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ON ONE NONLOCAL BOUNDARY VALUE PROBLEM FOR QUASILINEAR DIFFERENTIAL EQUATIONS

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Abstract. In the paper, a theorem on the existence and uniqueness of a generalized solution in the space $C_\alpha(\overline{G})$ is proved for quasilinear differential equations. The Bitsadze–Samarski boundary value problem is considered for a linear differential equation of first order. The existence of a generalized equation in the space $C^p_\alpha(\overline{G})$ is proved and an a priori estimate is obtained.

რეზიუმე. ნაშრომში დამტკიცებულია თეორემა კვაზიწრფივი დიფერენციალური განტოლებისათვის ბიწაძე-სამარსკის სასაზღვრო ამოცანის განზოგადოებული ამონახსნის არსებობისა და ერთადერთობის შესახებ $C_\alpha(\overline{G})$ სივრცეში. ასევე განხილულია ბიწაძე-სამარსკის სასაზღვრო ამოცანა პირველი რიგის წრფივი დიფერენციალური განტოლებისათვის, დამტკიცებულია ამ ამოცანის განზოგადოებული ამონახსნის არსებობა და ერთადერთობა $C^p_\alpha(\overline{G})$ სივრცეში და მიღებულია აპრიორული შეფასება.

Introduction

Nonlocal boundary value problems are quite an interesting generalization of classical problems and at the same time they are naturally obtained when constructing mathematical models in physics, engineering, sociology, ecology and so on [1]–[5].

The investigation of nonlocal problems for differential equations originated in the last century. Here we should in the first place refer to the works of T. Carleman, R. Beals, F. Browder and other works. The problems posed in [6]–[8] are the problems with nonlocal conditions, which are considered only on the boundary of the definition domain of a differential operator. In 1963, J. Cannon posed a nonlocal problem in his work [9] and

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thus gave an impetus to the development of a new trend in the investigation of nonlocal boundary value problems [10]–[13].

In 1969, the work of A. Bitsadze and A. Samarski [14] was published, which was dedicated to the investigation of a nonlocal problem of a new type. That problem arose in connection with the mathematical modeling of plasma processes. Intensive studies of Bitsadze–Samarski nonlocal problems and their various generalizations began in the 80ties of the last century (see the papers of D. G. Gordeziani, A. L. Skubachevski, V. P. Paneyakh, V. A. Ilyin, I. Moiseyev, G .V. Meladze, M. P. Sapagovas, D. V. Kapanadze, V. P. Mikhailov, A. K. Gushchin, G. Avalishvili, L. Gurevich [15]–[26] and other works). The papers of D. G. Sapagovas, G. K. Berikelashvili are certainly interesting from the standpoint of application and numerical methods (see e.g. [11], [20]). The algorithm of reducing nonlocal problems of the Bitsadze–Samarski type to an iteration sequence of Dirichlet problems is investigated in D. G. Gordeziani's papers [15], [16].

Problems of the existence and uniqueness of a generalized analytic function for the Riemann–Hilbert problem are investigated in I.N. Vekua's monograph [27]. Problems of the existence and uniqueness of a generalized solution for quasilinear equations of first order on the plane with Riemann–Hilbert boundary conditions are considered in [28]. In [29], [30] the Bitsadze–Samarski nonlocal boundary value problem is considered for quasilinear differential equations.

In the present paper, a theorem on the existence and uniqueness of a generalized solution in the space $C_\alpha(\overline{G})$ is proved for quasilinear differential equations of first order with nonlocal boundary conditions. Also, the Bitsadze–Samarski nonlocal boundary value problem is considered for a linear differential equation of first order. It is shown that there exists a generalized solution in the space $C^p_\alpha(\overline{G})$ and the a priori estimate is obtained. Nonlocal boundary value problems are investigated by using the iteration algorithm of reducing the considered problem to a sequence of Riemann–Hilbert problems for generalized analytic functions. This method enables one not only to solve the problem numerically, but also to prove the existence of a solution.

1. Existence of a Generalized Solution of the Bitsadze–Samarski Nonlocal Boundary Value Problem

Let G be the bounded domain of the complex plane E with boundary Γ which is a closed simple Lyapunov curve (i.e. the angle formed by the tangent to this curve with the constant direction is Holder-continuous).

Denote by γ the part of the boundary Γ, which is an open Liapunov curve with the parametric equation $z = z(s)$, $0 \le s \le \delta$. Let γ_0 be the diffeomorphic image $z_0 = I(z)$ of γ, which lies in the domain G, with the parametric equation $z_0 = z_0(s)$, $0 \le s \le \delta$. Lett γ_0 intersect Γ, but not tangentially to it, $z = x + iy \in G$, $w = w_1 + iw_2$. Assume that

$$\partial_{\overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

is the generalized Sobolev derivative [27], $C(\overline{G})$ is the Banach space consisting of all continuous functions on $\overline{G}$. The norm in $C(\overline{G})$ is defined by the equality

$$\|f\|_{C(\overline{G})} = \max_{z\in\overline{G}} |f(z)|.$$

$C_\alpha(\overline{G})$ is assumed to be the set of all bounded functions satisfying the Holder condition with index α. The norm in $C_\alpha(\overline{G})$ is defined by the equality

$$\|f\|_{C_\alpha(\overline{G})} = \max_{z\in\overline{G}} |f(z)| + \sup_{z_1,z_2\in\overline{G}} \frac{|f(z_1) - f(z_2)|}{|z_1 - z_2|^\alpha}\,.$$

