha, Y_{T'}^\alpha \notin \{\varnothing\}$ and $Y_T^\alpha \supseteq \varphi(T)$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$;*

(c) *$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$ for some $T, T', T'' \in D$, $T \subset T' \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and $Y_T^\alpha \supseteq \varphi(T)$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_{T''}^\alpha \cap \varphi(T'') \neq \varnothing$;*

(d) *$\alpha = (Y_7^\alpha \times Z_7) \cup (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_0^\alpha \times \breve{D})$ for some $T, T' \in D$, $Z_7 \subset T \subset T' \subset \breve{D}$, $Y_7^\alpha, Y_T^\alpha, Y_{T'}^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_7^\alpha \cup Y_T^\alpha \supseteq \varphi(T)$, $Y_7^\alpha \cup Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cap \varphi(T) \neq \varnothing$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_0^\alpha \cap \varphi(\breve{D}) \neq \varnothing$;*

(e) *$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_Z^\alpha \times Z) \cup (Y_{T' \cup Z}^\alpha \times (T' \cup Z))$ for some $T, T', Z \in D$, $T \subset T'$, $T \subset Z$, $T' \setminus Z \neq \varnothing$, $Z \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_Z^\alpha \notin \{\varnothing\}$ and $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq \varphi(T')$, $Y_T^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z)$, $Y_{T'}^\alpha \cap \varphi(T') \neq \varnothing$, $Y_Z^\alpha \cap \varphi(Z) \neq \varnothing$;*

(f) *$\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$ for some $Y_7^\alpha, Y_6^\alpha, Y_4^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and $Y_7^\alpha \cup Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_4^\alpha \supseteq \varphi(Z_4)$, $Y_6^\alpha \cap \varphi(Z_6) \neq \varnothing$, $Y_4^\alpha \cap \varphi(Z_4) \neq \varnothing$, $Y_0^\alpha \cap \varphi(\breve{D}) \neq \varnothing$;*

(g) *$\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$ for some $Y_7^\alpha, Y_6^\alpha, Y_3^\alpha, Y_1^\alpha \notin \{\varnothing\}$ and $Y_7^\alpha \cup Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_3^\alpha \supseteq \varphi(Z_3)$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_1^\alpha \supseteq \varphi(Z_1)$, $Y_6^\alpha \cap \varphi(Z_6) \neq \varnothing$, $Y_3^\alpha \cap \varphi(Z_3) \neq \varnothing$, $Y_1^\alpha \cap \varphi(Z_1) \neq \varnothing$;*

(h) *$\alpha = (Y_7^\alpha \times Z_7) \cup (Y_6^\alpha \times Z_6) \cup (Y_4^\alpha \times Z_4) \cup (Y_3^\alpha \times Z_3) \cup (Y_1^\alpha \times Z_1) \cup (Y_0^\alpha \times \breve{D})$ for some $Y_7^\alpha, Y_6^\alpha, Y_4^\alpha, Y_3^\alpha \notin \{\varnothing\}$ and $Y_7^\alpha \supseteq \varphi(Z_7)$, $Y_7^\alpha \cup Y_6^\alpha \supseteq \varphi(Z_6)$, $Y_7^\alpha \cup Y_4^\alpha \supseteq \varphi(Z_4)$, $Y_7^\alpha \cup Y_6^\alpha \cup Y_3^\alpha \supseteq \varphi(Z_3)$, $Y_6^\alpha \cap \varphi(Z_6) \neq \varnothing$, $Y_4^\alpha \cap \varphi(Z_4) \neq \varnothing$, $Y_3^\alpha \cap \varphi(Z_3) \neq \varnothing$.*

Proof. In this case, from Lemma 2.2 (see [11]) it follows that diagrams 1–8 given in Figure 2 (see [11]) exhaust all diagrams of XI-subsemilattices of the semilattices D. A quasinormal representation of regular elements of the semigroup $B_X(D)$, which are defined by these XI-semilattices, may have one of the forms listed above. Then the validity of the theorem immediately follows from Theorems 13.1.1, 13.3.1 and 13.9.1 (see [4] or [5]). □

Lemma 2. *If $|\Omega(Q)| = m_0$, then the following propositions are valid:*

(a) $|R(Q_1)| = 1$;

(b) $|R(Q_2)| = m_0 \cdot (2^{|\overline{T}'\setminus\overline{T}|} - 1) \cdot 2^{|X\setminus\overline{T}'|}$;

(c) $|R(Q_3)| = m_0 \cdot (2^{|\overline{T}'\setminus\overline{T}|} - 1) \cdot (3^{|\overline{T}''\setminus\overline{T}'|} - 2^{|\overline{T}''\setminus\overline{T}'|}) \cdot 3^{|X\setminus\overline{T}''|}$;

(d) $|R(Q_4)| = m_0 \cdot (2^{|\overline{T}\setminus Z_7|} - 1) \cdot (3^{|\overline{T}'\setminus\overline{T}|} - 2^{|\overline{T}'\setminus\overline{T}|}) \cdot (4^{|\breve{D}\setminus\overline{T}'|} - 3^{|\breve{D}\setminus\overline{T}'|}) \cdot 4^{|X\setminus\breve{D}|}$;

(e) $|R(Q_5)| = 2 \cdot m_0 \cdot (2^{|\overline{T}'\setminus\overline{Z}|} - 1) \cdot (2^{|\overline{Z}\setminus\overline{T}'|} - 1) \cdot 4^{|X\setminus(\overline{T}'\cup\overline{Z})|}$;

(f) $|R(Q_6)| = 2 \cdot m_0 \cdot (2^{|\overline{Z}_6\setminus\overline{Z}_4|} - 1) \cdot (2^{|\overline{Z}_4\setminus\overline{Z}_6|} - 1) \cdot (5^{|\breve{D}\setminus\overline{Z}_1|} - 4^{|\breve{D}\setminus\overline{Z}_1|}) \cdot 5^{|X\setminus\breve{D}|}$;

(g) $|R(Q_7)| = 2 \cdot m_0 \cdot (2^{|\overline{Z}_6\setminus\overline{Z}_7|} - 1) \cdot 2^{|(\overline{Z}_3\cap\overline{Z}_1)\setminus\overline{Z}_6|} \cdot (3^{|\overline{Z}_3\setminus\overline{Z}_1|} - 2^{|\overline{Z}_3\setminus\overline{Z}_1|}) \cdot$
$\cdot (3^{|\overline{Z}_1\setminus\overline{Z}_3|} - 2^{|\overline{Z}_1\setminus\overline{Z}_3|}) \cdot 5^{|X\setminus\breve{D}|}$;

(h) $|R(Q_8)| = m_0 \cdot (2^{|\overline{Z}_4\setminus\overline{Z}_3|} - 1) \cdot (2^{|\overline{Z}_6\setminus\overline{Z}_4|} - 1) \cdot (3^{|\overline{Z}_3\setminus\overline{Z}_1|} - 2^{|\overline{Z}_3\setminus\overline{Z}_1|}) \cdot$
$\cdot 6^{|X\setminus\breve{D}|}$.

Proof. The propositions (a), (b), (c) and (d) immediately follow from Theorem 13.1.2, while the propositions (e), (f), (g) and (h) follow from Corollaries 13.3.4, 13.3.5, 13.3.6 and 13.3.7 (see [4] or [5]). □

1. Lemma 3. *Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. If X is a finite set, then*

$$|R^*(Q_1)| = 8.$$

Proof. According to the definition of the semilattice D we have

$$Q_1\theta_{\mathrm{XI}} = \Big\{\{Z_7\}, \{Z_6\}, \{Z_5\}, \{Z_4\}, \{Z_3\}, \{Z_2\}, \{Z_1\}, \{\breve{D}\}\Big\}.$$

Assume that

$$D_1' = \{Z_7\},\ \ D_2' = \{Z_6\},\ \ D_3' = \{Z_5\},\ \ D_4' = \{Z_4\},$$
$$D_5' = \{Z_3\},\ \ D_6' = \{Z_2\},\ \ D_7' = \{Z_1\},\ \ D_8' = \{\breve{D}\}.$$

Then from Theorem 6.3.6 (see [4] or [5]) we obtain

$$|R^*(Q_1)| = |R(D_1')| + |R(D_2')| + |R(D_3')| + |R(D_4')| +$$
$$+ |R(D_5')| + |R(D_6')| + |R(D_7')| + |R(D_8')|.$$

By the proposition (a) of Lemma 2, from this equality we obtain

$$|R^*(Q_1)| = 1+1+1+1+1+1+1+1 = 8. \qquad \square$$

2. By the definition of the semilattice D we have

$$Q_2\theta_{\mathrm{XI}} = \Big\{\{Z_7,\breve{D}\},\{Z_5,Z_1\},\{Z_2,\breve{D}\},\{Z_1,\breve{D}\},\\ \{Z_7,Z_6\},\{Z_7,Z_4\},\{Z_7,Z_3\},\{Z_7,Z_1\},\{Z_6,Z_3\},\{Z_6,Z_1\},\\ \{Z_6,\breve{D}\},\{Z_5,\breve{D}\},\{Z_4,Z_1\},\{Z_4,\breve{D}\},\{Z_3,\breve{D}\}\Big\}.$$

Now, if

$$D'_1=\{Z_7,\breve{D}\},\quad D'_2=\{Z_5,\breve{D}\},\quad D'_3=\{Z_2,\breve{D}\},\quad D'_4=\{Z_1,\breve{D}\},\\ D'_5=\{Z_7,Z_6\},\quad D'_6=\{Z_7,Z_4\},\quad D'_7=\{Z_7,Z_3\},\quad D'_8=\{Z_7,Z_1\},\\ D'_9=\{Z_6,Z_3\},\quad D'_{10}=\{Z_6,Z_1\},\quad D'_{11}=\{Z_6,\breve{D}\},\quad D'_{12}=\{Z_5,Z_1\},\\ D'_{13}=\{Z_4,Z_1\},\quad D'_{14}=\{Z_4,\breve{D}\},\quad D'_{15}=\{Z_3,\breve{D}\},$$

then from Theorem 6.3.6 we obtain (see [4] or [5]):

$$R^*(Q_2) = \bigcup_{i=1}^{15} R(D'_i). \tag{2}$$

Lemma 3. *Let* $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_4(X,8)$. *Then*

$$|R^*(Q_2)| = |R(D'_1)|+|R(D'_2)|+|R(D'_3)|-|R(D'_4)|.$$

Proof. Assume that $D'=\{Z,Z'\}\in Q_2\theta_{\mathrm{XI}}$, then $Z,Z'\in D$ and $Z\subset Z'$. If $\alpha\in R(D')$, then the quasinormal representation of a binary relation α has the form $\alpha=(Y_T^\alpha\times T)\cup(Y_{T'}^\alpha\times T')$, where $T,T'\in D$, $T\subset T'$, $Y_T^\alpha, Y_{T'}^\alpha\notin\{\varnothing\}$, and, by the proposition (b) of Theorem 4.1 (see [11]), satisfies the conditions $Y_T^\alpha\supseteq Z$ and $Y_{T'}^\alpha\cap Z'\neq\varnothing$. Since Z_7, Z_5 and Z_2 are minimal elements of the semilattice D, we have $Z\supseteq Z_7$ or $Z\supseteq Z_5$ or $Z\supseteq Z_2$.

On the other hand, $\breve{D}$ is a maximal element of the semilattice D and therefore $Z'\subseteq\breve{D}$. Hence, in the considered case, only one of the following three conditions is fulfilled:

$$Y_T^\alpha\supseteq Z_7 \text{ and } Y_{T'}^\alpha\cap\breve{D}\neq\varnothing$$

or

$$Y_T^\alpha\supseteq Z_5 \text{ and } Y_{T'}^\alpha\cap\breve{D}\neq\varnothing$$

or

$$Y_T^\alpha\supseteq Z_2 \text{ and } \mathrm{Y}_{\mathrm{T}'}^\alpha\cap\breve{D}\neq\varnothing,$$

i.e. $\alpha\in R(D'_1)$ or $\alpha\in R(D'_2)$, or $\alpha\in R(D'_3)$. Hence, using equality (2), we obtain

$$R^*(Q_2) = R(D'_1)\cup R(D'_2)\cup R(D'_3). \tag{3}$$

Now assume that $\alpha \in R(D_1') \cap R(D_2')$, then

$$\begin{aligned} &Y_T^\alpha \supseteq Z_7, \quad Y_{T'}^\alpha \cap \breve{D} \neq \varnothing,\\ &Y_T^\alpha \supseteq Z_5, \quad Y_{T'}^\alpha \cap \breve{D} \neq \varnothing. \end{aligned} \tag{4}$$

The last conditions imply that $Y_T^\alpha \supseteq Z_7 \cup Z_5 = Z_1$, $\breve{D} \cap Y_{T'}^\alpha \neq \varnothing$, i.e. $\alpha \in R(D_4')$. Therefore the following inclusion is true:

$$R(D_1') \cap R(D_2') \subseteq R(D_4').$$

Now, if we assume that $\alpha \in R(D_4')$, then we get $Y_T^\alpha \supseteq Z_1$ and $\mathrm{Y}_{\mathrm{T}'}^\alpha \cap \breve{D} \neq \varnothing$. These two inclusions imply the validity of the third inclusion, i.e. $\alpha \in R(D_1') \cap R(D_2')$, therefore $R(D_4') \subseteq R(D_1') \cap R(D_2')$. Thus the equality $R(D_1') \cap R(D_2') = R(D_4')$ is valid.

Assume that $\alpha \in R(D_1') \cap R(D_3')$, then

$$\begin{aligned} &Y_T^\alpha \supseteq Z_7, \quad Y_{T'}^\alpha \cap \breve{D} \neq \varnothing,\\ &Y_T^\alpha \supseteq Z_2, \quad Y_{T'}^\alpha \cap \breve{D} \neq \varnothing. \end{aligned} \tag{5}$$

From these conditions we obtain $Y_T^\alpha \supseteq Z_7 \cup Z_2 = \breve{D}$, $\breve{D} \cap Y_{T'}^\alpha \neq \varnothing$, i.e. $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq \breve{D} \cap Y_{T'}^\alpha \neq \varnothing$. The latter inequality contradicts the quasinormality of the representation of the binary relation α. Thus we have $R(D_1') \cap R(D_3') = \varnothing$. $R(D_2') \cap R(D_3') = \varnothing$ is proved analogously. Therefore the following equalities are valid:

$$\begin{aligned} &R(D_1') \cap R(D_2') = R(D_4'),\\ &R(D_1') \cap R(D_3') = \varnothing,\\ &R(D_2') \cap R(D_3') = \varnothing. \end{aligned} \tag{6}$$

From the second and fifth equalities we obtain

$$\begin{aligned} |R^*(Q_2)| = &|R(D_1') \cup R(D_2') \cup R(D_3')| = |R(D_1')| + |R(D_2')| + |R(D_3')| -\\ &- |R(D_1') \cap R(D_2')| - |R(D_1') \cap R(D_3')| - |R(D_2') \cap R(D_3')| +\\ &+ |R(D_1') \cap R(D_2') \cap R(D_3')| =\\ = &|R(D_1')| + |R(D_2')| + |R(D_3')| - |R(D_4')|. \end{aligned}$$

□

Lemma 4. *Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. If X is a finite set, then*

$$|R^*(Q_2)| = 15 \cdot \left(2^{|\breve{D} \setminus Z_7|} + 2^{|\breve{D} \setminus Z_5|} + 2^{|\breve{D} \setminus Z_2|} - 2^{|\breve{D} \setminus Z_1|} - 2\right) \cdot 2^{|X \setminus \breve{D}|}.$$

Proof. By Lemma 3 the following equality is true:

$$\begin{aligned}|R^*(Q_2)| =&|R(D_1')| + |R(D_2')| + |R(D_3')| - |R(D_4')| = \\ &=15 \cdot (2^{|\breve{D}\setminus Z_7|} - 1) \cdot 2^{|X\setminus\breve{D}|} + 15 \cdot (2^{|\breve{D}\setminus Z_5|} - 1) \cdot 2^{|X\setminus\breve{D}|} + \\ &\quad 15 \cdot (2^{|\breve{D}\setminus Z_2|} - 1) \cdot 2^{|X\setminus\breve{D}|} - 15 \cdot (2^{|\breve{D}\setminus Z_1|} - 1) \cdot 2^{|X\setminus\breve{D}|} = \\ &=15 \cdot (2^{|\breve{D}\setminus Z_7|} + 2^{|\breve{D}\setminus Z_5|} + 2^{|\breve{D}\setminus Z_2|} - 2^{|\breve{D}\setminus Z_1|} - 2) \cdot 2^{|X\setminus\breve{D}|}. \quad \square\end{aligned}$$

3. According to the definition of the semilattice D we have

$$\begin{aligned}Q_3\theta_{\mathrm{XI}} =\Big\{&\{Z_7, Z_6, \breve{D}\}, \{Z_7, Z_4, \breve{D}\}, \{Z_7, Z_3, \breve{D}\}, \{Z_7, Z_1, \breve{D}\}, \\ &\{Z_5, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_3\}, \{Z_7, Z_4, Z_1\}, \{Z_7, Z_6, Z_1\}, \\ &\{Z_6, Z_1, \breve{D}\}, \{Z_6, Z_3, \breve{D}\}, \{Z_4, Z_1, \breve{D}\}\Big\}.\end{aligned}$$

Now, if

$$\begin{gathered}D_1' = \{Z_7, Z_6, \breve{D}\},\ \ D_2' = \{Z_7, Z_4, \breve{D}\},\ \ D_3' = \{Z_7, Z_3, \breve{D}\}, \\ D_4' = \{Z_7, Z_1, \breve{D}\},\ \ D_5' = \{Z_5, Z_1, \breve{D}\},\ \ D_6' = \{Z_7, Z_6, Z_3\}, \\ D_7' = \{Z_7, Z_4, Z_1\},\ \ D_8' = \{Z_7, Z_6, Z_1\},\ \ D_9' = \{Z_6, Z_1, \breve{D}\}, \\ D_{10}' = \{Z_6, Z_3, \breve{D}\},\ \ D_{11}' = \{Z_4, Z_1, \breve{D}\},\end{gathered}$$

then by Theorem 6.3.6 (see [4] or [5]) we obtain

$$R^*(Q_3) = \bigcup_{i=1}^{11} R(D_i'). \tag{7}$$

Lemma 5. *Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. Then*

$$\begin{aligned}|R^*(Q_3)| = &|R(D_1')| + |R(D_2')| + |R(D_3')| + |R(D_4')| + |R(D_5')| - \\ &- |R(D_1') \cap R(D_3')| - |R(D_1') \cap R(D_4')| - |R(D_2') \cap R(D_4')|.\end{aligned}$$

Proof. Assume that $D' = \{Y, Y', Y''\}$ is any element of the set $Q_3\theta_{\mathrm{XI}}$ and $\alpha \in R(D')$. Then a quasinormal representation of the binary relation α has the form $\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'')$ for some $T, T', T'' \in D$, $T \subset T' \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$ and, by the proposition (c) of Theorem 1, satisfies the following conditions: $Y_T^\alpha \supseteq Y$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y'$, $Y_{T'}^\alpha \cap Y' \neq \varnothing$, $Y_{T''}^\alpha \cap Y'' \neq \varnothing$. According to the definition of semilattices of the class $\Sigma_4(X, 8)$ we have $Y \supseteq Z_7$ or $Y \supseteq Z_5$ and $Y'' \subseteq \breve{D}$. Hence the conditions $Y_T^\alpha \supseteq Y$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y'$, $Y_{T'}^\alpha \cap Y' \neq \varnothing$, $Y_{T''}^\alpha \cap Y'' \neq \varnothing$ imply

$$Y_T^\alpha \supseteq Z_7, \quad Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y', \quad Y_{T'}^\alpha \cap Y' \neq \varnothing, \quad Y_{T''}^\alpha \cap \breve{D} \neq \varnothing$$

or

$$Y_T^\alpha \supseteq Z_5, \quad Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y', \quad Y_{T'}^\alpha \cap Y' \neq \varnothing, \quad Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$

i.e., $\alpha \in R(\{Z_7, Y', \breve{D}\})$ or $\alpha \in R(\{Z_5, Y', \breve{D}\})$. Taking these conditions into account, from equality (7) we obtain

$$R^*(Q_3) = R(D_1') \cup R(D_2') \cup R(D_3') \cup R(D_4') \cup R(D_5'). \tag{8}$$

Now, assume that $D' = \{Y, Y', Y''\}$ and $D'' = \{Y_1, Y_1', Y_1''\}$ are any elements of the set $\{D_1', D_2', D_3', D_4', D_5'\}$, $D' \neq D''$ and $\alpha \in R(D') \cap R(D'')$. Then, concurrently, the following conditions take place:

$$Y_T^\alpha \supseteq Y, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y', \ \ Y_{T'}^\alpha \cap Y' \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Y_1, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1', \ \ Y_{T'}^\alpha \cap Y_1' \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$

whence we obtain $Y_T^\alpha \supseteq Y \cup Y_1$, $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y' \cup Y_1'$.

Let us consider the following cases:

(1) $D' \in \{D_1', D_2', D_3', D_4', D_5'\}$ and $D'' = D_5'$. Then $Y_T^\alpha \supseteq Y \cup Y_1 = Z_7 \cup Z_5 = Z_1$. By assumption, $D'' = D_5'$ and therefore

$$Y_T^\alpha \supseteq Z_5, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_1, \ \ Y_{T'}^\alpha \cap Z_1 \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing.$$

Thus $Y_T^\alpha \cap Y_{T'}^\alpha \supseteq Z_1 \cap Y_{T'}^\alpha \neq \varnothing$, which contradicts the condition of quasinormality of the binary relation α. The obtained contradiction shows that $R(D') \cap R(D_5') = \varnothing$. Therefore

$$\begin{aligned} R(D_1') \cap R(D_5') = \varnothing, \ \ R(D_2') \cap R(D_5') = \varnothing, \\ R(D_3') \cap R(D_5') = \varnothing, \ \ R(D_4') \cap R(D_5') = \varnothing. \end{aligned} \tag{9}$$

(2) Now let $\alpha \in R(D_1') \cap R(D_2'')$, then according to the proposition (c) of Theorem 1

$$Y_T^\alpha \supseteq Z_7, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_6, \ \ Y_{T'}^\alpha \cap Z_6 \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_7, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4, \ \ Y_{T'}^\alpha \cap Z_4 \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$

i.e. the latter conditions are fulfilled if and only if

$$\begin{gathered} Y_T^\alpha \supseteq Z_7, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_1, \ \ Y_{T'}^\alpha \cap Z_6 \neq \varnothing, \\ Y_{T'}^\alpha \cap Z_4 \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing. \end{gathered} \tag{10}$$

On the other hand, if $\alpha \in R(D_1') \cap R(D_2') \cap R(D_4'')$, then

$$Y_T^\alpha \supseteq Z_7, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_6, \ \ Y_{T'}^\alpha \cap Z_6 \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_7, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4, \ \ Y_{T'}^\alpha \cap Z_4 \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,$$
$$Y_T^\alpha \supseteq Z_7, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_1, \ \ Y_{T'}^\alpha \cap Z_1 \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing.$$

i.e. the latter conditions are fulfilled if and only if

$$\begin{gathered} Y_T^\alpha \supseteq Z_7, \ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_1, \ \ Y_{T'}^\alpha \cap Z_6 \neq \varnothing, \\ Y_{T'}^\alpha \cap Z_4 \neq \varnothing, \ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing. \end{gathered} \tag{11}$$

From conditions (10) and (11) it follows that

$$R(D_1') \cap R(D_2'') = R(D_1') \cap R(D_2') \cap R(D_4'').$$

(3) Now let $\alpha \in R(D_2') \cap R(D_3'')$, then according to the proposition (c) of Theorem 1

$$\begin{aligned} &Y_T^\alpha \supseteq Z_7,\ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4,\ \ Y_{T'}^\alpha \cap Z_4 \neq \varnothing,\ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,\\ &Y_T^\alpha \supseteq Z_7,\ \ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_3,\ \ Y_{T'}^\alpha \cap Z_3 \neq \varnothing,\ \ Y_{T''}^\alpha \cap \breve{D} \neq \varnothing,\end{aligned}$$

i.e. $Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4 \cup Z_3 = \breve{D}$, $(Y_T^\alpha \cup Y_{T'}^\alpha) \cap Y_{T''}^\alpha \supseteq \breve{D} \cap Y_{T''}^\alpha \neq \varnothing$. The latter inequality contradicts the condition of quasinormality of the binary relation α. Therefore $R(D_2') \cap R(D_3'') = \varnothing$.

In an analogous manner we prove that $R(D_3') \cap R(D_4'') = \varnothing$. Thus the following equalities are valid:

$$\begin{gathered} R(D_1') \cap R(D_2'') = R(D_1') \cap R(D_2') \cap R(D_4''), \qquad (12)\\ R(D_1') \cap R(D_5') = \varnothing,\ \ R(D_2') \cap R(D_5') = \varnothing,\ \ R(D_3') \cap R(D_5') = \varnothing,\\ R(D_4') \cap R(D_5') = \varnothing,\ \ R(D_2') \cap R(D_3'') = \varnothing,\ \ R(D_3') \cap R(D_4'') = \varnothing.\end{gathered}$$

Using equalities (8) and (12) we obtain

$$\begin{aligned} |R^*(Q_3)| =& |R(D_1') \cup R(D_2') \cup R(D_3') \cup R(D_4') \cup R(D_5')| =\\ =& |R(D_1')| + |R(D_2')| + |R(D_3')| + |R(D_4')| + |R(D_5')| -\\ &- |R(D_1') \cap R(D_3')| - |R(D_1') \cap R(D_4')| - |R(D_2') \cap R(D_4')|. \ \square\end{aligned}$$

Lemma 6. *Let $D' = \{Z_7, Y', \breve{D}\}$ and $D'' = \{Z_7, Y_1', \breve{D}\}$, where $Y_1' \supseteq Y'$. If a quasinormal representation of the binary relation α of the semigroup $B_X(D)$ has the form $\alpha = (Y_7^\alpha \times Z_7) \cup (Y_T^\alpha \times T) \cup (Y_0^\alpha \times \breve{D})$ for some $T \in D$, $Z_7 \subset T \subset \breve{D}$ and $Y_7^\alpha, Y_T^\alpha, Y_0^\alpha \notin \{\varnothing\}$, then $\alpha \in R(D') \cap R(D'')$ if and only if*

$$Y_7^\alpha \supseteq Z_7,\ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Y_1',\ \ Y_T^\alpha \cap Y' \neq \varnothing,\ \ Y_0^\alpha \cap \breve{D} \neq \varnothing.$$

Proof. If $\alpha \in R(D') \cap R(D'')$, then

$$\begin{aligned} &Y_7^\alpha \supseteq Z_7,\ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Y',\ \ Y_T^\alpha \cap Y' \neq \varnothing,\ \ Y_0^\alpha \cap \breve{D} \neq \varnothing,\\ &Y_7^\alpha \supseteq Z_7,\ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Y_1',\ \ Y_T^\alpha \cap Y_1' \neq \varnothing,\ \ Y_0^\alpha \cap \breve{D} \neq \varnothing.\end{aligned} \qquad (13)$$

These conditions imply

$$Y_7^\alpha \supseteq Z_7,\ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Y_1',\ \ Y_T^\alpha \cap Y' \neq \varnothing,\ \ Y_0^\alpha \cap \breve{D} \neq \varnothing, \qquad (14)$$

because the inclusion $Y_1' \supseteq Y'$ is valid by assumption.

On the other hand, if conditions (14) are fulfilled, then conditions (13) are valid, too, i.e. the condition $\alpha \in R(D') \cap R(D'')$, too, is fulfilled. $\square$

Lemma 7. *Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_4(X,8)$. If X is a finite set, then the following equalities are valid:*

$$\left|R(D_1')\cap R(D_3')\right|=11\cdot 2^{|Z_3\setminus Z_6|}\cdot(2^{|Z_6\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|})\cdot 3^{|X\setminus\breve{D}|},$$

$$\left|R(D_1')\cap R(D_4')\right|=11\cdot 2^{|Z_1\setminus Z_6|}\cdot(2^{|Z_6\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|},$$

$$\left|R(D_2')\cap R(D_4')\right|=11\cdot 2^{|Z_1\setminus Z_4|}\cdot(2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|}.$$

Proof. Let $D'=\{Z_7,Y',\breve{D}\}$ and $D''=\{Z_7,Y_1',\breve{D}\}$, where $D'\neq D''$ and $Y_1'\supseteq Y'$. Assume that $\alpha\in R(D')\cap R(D'')$ and the quasinormal representation of the binary relation α of the semilattice $B_X(D)$ has the form $\alpha=(Y_7^\alpha\times Z_7)\cup(Y_T^\alpha\times T)\cup(Y_0^\alpha\times\breve{D})$ for some $T\in D$, $Z_7\subset T\subset\breve{D}$ and $Y_7^\alpha,Y_T^\alpha,Y_0^\alpha\notin\{\varnothing\}$. Then, using Lemma 6, we obtain

$$Y_7^\alpha\supseteq Z_7,\ \ Y_7^\alpha\cup Y_T^\alpha\supseteq Y_1',\ \ Y_T^\alpha\cap Y'\neq\varnothing,\ \ Y_0^\alpha\cap\breve{D}\neq\varnothing. \tag{15}$$

Now assume that f_α is such a mapping of the set X in the semilattice D that $f_\alpha(t)=t\alpha$ for any $t\in X$. Assume that $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ are the restrictions of the mapping f_α on the sets Z_7, $Y_1'\setminus Z_7$, $\breve{D}\setminus Y_1'$ and $X\setminus\breve{D}$, respectively. It is clear that the elements of the set $\{Z_7,Y_1'\setminus Z_7,\breve{D}\setminus Y_1',X\setminus\breve{D}\}$ do not intersect pairwise and their union is equal to the set X.

Now let us consider separately each of the mappings $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$.

(1) If $t\in Z_7$, then by conditions (15) we have $t\in Y_7^\alpha$. Hence, by the definition of the set Y_7^α, we obtain $t\alpha=Z_7$. Thus $f_{0\alpha}(t)=Z_7$ for every $t\in Z_7$.

(2) If $t\in Y_1'\setminus Z_7$, then by conditions (15) we have

$$t\in Y_1'\setminus Z_7\subseteq Y_1'\subseteq Y_7^\alpha\cup Y_T^\alpha.$$

From this, according to the definition of the sets Y_7^α and Y_T^α, we obtain $t\alpha\in\{Z_7,T\}$, i.e. $f_{1\alpha}(t)\in\{Z_7,T\}$ for any $t\in Y_1'\setminus Z_7$.

By assumption, $Y_T^\alpha\cap Y'\neq\varnothing$. Therefore $t_1\alpha=T$ for some $t_1\in Y'$. If $t_1\in Z_7$, then by conditions (15) we have $t_1\alpha=Z_7$. The latter condition contradicts the equality $t_1\alpha=T$ since $T\neq Z_7$. The obtained contradiction shows that $t_1\in Y'\setminus Z_7$.

Thus $f_{1\alpha}(t_1)=T$ for some $t_1\in Y'\setminus Z_7$.

(3) If $t\in\breve{D}\setminus Y_1'$, then by conditions (15) we obtain

$$t\in\breve{D}\setminus Y_1'\subseteq\breve{D}\subseteq X=Y_7^\alpha\cup Y_T^\alpha\cup Y_0^\alpha,$$

whence, according to the definition of the sets Y_7^α, Y_T^α and Y_0^α we get $t\alpha\in\{Z_7,T,\breve{D}\}$, i.e., $f_{2\alpha}(t)\in\{Z_7,T,\breve{D}\}$ for any $t\in\breve{D}\setminus Y_1'$.

By assumption, $Y_0^\alpha\cap\breve{D}\neq\varnothing$. Therefore $t_0\alpha=\breve{D}$ for some $t_0\in\breve{D}$. If $t_0\in Y_1'$, then, using conditions (15) we have $t_0\alpha\in\{Z_7,T\}$. The latter condition contradicts the equality $t_0\alpha=\breve{D}$ since $\breve{D}\notin\{Z_7,T\}$. The obtained contradiction shows that $t_0\in\breve{D}\setminus Y_1'$.

Thus $f_{2\alpha}(t_0) = \breve{D}$ for some $t_0 \in \breve{D} \setminus Y_1'$.

(4) If $t \in X \setminus \breve{D}$, then by conditions (15) we have

$$t \in X \setminus \breve{D} \subseteq X = Y_7^\alpha \cup Y_T^\alpha \cup Y_0^\alpha,$$

which, according to the definition of the sets Y_7^α, Y_T^α and Y_0^α, implies $t\alpha \in \{Z_7, T, \breve{D}\}$, i.e., $f_{3\alpha}(t) \in \{Z_7, T, \breve{D}\}$ for any $t \in X \setminus \breve{D}$.

Thus, for every $\alpha \in R(D') \cap R(D'')$ there exists a system of mappings $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$. Also, to different binary relations there correspond different systems.

Now assume that

$$f_0 : Z_7 \to \{Z_7\}, \quad f_1 : Y_1' \setminus Z_7 \to \{Z_7, T\},$$
$$f_2 : \breve{D} \setminus Y_1' \to \{Z_7, T, \breve{D}\}, \quad f_3 : X \setminus \breve{D} \to \{Z_7, T, \breve{D}\}$$

are the mappings satisfying the following conditions:

(5) $f_0(t) = Z_7$ for every $t \in Z_7$;

(6) $f_1(t) \in \{Z_7, T\}$ for every $t \in Y_1' \setminus Z_7$ and $f_{1\alpha}(t_1) = T$ for some $t_1 \in Y' \setminus Z_7$;

(7) $f_2(t) \in \{Z_7, T, \breve{D}\}$ for every $t \in \breve{D} \setminus Y_1'$ and $f_{2\alpha}(t_0) = \breve{D}$ for some $t_0 \in \breve{D} \setminus Y_1'$;

(8) $f_3(t) \in \{Z_7, T, \breve{D}\}$ for every $t \in X \setminus \breve{D}$.

Now let us define the mapping $f : X \to D$ as follows:

$$f(t) = \begin{cases} f_0(t), & \text{if } t \in Z_7, \\ f_1(t), & \text{if } t \in Y_1' \setminus Z_7, \\ f_2(t), & \text{if } t \in \breve{D} \setminus Y_1', \\ f_3(t), & \text{if } t \in X \setminus \breve{D}. \end{cases}$$

To this mapping we put into correspondence the binary relation $\beta = \bigcup\limits_{x \in X} (\{x\} \times f(x))$ of the semigroup $B_X(D)$.

Now, if $Y_7^\beta = \{t | \ t\beta = Z_7\}$, $Y_T^\beta = \{t | \ t\beta = T\}$, $Y_0^\beta = \{t | \ t\beta = \breve{D}\}$, then we write the relation β in the form

$$\beta = (Y_7^\beta \times Z_7) \cup (Y_T^\beta \times T) \cup (Y_0^\beta \times \breve{D})$$

and it satisfies the conditions

$$Y_7^\beta \supseteq Z_7, \;\; Y_7^\beta \cup Y_T^\beta \supseteq Y_1', \;\; Y_T^\beta \cap Y' \neq \varnothing, \;\; Y_0^\beta \cap \breve{D} \neq \varnothing.$$

(By assumption, $f_1(t_1) = T$ for some $t_1 \in Y' \setminus Z_7$ and $f_2(t_0) = \breve{D}$ for some $t_0 \in \breve{D} \setminus Y_1'$.) Therefore, by virtue of Lemma 6, $\beta \in R(D') \cap R(D'')$.

Let us assume that there exists a one-to-one correspondence between the system of binary relations α taken from $R(D') \cap R(D'')$ and the system of mappings $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$.

According to Theorem 1.18.2 (see [4] or [5]), the number of systems of mappings is equal to

$$1,\ 2^{|(Y_1'\setminus Z_7)\setminus(Y'\setminus Z_7)|}\cdot(2^{|Y'\setminus Z_7|}-1),\ 3^{|\breve{D}\setminus Y_1'|}-2^{|\breve{D}\setminus Y_1'|},\ 3^{|X\setminus\breve{D}|}.$$

Note that the number

$$2^{|Y_1'\setminus Y'|}\cdot(2^{|Y'\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Y_1'|}-2^{|\breve{D}\setminus Y_1'|})\cdot 3^{|X\setminus\breve{D}|}$$

does not depend on a choice of chains $T\subset T'\subset T''$ $(T,T',T''\in D)$ and since the number of differing three-element chains is equal to eleven, we have

$$\big|R(D')\cap R(D'')\big| = 11\cdot 2^{|Y_1'\setminus Y'|}\cdot(2^{|Y'\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Y_1'|}-2^{|\breve{D}\setminus Y_1'|})\cdot 3^{|X\setminus\breve{D}|}.$$

Hence we obtain

$$\begin{aligned}
\big|R(D_1')\cap R(D_3')\big| &= 11\cdot 2^{|Z_3\setminus Z_6|}\cdot(2^{|Z_6\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|})\cdot 3^{|X\setminus\breve{D}|},\\
\big|R(D_1')\cap R(D_4')\big| &= 11\cdot 2^{|Z_1\setminus Z_6|}\cdot(2^{|Z_6\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|},\\
\big|R(D_2')\cap R(D_4')\big| &= 11\cdot 2^{|Z_1\setminus Z_4|}\cdot(2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|}.\ \square
\end{aligned}$$

Lemma 8. *Let $D=\{Z_7,Z_6,Z_5,Z_4,Z_3,Z_2,Z_1,\breve{D}\}\in\Sigma_4(X,8)$. If X is a finite set, then*

$$\begin{aligned}
|R^*(Q_3)| = {}& 11\cdot(2^{|Z_6\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_6|}-2^{|\breve{D}\setminus Z_6|})\cdot 3^{|X\setminus\breve{D}|}+\\
&+11\cdot(2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_4|}-2^{|\breve{D}\setminus Z_4|})\cdot 3^{|X\setminus\breve{D}|}+\\
&+11\cdot(2^{|Z_3\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|})\cdot 3^{|X\setminus\breve{D}|}+\\
&+11\cdot(2^{|Z_1\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|}+\\
&+11\cdot(2^{|Z_1\setminus Z_5|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|}-\\
&-11\cdot 2^{|Z_3\setminus Z_6|}\cdot(2^{|Z_6\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_3|}-2^{|\breve{D}\setminus Z_3|})\cdot 3^{|X\setminus\breve{D}|}-\\
&-11\cdot 2^{|Z_1\setminus Z_6|}\cdot(2^{|Z_6\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|}-\\
&-11\cdot 2^{|Z_1\setminus Z_4|}\cdot(2^{|Z_4\setminus Z_7|}-1)\cdot(3^{|\breve{D}\setminus Z_1|}-2^{|\breve{D}\setminus Z_1|})\cdot 3^{|X\setminus\breve{D}|}.
\end{aligned}$$

Proof. The following equality holds by virtue of Lemma 5:

$$\begin{aligned}|R^*(Q_3)| &= |R(D_1')| + |R(D_2')| + |R(D_3')| + |R(D_4')| + |R(D_5')| - \\ &\quad - |R(D_1') \cap R(D_3')| - |R(D_1') \cap R(D_4')| - |R(D_2') \cap R(D_4')| = \\ &= 11 \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot (3^{|\breve{D} \setminus Z_6|} - 2^{|\breve{D} \setminus Z_6|}) \cdot 3^{|X \setminus \breve{D}|} + \\ &\quad + 11 \cdot (2^{|Z_4 \setminus Z_7|} - 1) \cdot (3^{|\breve{D} \setminus Z_4|} - 2^{|\breve{D} \setminus Z_4|}) \cdot 3^{|X \setminus \breve{D}|} + \\ &\quad + 11 \cdot (2^{|Z_3 \setminus Z_7|} - 1) \cdot (3^{|\breve{D} \setminus Z_3|} - 2^{|\breve{D} \setminus Z_3|}) \cdot 3^{|X \setminus \breve{D}|} + \\ &\quad + 11 \cdot (2^{|Z_1 \setminus Z_7|} - 1) \cdot (3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}) \cdot 3^{|X \setminus \breve{D}|} + \\ &\quad + 11 \cdot (2^{|Z_1 \setminus Z_5|} - 1) \cdot (3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}) \cdot 3^{|X \setminus \breve{D}|} - \\ &\quad - 11 \cdot 2^{|Z_3 \setminus Z_6|} \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot (3^{|\breve{D} \setminus Z_3|} - 2^{|\breve{D} \setminus Z_3|}) \cdot 3^{|X \setminus \breve{D}|} - \\ &\quad - 11 \cdot 2^{|Z_1 \setminus Z_6|} \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot (3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}) \cdot 3^{|X \setminus \breve{D}|} - \\ &\quad - 11 \cdot 2^{|Z_1 \setminus Z_4|} \cdot (2^{|Z_4 \setminus Z_7|} - 1) \cdot (3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}) \cdot 3^{|X \setminus \breve{D}|}. \qquad \square\end{aligned}$$

4. By the definition of a semilattice of unions we have

$$Q_4\theta_{\mathrm{XI}} = \Big\{\{Z_7, Z_4, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_1, \breve{D}\}, \{Z_7, Z_6, Z_3, \breve{D}\}\Big\}.$$

Now, if

$$D_1' = \{Z_7, Z_4, Z_1, \breve{D}\}, \;\; D_2' = \{Z_7, Z_6, Z_1, \breve{D}\}, \;\; D_3' = \{Z_7, Z_6, Z_3, \breve{D}\},$$

then by Theorem 6.3.6 (see [4] or [5]) we obtain

$$R^*(Q_4) = R(D_1') \cup R(D_2') \cup R(D_3'). \tag{16}$$

Lemma 9. *Let* $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. *Then*

$$|R^*(Q_4)| = |R(D_1')| + |R(D_2')| + |R(D_3')|.$$

Proof. Assume that $D' = \{Z_7, Y, Z, \breve{D}\}$ and $D'' = \{Z_7, Y_1', Z_1', \breve{D}\}$ are any elements from the set $Q_4\theta_{\mathrm{XI}}$ and $\alpha \in R(D') \cap R(D'')$. Then a quasinormal representation of the binary representation α has the form

$$\alpha = (Y_7^\alpha \times Z_7) \cup (Y_T^\alpha \times T) \cup (Y_Z^\alpha \times Z) \cup (Y_0^\alpha \times \breve{D})$$

for some $T, Z \in D$, $Z_7 \subset T \subset Z \subset \breve{D}$, $Y_7^\alpha, Y_T^\alpha, Y_Z^\alpha, Y_0^\alpha \notin \{\varnothing\}$ and

$$\begin{gathered} Y_7^\alpha \supseteq \varphi(Z_7), \quad Y_7^\alpha \cup Y_T^\alpha \supseteq \varphi(T), \quad Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq \varphi(Z), \\ Y_T^\alpha \cap \varphi(T) \neq \varnothing, \quad Y_Z^\alpha \cap \varphi(Z) \neq \varnothing, \quad Y_0^\alpha \cap \varphi(\breve{D}) \neq \varnothing. \end{gathered}$$

Now, if $\alpha \in R(D_1') \cap R(D_2')$, then, concurrently, the following condition takes place

$$Y_7^\alpha \supseteq Z_7, \ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Z_4, \ \ Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_1,$$
$$Y_T^\alpha \cap Z_4 \neq \varnothing, \ \ Y_Z^\alpha \cap Z_1 \neq \varnothing, \ \ Y_0^\alpha \cap \breve{D} \neq \varnothing;$$
$$Y_T^\alpha \supseteq Z_7, \ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Z_6, \ \ Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_1,$$
$$Y_T^\alpha \cap Z_6 \neq \varnothing, \ \ Y_Z^\alpha \cap Z_1 \neq \varnothing, \ \ Y_0^\alpha \cap \breve{D} \neq \varnothing.$$

whence we obtain

$$Y_7^\alpha \cup Y_T^\alpha \supseteq Z_4 \cup Z_6 = Z_1, \quad (Y_7^\alpha \cup Y_T^\alpha) \cap Y_Z^\alpha \supseteq Z_1 \cap Y_Z^\alpha \neq \varnothing.$$

But the inequality $(Y_7^\alpha \cup Y_T^\alpha) \cap Y_Z^\alpha \neq \varnothing$ contradicts the condition of the quasinormal representation of the binary relation α. The obtained contradiction shows that $R(D_1') \cap R(D_2') = \varnothing$.

Now assume that $\alpha \in R(D_1') \cap R(D_3')$, then, concurrently, the following condition takes place

$$Y_7^\alpha \supseteq Z_7, \ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Z_4, \ \ Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_1,$$
$$Y_T^\alpha \cap Z_4 \neq \varnothing, \ \ Y_Z^\alpha \cap Z_1 \neq \varnothing, \ \ Y_0^\alpha \cap \breve{D} \neq \varnothing;$$
$$Y_T^\alpha \supseteq Z_7, \ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Z_6, \ \ Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_3,$$
$$Y_T^\alpha \cap Z_6 \neq \varnothing, \ \ Y_Z^\alpha \cap Z_3 \neq \varnothing, \ \ Y_0^\alpha \cap \breve{D} \neq \varnothing.$$

whence we obtain

$$Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_1 \cup Z_3 = \breve{D}, \quad (Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha) \cap Y_0^\alpha \supseteq \breve{D} \cap Y_0^\alpha \neq \varnothing.$$

But the inequality $(Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha) \cap Y_0^\alpha \neq \varnothing$ contradicts the condition of quasinormal representation of the binary relation α. The obtained contradiction shows that $R(D_1') \cap R(D_3') = \varnothing$.

Finally, assume that $\alpha \in R(D_2') \cap R(D_3')$, then, concurrently, the following condition takes place

$$Y_7^\alpha \supseteq Z_7, \ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Z_6, \ \ Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_1,$$
$$Y_T^\alpha \cap Z_6 \neq \varnothing, \ \ Y_Z^\alpha \cap Z_1 \neq \varnothing, \ \ Y_0^\alpha \cap \breve{D} \neq \varnothing;$$
$$Y_T^\alpha \supseteq Z_7, \ \ Y_7^\alpha \cup Y_T^\alpha \supseteq Z_6, \ \ Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_3,$$
$$Y_T^\alpha \cap Z_6 \neq \varnothing, \ \ Y_Z^\alpha \cap Z_3 \neq \varnothing, \ \ Y_0^\alpha \cap \breve{D} \neq \varnothing,$$

from which we obtain

$$Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha \supseteq Z_1 \cup Z_3 = \breve{D}, \quad (Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha) \cap Y_0^\alpha \supseteq \breve{D} \cap Y_0^\alpha \neq \varnothing.$$

But the inequality $(Y_7^\alpha \cup Y_T^\alpha \cup Y_Z^\alpha) \cap Y_0^\alpha \neq \varnothing$ contradicts the condition of quasinormal representation of the binary relation α. The obtained contradiction shows that $R(D_2') \cap R(D_3') = \varnothing$.

Now let us assume that

$$R(D_1') \cap R(D_2') = \varnothing, \ \ R(D_1') \cap R(D_3') = \varnothing, \ \ R(D_2') \cap R(D_3') = \varnothing. \tag{17}$$

Using equalities (16) and (17), we obtain

$$|R^*(Q_4)| = |R(D_1')| + |R(D_2')| + |R(D_3')|. \qquad \square$$

Lemma 10. *Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. If X is a finite set, then*

$$\begin{aligned}|R^*(Q_4)| = & 3 \cdot (2^{|Z_4 \setminus Z_7|} - 1) \cdot (3^{|Z_1 \setminus Z_4|} - 2^{|Z_1 \setminus Z_4|}) \cdot (3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}) \cdot 4^{|X \setminus \breve{D}|} + \\ & + 3 \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot (3^{|Z_1 \setminus Z_6|} - 2^{|Z_1 \setminus Z_6|}) \cdot (3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}) \cdot 4^{|X \setminus \breve{D}|} + \\ & + 3 \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot (3^{|Z_3 \setminus Z_6|} - 2^{|Z_3 \setminus Z_6|}) \cdot (3^{|\breve{D} \setminus Z_3|} - 2^{|\breve{D} \setminus Z_3|}) \cdot 4^{|X \setminus \breve{D}|}.\end{aligned}$$

Proof. By virtue of Lemma 10 we have the equality

$$|R^*(Q_4)| = |R(D_1')| + |R(D_2')| + |R(D_3')|.$$

Using this inequality and performing some transformation we obtain

$$\begin{aligned}|R^*(Q_4)| &= |R(D_1')| + |R(D_2')| + |R(D_3')| = \\ &= 3 \cdot (2^{|Z_4 \setminus Z_7|} - 1) \cdot (3^{|Z_1 \setminus Z_4|} - 2^{|Z_1 \setminus Z_4|}) \cdot (3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}) \cdot 4^{|X \setminus \breve{D}|} + \\ &+ 3 \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot (3^{|Z_1 \setminus Z_6|} - 2^{|Z_1 \setminus Z_6|}) \cdot (3^{|\breve{D} \setminus Z_1|} - 2^{|\breve{D} \setminus Z_1|}) \cdot 4^{|X \setminus \breve{D}|} + \\ &+ 3 \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot (3^{|Z_3 \setminus Z_6|} - 2^{|Z_3 \setminus Z_6|}) \cdot (3^{|\breve{D} \setminus Z_3|} - 2^{|\breve{D} \setminus Z_3|}) \cdot 4^{|X \setminus \breve{D}|}. \qquad \square\end{aligned}$$

5. By the definition of a semilattice of unions we have

$$Q_5 \theta_{\mathrm{XI}} = \Big\{ \{Z_7, Z_6, Z_4, Z_1\}, \{Z_7, Z_4, Z_3, \breve{D}\}, \{Z_6, Z_3, Z_1, \breve{D}\}, \\ \{Z_7, Z_3, Z_1, \breve{D}\} \Big\}.$$

Now if

$$D_1' = \{Z_7, Z_6, Z_4, Z_1\}, \quad D_2' = \{Z_7, Z_4, Z_6, Z_1\}, \quad D_3' = \{Z_7, Z_4, Z_3, \breve{D}\},$$
$$D_4' = \{Z_7, Z_3, Z_4, \breve{D}\}, \quad D_5' = \{Z_7, Z_3, Z_1, \breve{D}\}, \quad D_6' = \{Z_7, Z_1, Z_3, \breve{D}\},$$
$$D_7' = \{Z_6, Z_3, Z_1, \breve{D}\}, \quad D_8' = \{Z_6, Z_1, Z_3, \breve{D}\},$$

then by Theorem 6.3.6 (see [4] or [5]) we obtain

$$R^*(Q_5) = \bigcup_{i=1}^{8} R(D_i'). \tag{18}$$

Lemma 11. *Let $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. Then the following equality is valid:*

$$\begin{aligned}|R^*(Q_5)| = & |R(D_1')| + |R(D_2')| + |R(D_3')| + \\ & + |R(D_4')| + |R(D_5')| + |R(D_6')| - \big|R(D_1') \cap R(D_4')\big| - \\ & - \big|R(D_2') \cap R(D_3')\big| - \big|R(D_3') \cap R(D_6')\big| - \big|R(D_4') \cap R(D_5')\big|.\end{aligned}$$

Proof. Assume that $D' = \{Z, Z', Z'', Z' \cup Z''\}$ is any element of the set $Q_5\theta_{\mathrm{XI}}$ and $\alpha \in R(D')$. Then the quasinormal representation of the binary relation α of the semigroup $B_X(D)$ has the form

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T'')),$$

where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $T' \setminus T'' \neq \varnothing$, $T'' \setminus T' \neq \varnothing$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$, and, by the proposition (e) of Theorem 1 we have

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z', \;\; Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z'', \;\; Y_{T'}^\alpha \cap Z' \neq \varnothing, \;\; Y_{T''}^\alpha \cap Z'' \neq \varnothing.$$

Therefore the following inclusions are true:

$$R(D_7') \subseteq R(D_5'), \quad R(D_8') \subseteq R(D_6'). \tag{19}$$

Using equalities (18) and (19), we get that

$$R^*(Q_5) = R(D_1') \cup R(D_2') \cup R(D_3') \cup R(D_4') \cup R(D_5') \cup R(D_6'). \tag{20}$$

Further, we show the validity of the following equalities:

$$\begin{gathered} R(D_1') \cap R(D_2') = \varnothing, \;\; R(D_1') \cap R(D_3') = \varnothing, \;\; R(D_1') \cap R(D_5') = \varnothing, \\ R(D_1') \cap R(D_6') = \varnothing, \;\; R(D_2') \cap R(D_4') = \varnothing, \;\; R(D_2') \cap R(D_5') = \varnothing, \\ R(D_2') \cap R(D_6') = \varnothing, \;\; R(D_3') \cap R(D_4') = \varnothing, \;\; R(D_3') \cap R(D_5') = \varnothing, \\ R(D_4') \cap R(D_6') = \varnothing, \;\; R(D_5') \cap R(D_6') = \varnothing. \end{gathered} \tag{21}$$

Indeed, assume that $D' = \{Z_7, Y, Y', Y \cup Y'\}$ and $D'' = \{Z_7, Y_1, Y_1', Y_1 \cup Y_1'\}$ are such elements of the set $\{D_1', D_2', D_3', D_4', D_5', D_6'\}$ that $D' \neq D''$ and $\alpha \in R(D') \cap R(D'')$. Then the following inclusions and inequalities are valid:

$$\begin{gathered} Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y, \;\; Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y', \;\; Y_{T'}^\alpha \cap Y \neq \varnothing, \;\; Y_{T''}^\alpha \cap Y' \neq \varnothing, \\ Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1, \;\; Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_1', \;\; Y_{T'}^\alpha \cap Y_1 \neq \varnothing, \;\; Y_{T''}^\alpha \cap Y_1' \neq \varnothing. \end{gathered}$$

Using the latter conditions we obtain

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y \cup Y_1, \;\; Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y' \cup Y_1'.$$

Now, for the semilattices D' and D'' lets us consider the following cases:

1) $Y \cup Y_1 = \breve{D}$. Then $(Y_T^\alpha \cup Y_{T'}^\alpha) \cap Y_{T''}^\alpha \supseteq \breve{D} \cap Y_{T''}^\alpha \supseteq Y' \cap Y_{T''}^\alpha \neq \varnothing$. But this inequality contradicts the quasinormality of the representation of the binary relation α. Therefore $R(D') \cap R(D'') = \varnothing$. From this and the definition of the given semilattice it follows that the following equalities are valid:

$$\begin{gathered} R(D_2') \cap R(D_4') = \varnothing, \;\; R(D_2') \cap R(D_5') = \varnothing, \;\; R(D_3') \cap R(D_4') = \varnothing, \\ R(D_3') \cap R(D_5') = \varnothing, \;\; R(D_4') \cap R(D_6') = \varnothing, \;\; R(D_5') \cap R(D_6') = \varnothing. \end{gathered}$$

2) $Y_1 \cup Y_1' = \breve{D}$. Then $(Y_T^\alpha \cup Y_{T''}^\alpha) \cap Y_{T'}^\alpha \supseteq \breve{D} \cap Y_{T'}^\alpha \supseteq Y \cap Y_{T'}^\alpha \neq \varnothing$. But this inequality contradicts the quasinormality of the representation of the binary relation α. Therefore $R(D') \cap R(D'') = \varnothing$. From this and the

definition of the given semilattice it follows that the following equalities are valid:

$$R(D_1') \cap R(D_3') = \varnothing, \quad R(D_1') \cap R(D_6') = \varnothing, \quad R(D_3') \cap R(D_4') = \varnothing,$$
$$R(D_3') \cap R(D_5') = \varnothing, \quad R(D_4') \cap R(D_6') = \varnothing, \quad R(D_5') \cap R(D_6') = \varnothing.$$

3) $\alpha \in R(D_1') \cap R(D_2')$. Then the following inclusions and inequalities are true:

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_6, \quad Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_4, \quad Y_{T'}^\alpha \cap Z_6 \neq \varnothing, \quad Y_{T''}^\alpha \cap Z_4 \neq \varnothing,$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4, \quad Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_6, \quad Y_{T'}^\alpha \cap Z_4 \neq \varnothing, \quad Y_{T''}^\alpha \cap Z_6 \neq \varnothing.$$

Hence we obtain $(Y_T^\alpha \cup Y_{T''}^\alpha) \cap Y_{T'}^\alpha \supseteq Z_1 \cap Y_{T'}^\alpha \supseteq Z_4 \cap Y_{T'}^\alpha \neq \varnothing$. But this inequality contradicts the quasinormality of the representation of the binary relation α. Therefore $R(D_1') \cap R(D_2') = \varnothing$.

4) $\alpha \in R(D_1') \cap R(D_5')$. In this case, the following inclusions and inequalities are valid:

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_6, \quad Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_4, \quad Y_{T'}^\alpha \cap Z_6 \neq \varnothing, \quad Y_{T''}^\alpha \cap Z_4 \neq \varnothing,$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_3, \quad Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_1, \quad Y_{T'}^\alpha \cap Z_3 \neq \varnothing, \quad Y_{T''}^\alpha \cap Z_1 \neq \varnothing.$$

Hence we obtain $(Y_T^\alpha \cup Y_{T''}^\alpha) \cap Y_{T'}^\alpha \supseteq Z_3 \cap Y_{T'}^\alpha \supseteq Z_6 \cap Y_{T'}^\alpha \neq \varnothing$. But this inequality contradicts the quasinormality of the representation of the binary relation α. Therefore $R(D_1') \cap R(D_5') = \varnothing$.

5) $\alpha \in R(D_2') \cap R(D_6')$. In this case, the following inclusions and inequalities are valid:

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_4, \quad Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_6, \quad Y_{T'}^\alpha \cap Z_4 \neq \varnothing, \quad Y_{T''}^\alpha \cap Z_6 \neq \varnothing,$$
$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Z_1, \quad Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Z_3, \quad Y_{T'}^\alpha \cap Z_1 \neq \varnothing, \quad Y_{T''}^\alpha \cap Z_3 \neq \varnothing.$$

Hence we obtain $(Y_T^\alpha \cup Y_{T'}^\alpha) \cap Y_{T''}^\alpha \supseteq Z_1 \cap Y_{T''}^\alpha \supseteq Z_6 \cap Y_{T''}^\alpha \neq \varnothing$. But this inequality contradicts the quasinormality of the representation of the binary relation α. Therefore $R(D_2') \cap R(D_6') = \varnothing$.

Now, using equalities (20) and (21) we obtain

$$\begin{aligned} |R^*(Q_5)| &= |R(D_1')| + |R(D_2')| + |R(D_3')| + \\ &\quad + |R(D_4')| + |R(D_5')| + |R(D_6')| - \big|R(D_1') \cap R(D_4')\big| - \\ &\quad - \big|R(D_2') \cap R(D_3')\big| - \big|R(D_3') \cap R(D_6')\big| - \big|R(D_4') \cap R(D_5')\big|. \ \square \end{aligned}$$

Lemma 12. *Assume that $D' = \{Z_7, Y, Y', Y \cup Y'\}$ and $D'' = \{Z_7, Y_1, Y_1', Y_1 \cup Y_1'\}$ are elements of the set $\{D_1', D_2', D_3', D_4', D_5', D_6'\}$ such that $D' \neq D''$, $Y_1 \supseteq Y$, $Y_1' \supseteq Y'$ and a quasinormal representation of the binary relation α of the semigroup $B_X(D)$ has the form*

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T'')),$$

where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$. *Then* $\alpha \in R(D') \cap R(D'')$ *if and only if*

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1,\ \ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_1',\ \ Y_{T'}^\alpha \cap Y \neq \varnothing,\ \ Y_{T''}^\alpha \cap Y' \neq \varnothing.$$

Proof. Assume that $\alpha \in R(D') \cap R(D'')$. Then by the proposition (e) of Theorem 1 we have

$$\begin{aligned} &Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y,\ \ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y',\ \ Y_{T'}^\alpha \cap Y \neq \varnothing,\ \ Y_{T''}^\alpha \cap Y' \neq \varnothing;\\ &Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1,\ \ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_1',\ \ Y_{T'}^\alpha \cap Y_1 \neq \varnothing,\ \ Y_{T''}^\alpha \cap Y_1' \neq \varnothing. \end{aligned} \tag{22}$$

Using these conditions and the inclusions $Y_1 \supseteq Y$, $Y_1' \supseteq Y'$, we obtain

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1,\ \ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_1',\ \ Y_{T'}^\alpha \cap Y \neq \varnothing,\ \ Y_{T''}^\alpha \cap Y' \neq \varnothing. \tag{23}$$

If we now assume that conditions (23) are valid, then so are conditions (22), too, and therefore $\alpha \in R(D') \cap R(D'')$.

The lemma is proved. □

Lemma 13. *Assume that* $D' = \{Z_7, Y, Y', Y \cup Y'\}$ *and* $D'' = \{Z_7, Y_1, Y_1', Y_1 \cup Y_1'\}$ *are elements of the set* $\{D_1', D_2', D_3', D_4', D_5', D_6'\}$ *such that* $D' \neq D''$, $Y_1 \supseteq Y$, $Y_1' \supseteq Y'$ *if* X *is α finite set. Then the following equalities are valid:*

$$\begin{aligned} \left|R(D_1') \cap R(D_4'')\right| &= 4 \cdot 2^{|Z_3 \setminus Z_1|} \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot 4^{|X \setminus \breve{D}|},\\ \left|R(D_2') \cap R(D_3')\right| &= 4 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot 2^{|Z_3 \setminus Z_1|} \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot 4^{|X \setminus \breve{D}|},\\ \left|R(D_3') \cap R(D_6')\right| &= 4 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot (2^{|Z_3 \setminus Z_1|} - 1) \cdot 4^{|X \setminus \breve{D}|},\\ \left|R(D_4') \cap R(D_5')\right| &= 4 \cdot (2^{|Z_3 \setminus Z_1|} - 1) \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot 4^{|X \setminus \breve{D}|} \end{aligned}$$

Proof. Assume that $D' = \{Z_7, Y, Y', Y \cup Y'\}$ and $D'' = \{Z_7, Y_1, Y_1', Y_1 \cup Y_1'\}$ are elements of the set $\{D_1', D_2', D_3', D_4', D_5', D_6'\}$ such that $D' \neq D''$, $Y_1 \supseteq Y$, $Y_1' \supseteq Y'$. If $\alpha \in R(D') \cap R(D'')$ and the quasinormal representation of the binary relation α of the semigroup $B_X(D)$ has the form

$$\alpha = (Y_T^\alpha \times T) \cup (Y_{T'}^\alpha \times T') \cup (Y_{T''}^\alpha \times T'') \cup (Y_{T' \cup T''}^\alpha \times (T' \cup T'')),$$

where $T, T', T'' \in D$, $T \subset T'$, $T \subset T''$, $Y_T^\alpha, Y_{T'}^\alpha, Y_{T''}^\alpha \notin \{\varnothing\}$, then by Lemma 12 we have

$$Y_T^\alpha \cup Y_{T'}^\alpha \supseteq Y_1,\ \ Y_T^\alpha \cup Y_{T''}^\alpha \supseteq Y_1',\ \ Y_{T'}^\alpha \cap Y \neq \varnothing,\ \ Y_{T''}^\alpha \cap Y' \neq \varnothing. \tag{24}$$

Now assume that f_α is a mapping of the set X in the semilattice D such that $f_\alpha(t) = t\alpha$ for any $t \in X$. Suppose that $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ are the restrictions of the mapping f_α on the sets $Y_1 \cap Y_1'$, $Y_1 \setminus Y_1'$, $Y_1' \setminus Y_1$ and $X \setminus (Y_1 \cup Y_1')$, respectively. It is clear that the elements of the set $\{Y_1 \cap Y_1', Y_1 \setminus Y_1', Y_1' \setminus Y_1, X \setminus (Y_1 \cup Y_1')\}$ do not intersect pairwise and their union is equal to the set X.

Let us now consider each of the mappings $f_{0\alpha}$, $f_{1\alpha}$, $f_{2\alpha}$ and $f_{3\alpha}$ separately.

1) If $t \in Y_1 \cap Y_1'$, then we have $t \in Y_1 \cap Y_1' \subseteq (Y_T^\alpha \cup Y_{T'}^\alpha) \cap (Y_T^\alpha \cup Y_{T''}^\alpha) = Y_T^\alpha$, i.e. $t \in Y_T^\alpha$. Hence, by the definition of the set Y_T^α, we obtain $t\alpha = T$.

Thus $f_{0\alpha}(t) = T$ for every $t \in Y_1 \cap Y_1'$.

2) If $t \in Y_1 \setminus Y_1'$, then by condition (24) we have

$$t \in Y_1 \setminus Y_1' \subseteq Y_1 \subseteq Y_T^\alpha \cup Y_{T'}^\alpha.$$

Hence, by the definition of the sets Y_T^α and $Y_{T'}^\alpha$, we obtain $t\alpha \in \{T, T'\}$, i.e. $f_{1\alpha}(t) \in \{T, T'\}$ for any $t \in Y_1 \setminus Y_1'$.

By assumption, $Y_{T'}^\alpha \cap Y \neq \varnothing$. Therefore $t_1\alpha = T'$ for some $t_1 \in Y$. If $t_1 \in Y_1'$, then by conditions (24) we have $t_1\alpha \in \{T, T''\}$. The latter condition contradicts the equality $t_1\alpha = T'$, since $T' \notin \{T, T''\}$. The obtained contradiction shows that $t_1 \in Y \setminus Y_1'$.

Thus $f_{1\alpha}(t_1) = T'$ for some $t_1 \in Y \setminus Y_1'$.

3) If $t \in Y_1' \setminus Y_1$, then we have $t \in Y_1' \setminus Y_1 \subseteq Y_1' \subseteq Y_T^\alpha \cup Y_{T''}^\alpha$. Hence, by the definition of the sets Y_T^α and $Y_{T''}^\alpha$ we obtain $t\alpha \in \{T, T''\}$, i.e., $f_{2\alpha}(t) \in \{T, T''\}$ for any $t \in Y_1' \setminus Y_1$.

By assumption, $Y_{T''}^\alpha \cap Y' \neq \varnothing$. Therefore $t_0\alpha = T''$ for some $t_0 \in Y'$. If $t_0 \in Y_1$, then by conditions (24) we have $t_0\alpha \in \{T, T'\}$. The latter condition contradicts the equality $t_0\alpha = T''$ since $T'' \notin \{T, T'\}$. The obtained contradiction shows that $t_0 \in Y' \setminus Y_1$.

Thus $f_{2\alpha}(t_0) = T''$ for some $t_0 \in Y' \setminus Y_1$.

4) If $t \in X \setminus (Y_1 \cup Y_1')$, then by conditions (24) we have

$$t \in X \setminus (Y_1 \cup Y_1') \subseteq X = Y_T^\alpha \cup Y_{T'}^\alpha \cup Y_{T''}^\alpha \cup Y_{T' \cup T''}^\alpha.$$

Hence, by the definition of the sets Y_T^α, $Y_{T'}^\alpha$, $Y_{T''}^\alpha$ and $Y_{T' \cup T''}^\alpha$ we obtain $t\alpha \in \{T, T', T'', T' \cup T''\}$, i.e., $f_{3\alpha}(t) \in \{T, T', T'', T' \cup T''\}$ for some $t \in X \setminus (Y_1 \cup Y_1')$.

Thus for every $\alpha \in R(D') \cap R(D'')$ there exists a system of mappings $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$. Moreover, to different binary relations there correspond systems of different mappings.

Now assume that

$$f_0 : Y_1 \cap Y_1' \to \{T\}, \quad f_1 : Y_1 \setminus Y_1' \to \{T, T'\},$$
$$f_2 : Y_1' \setminus Y_1 \to \{T, T''\}, \quad f_3 : X \setminus (Y_1 \cup Y_1') \to \{T, T', T'', T' \cup T''\}$$

are the mappings which satisfy the following conditions:

5) $f_0(t) = T$ for every $t \in Y_1 \cap Y_1'$;

6) $f_1(t) \in \{T, T'\}$ for every $t \in Y_1 \setminus Y_1'$and $f_{1\alpha}(t_1) = T'$ for some $t_1 \in Y \setminus Y_1'$;

7) $f_2(t) \in \{T, T''\}$for every $t \in Y_1' \setminus Y_1$ and $f_{2\alpha}(t_0) = T''$ for some $t_0 \in Y' \setminus Y_1$;

8) $f_3(t) \in \{T, T', T'', T' \cup T''\}$ for every $t \in X \setminus (Y_1 \cup Y_1')$.

Now let us define $f : X \to D$ as follows:

$$f(t) = \begin{cases} f_0(t), & \text{if } t \in Y_1 \cap Y_1', \\ f_1(t), & \text{if } t \in Y_1 \setminus Y_1', \\ f_2(t), & \text{if } t \in Y_1' \setminus Y_1, \\ f_3(t), & \text{if } t \in X \setminus (Y_1 \cup Y_1'). \end{cases}$$

To this mapping we put into correspondence the binary relation $\beta = \bigcup\limits_{x \in X} (\{x\} \times f(x))$ of the semigroup $B_X(D)$.

Now if $Y_T^\beta = \{t|\ \ t\beta = T\}$, $Y_{T'}^\beta = \{t|\ \ t\beta = T'\}$, $Y_{T''}^\beta = \{t|\ \ t\beta = T''\}$, $Y_{T' \cup T''}^\beta = \{t|\ \ t\beta = T' \cup T''\}$, then the relation β can be written in the form

$$\beta = (Y_T^\beta \times T) \cup (Y_{T'}^\beta \times T') \cup (Y_{T''}^\beta \times T'') \cup (Y_{T' \cup T''}^\beta \times (T' \cup T'')).$$

And it satisfies the conditions

$$Y_T^\beta \cup Y_{T'}^\beta \supseteq Y_1, \ \ Y_T^\beta \cup Y_{T''}^\beta \supseteq Y_1', \ \ Y_{T'}^\beta \cap Y \neq \varnothing, \ \ Y_{T''}^\beta \cap Y' \neq \varnothing.$$

(By assumption, $f_1(t_1) = T'$ for some $t_1 \in Y \setminus Y_1'$ and $f_2(t_0) = T''$ for some $t_0 \in Y' \setminus Y_1$.) Therefore, by Lemma 12, $\beta \in R(D') \cap R(D'')$.

We have obtained that there exists a one-to-one correspondence between the binary relations α from the set $R(D') \cap R(D'')$ and the system of mappings $(f_{0\alpha}, f_{1\alpha}, f_{2\alpha}, f_{3\alpha})$.

According to Theorem 1.18.2 (see [4] or [5]), the number of systems of mappings is equal to

$$1,\ 2^{|(Y_1 \setminus Y_1') \setminus (Y \setminus Y_1')|} \cdot (2^{|Y \setminus Y_1'|} - 1),\ 2^{|(Y_1' \setminus Y_1) \setminus (Y' \setminus Y_1)|} \cdot (2^{|Y' \setminus Y_1|} - 1),\ 4^{|X \setminus (Y_1 \cup Y_1')|}.$$

Note that the number

$$2^{|Y_1 \setminus (Y \cup Y_1')|} \cdot (2^{|Y \setminus Y_1'|} - 1) \cdot 2^{|Y_1' \setminus (Y' \cup Y_1)|} \cdot (2^{|Y' \setminus Y_1|} - 1) \cdot 4^{|X \setminus (Y_1 \cup Y_1')|}$$

does not depend on a choice of a semilattice $\{T, T', T'', T' \cup T''\}$ defined by item 5), and since the number of semilattices defined in this manner is equal to four, we obtain

$$\begin{aligned} &\left| R(D') \cap R(D'') \right| = \\ &\quad = 4 \cdot 2^{|Y_1 \setminus (Y \cup Y_1')|} \cdot (2^{|Y \setminus Y_1'|} - 1) \cdot 2^{|Y_1' \setminus (Y' \cup Y_1)|} \cdot (2^{|Y' \setminus Y_1|} - 1) \cdot 4^{|X \setminus (Y_1 \cup Y_1')|}. \end{aligned}$$

Hence we have

$$\begin{aligned} \left| R(D_1') \cap R(D_4'') \right| &= 4 \cdot 2^{|Z_3 \setminus Z_1|} \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot 4^{|X \setminus \breve{D}|}, \\ \left| R(D_2') \cap R(D_3') \right| &= 4 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot 2^{|Z_3 \setminus Z_1|} \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot 4^{|X \setminus \breve{D}|}, \\ \left| R(D_3') \cap R(D_6') \right| &= 4 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot (2^{|Z_3 \setminus Z_1|} - 1) \cdot 4^{|X \setminus \breve{D}|}, \\ \left| R(D_4') \cap R(D_5') \right| &= 4 \cdot (2^{|Z_3 \setminus Z_1|} - 1) \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot 4^{|X \setminus \breve{D}|}. \end{aligned}$$

□

Lemma 14. *Assume that* $D=\{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. *If* X *is a finite set, then*

$$\begin{aligned}|R^*(Q_5)| = {} & 8 \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot (2^{|Z_4 \setminus Z_6|} - 1) \cdot 4^{|X \setminus Z_1|} + \\ & + 8 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot 4^{|X \setminus \breve{D}|} + \\ & + 8 \cdot (2^{|Z_3 \setminus Z_1|} - 1) \cdot (2^{|Z_1 \setminus Z_3|} - 1) \cdot 4^{|X \setminus \breve{D}|} - \\ & - 8 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot 2^{|Z_3 \setminus Z_1|} \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot 4^{|X \setminus \breve{D}|} - \\ & - 8 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot (2^{|Z_3 \setminus Z_1|} - 1) \cdot 4^{|X \setminus \breve{D}|}.\end{aligned}$$

Proof. Using Lemmas 11 and 13 we obtain

$$\begin{aligned}|R^*(Q_5)| = {} & |R(D'_1)| + |R(D'_2)| + |R(D'_3)| + |R(D'_4)| + |R(D'_5)| + |R(D'_6)| - \\ & - \left|R(D'_2) \cap R(D'_3)\right| - \left|R(D'_3) \cap R(D'_6)\right| - \left|R(D'_4) \cap R(D'_5)\right| = \\ = {} & 8 \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot (2^{|Z_4 \setminus Z_6|} - 1) \cdot 4^{|X \setminus Z_1|} + \\ & + 8 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot (2^{|Z_3 \setminus Z_4|} - 1) \cdot 4^{|X \setminus \breve{D}|} + \\ & + 8 \cdot (2^{|Z_3 \setminus Z_1|} - 1) \cdot (2^{|Z_1 \setminus Z_3|} - 1) \cdot 4^{|X \setminus \breve{D}|} - \\ & - 8 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot 2^{|Z_3 \setminus Z_1|} \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot 4^{|X \setminus \breve{D}|} - \\ & - 8 \cdot (2^{|Z_4 \setminus Z_3|} - 1) \cdot (2^{|Z_3 \setminus Z_1|} - 1) \cdot 4^{|X \setminus \breve{D}|}. \qquad \square\end{aligned}$$

6. By the definition of the semilattice D of unions we have

$$Q_6\theta_{\mathrm{XI}} = \{\{Z_7, Z_6, Z_4, Z_1, \breve{D}\}\}.$$

Now if $D'_1 = \{Z_7, Z_6, Z_4, Z_1, \breve{D}\}$, then

$$R^*(Q_6) = R(D'_1) \ \text{ and } \ |R^*(Q_6)| = |R(D'_1)|.$$

Lemma 15. *Let* $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. *If* X *is a finite set, then*

$$|R^*(Q_6)| = 2 \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot (2^{|Z_4 \setminus Z_6|} - 1) \cdot (5^{|\breve{D} \setminus Z_1|} - 4^{|\breve{D} \setminus Z_1|}) \cdot 5^{|X \setminus \breve{D}|}.$$

Proof. In our case we have

$$\begin{aligned}|R^*(Q_6)| & = |R(D'_1)| = \\ & = 2 \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot (2^{|Z_4 \setminus Z_6|} - 1) \cdot (5^{|\breve{D} \setminus Z_1|} - 4^{|\breve{D} \setminus Z_1|}) \cdot 5^{|X \setminus \breve{D}|}. \qquad \square\end{aligned}$$

7. By the definition of the semilattice D of unions we have

$$Q_7\theta_{\mathrm{XI}} = \{\{Z_7, Z_6, Z_3, Z_1, \breve{D}\}\}.$$

Now if $D'_1 = \{Z_7, Z_6, Z_3, Z_1, \breve{D}\}$, then$R^*(Q_7) = R(D'_1)$ and $|R^*(Q_7)| = |R(D'_1)|$.

Lemma 16. *Let* $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. *If* X *is a finite set, then*

$$|R^*(Q_7)| = 2 \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot 2^{|(Z_3 \cap Z_1) \setminus Z_6|} \cdot (3^{|Z_3 \setminus Z_1|} - 2^{|Z_3 \setminus Z_1|}) \cdot \\ \cdot (3^{|Z_1 \setminus Z_3|} - 2^{|Z_1 \setminus Z_3|}) \cdot 5^{|X \setminus \breve{D}|}.$$

Proof. In our case we have

$$|R^*(Q_7)| = 2 \cdot (2^{|Z_6 \setminus Z_7|} - 1) \cdot 2^{|(Z_3 \cap Z_1) \setminus Z_6|} \cdot (3^{|Z_3 \setminus Z_1|} - 2^{|Z_3 \setminus Z_1|}) \cdot \\ \cdot (3^{|Z_1 \setminus Z_3|} - 2^{|Z_1 \setminus Z_3|}) \cdot 5^{|X \setminus \breve{D}|}.$$

□

8. By the definition of the semilattice D of unions we have

$$Q_8 \theta_{\mathrm{XI}} = \{\{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}\}.$$

Now if $D_1' = \{Z_7, Z_6, Z_4, Z_3, Z_1, \breve{D}\}$, then

$$R^*(Q_8) = R(D_1') \text{ and } |R^*(Q_8)| = |R(D_1')|.$$

Lemma 17. *Let* $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. *If* X *is a finite set, then*

$$|R^*(Q_8)| = (2^{|Z_4 \setminus Z_3|} - 1) \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot (3^{|Z_3 \setminus Z_1|} - 2^{|Z_3 \setminus Z_1|}) \cdot 6^{|X \setminus \breve{D}|}.$$

Proof. In our case we have

$$|R^*(Q_8)| = |R(D_1')| = (2^{|Z_4 \setminus Z_3|} - 1) \cdot (2^{|Z_6 \setminus Z_4|} - 1) \cdot \\ \cdot (3^{|Z_3 \setminus Z_1|} - 2^{|Z_3 \setminus Z_1|}) \cdot 6^{|X \setminus \breve{D}|}.$$

□

Theorem 2. *Assume that* $D = \{Z_7, Z_6, Z_5, Z_4, Z_3, Z_2, Z_1, \breve{D}\} \in \Sigma_4(X, 8)$. *If* X *is a finite set and* R_D *is a set of all regular elements of the semigroup* $B_X(D)$, *then*

$$|R_D| = \sum_{i=1}^{8} |R^*(Q_i)|.$$

Proof. We obtain the proof of this theorem from Theorem 1. □

Example 1. Let $X = \{1, 2, 3, 4, 5\} = \breve{D}$ and

$$P_1 = \{1\}, \ P_2 = \{2\}, \ P_3 = \{3\}, \ P_4 = \{4\}, \ P_5 = \{5\}, \ P_0 = P_6 = P_7 = \varnothing.$$

Then $\breve{D} = \{1, 2, 3, 4, 5\}$, $Z_1 = \{2, 3, 4, 5\}$, $Z_2 = \{1, 3, 4, 5\}$, $Z_3 = \{1, 2, 4, 5\}$, $Z_4 = \{2, 3, 5\}$, $Z_5 = \{2, 3, 4\}$, $Z_6 = \{2, 4, 5\}$, $Z_7 = \{2, 5\}$ and

$$D = \Big\{\{2, 5\}, \{2, 4, 5\}, \{2, 3, 4\}, \{2, 3, 5\}, \{1, 2, 4, 5\}, \{1, 3, 4, 5\}, \\ \{2, 3, 4, 5\}, \{1, 2, 3, 4, 5\}\Big\}.$$

In that case, we have

$$|R^*(Q_1)| = 8, \ \ |R^*(Q_2)| = 150, \ \ |R^*(Q_3)| = 121, \ \ |R^*(Q_4)| = 9,$$
$$|R^*(Q_5)| = 40, \quad |R^*(Q_6)| = 2, \quad |R^*(Q_7)| = 2, \quad |R^*(Q_8)| = 1,$$

i.e. $|R_D| = 333$.

Example 2. Let $X = \{1, 2, 3, 4, 5, 6, 7, 8\} = \breve{D}$ and

$$P_0 = \{8\}, \ \ P_1 = \{1\}, \ \ P_2 = \{2\}, \ \ P_3 = \{3\},$$
$$P_4 = \{4\}, \ \ P_5 = \{5\}, \ \ P_6 = \{6\}, \ \ P_7 = \{7\}.$$

Then

$$\breve{D}=\{1, 2, 3, 4, 5, 6, 7, 8\}, \ Z_1=\{2, 3, 4, 5, 6, 7, 8\}, \ Z_2=\{1, 3, 4, 5, 6, 7, 8\},$$
$$Z_3 = \{1, 2, 4, 5, 6, 7, 8\}, \quad Z_4 = \{2, 3, 5, 6, 7, 8\}, \quad Z_5 = \{2, 3, 4, 6, 7, 8\},$$
$$Z_6 = \{2, 4, 5, 7, 8\}, \ \ Z_7 = \{2, 5, 8\}$$

and

$$D = \Big\{\{2, 5, 8\}, \{2, 4, 5, 7, 8\}, \{2, 3, 4, 6, 7, 8\}, \{2, 3, 5, 6, 7, 8\},$$
$$\{1, 2, 4, 5, 6, 7, 8\}, \{1, 3, 4, 5, 6, 7, 8\}, \{2, 3, 4, 5, 6, 7, 8\},$$
$$\{1, 2, 3, 4, 5, 6, 7, 8\}\Big\}.$$

In this case we have

$$|R^*(Q_1)| = 8, \ \ |R^*(Q_2)| = 510, \ \ |R^*(Q_3)| = 935, \ \ |R^*(Q_4)| = 111,$$
$$|R^*(Q_5)| = 104, \quad |R^*(Q_6)| = 6, \quad |R^*(Q_7)| = 12, \quad |R^*(Q_8)| = 1,$$

i.e. $|R_D| = 1687$.

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