$L_p(\overline{G})$ is the Banach space consisting of all measurable functions on $\overline{G}$, which are summable over $\overline{G}$ with power $p \ge 1$. The norm in $L_p(\overline{G})$ is defined by the equality

$$\|f\|_{L_p(\overline{G})} = \left(\int_{\overline{G}} |f|^p\,dz\right)^{1/p}.$$

In the domain $\overline{G}$ we will consider the Bitsadze–Samarski boundary value problem [14] for quasilinear differential equations of first order

$$\partial_{\overline{z}} w = f(z, w, \overline{w}), \quad z \in G, \tag{1.1}$$

$$\operatorname{Re}[w(z)] = \phi(z), \quad z \in \Gamma\setminus\gamma, \quad \operatorname{Im}[w(z^*)] = c, \quad z^* \in \Gamma\setminus\gamma, \tag{1.2}$$

$$\operatorname{Re}[w(z(s))] = \sigma \operatorname{Re}\big[w(z_0(s))\big], \quad z(s)\in\gamma, \quad z_0(s)\in\gamma_0, \quad 0<\sigma=\text{const}\,. \tag{1.3}$$

It is assumed that the following conditions are fulfilled:

(A1) The function $f(z, w, \overline{w})$ is defined for $z \in G$, $|w| < R$, $f(z, 0, 0) \in L_p(\overline{G})$, $p > 2$, and

$$\big|f(z, w, \overline{w}) - f(z, w_0, \overline{w}_0)\big| \le L\big(|w - w_0| + |w - \overline{w}_0|\big).$$

(A2) $\phi(z) \in C_\alpha(\Gamma\setminus\gamma)$, $\alpha > 1/2$.

To prove the existence of a solution of problem (1.1)–(1.3), we consider the following iteration process:

$$\partial_{\overline{z}} w_n = f(z, w_n, \overline{w}_n), \quad z \in G, \tag{1.4}$$

$$\operatorname{Re}[w_n(z)] = \phi(z), \quad z \in \Gamma\setminus\gamma, \quad \operatorname{Im}\big[w_n(z^*)\big] = c, \quad z^* \in \Gamma\setminus\gamma, \tag{1.5}$$

$$\operatorname{Re}\big[w_n(z(s))\big] = \sigma \operatorname{Re}\big[w_{n-1}(z_0(s))\big], \quad z(s) \in \gamma, \quad z_0(s) \in \gamma_0, \quad n = 1, 2, 3, \dots, \tag{1.6}$$

where $w_0(z)$ is any function from $C_\alpha(\gamma)$ that continuously adjoins the values of $\phi(z)$ at the ends of the contour γ.

For every $n \in N$, problem (1.4)–(1.6) is a Dirichlet type problem and a regular generalized solution belongs to the space $C_\alpha(\overline{G})$ [27], [28].

Let us consider the function $v_n = w_{n+1} - w_n$. Then from (1.5)–(1.6) it follows that the function v_n is the solution of the following problem

$$\partial_{\overline{z}} v_n = f(z, w_{n+1}, \overline{w}_{n+1}) - f(z, w_n, \overline{w}_n) \equiv \\ \equiv F(z, w_n, \overline{w}_n, w_{n+1}, \overline{w}_{n+1}), \quad z \in G, \tag{1.7}$$

$$\mathrm{Re}[v_n(z)] = 0, \quad z \in \Gamma \setminus \gamma, \quad \mathrm{Im}\big[v_n(z^*)\big] = 0, \quad z^* \in \Gamma \setminus \gamma, \tag{1.8}$$

$$\mathrm{Re}\big[v_n(z(s))\big] = \sigma\, \mathrm{Re}\big[v_{n-1}(z_0(s))\big], \quad z(s) \in \gamma, \quad z_0(s) \in \gamma_0, \tag{1.9}$$
$$n = 1, 2, 3, \dots .$$

The solution of problem (1.8)–(1.9) can be reduced to the following nonlinear integral equation [28]

$$v^*(z) = \psi_n(z) + \varphi_n(z) - \\ - \frac{1}{\pi} \iint\limits_G \frac{F(\zeta, w_n(\zeta), \overline{w}_n(\zeta), w_{n+1}(\zeta), \overline{w}_{n+1}(\zeta))}{\zeta - z}\, d\xi\, d\eta, \tag{1.10}$$

where $\zeta = \xi + i\eta$, $\psi_n(z)$ is a holomorphic function that satisfies conditions (1.7)–(1.9), and $\varphi_n(z)$ is a holomorphic function such that the difference

$$\varphi_n(z) - \frac{1}{\pi} \iint\limits_G \frac{F(\zeta, w_n, \overline{w}_n, w_{n+1}, \overline{w}_{n+1})}{\zeta - z}\, d\xi\, d\eta$$

satisfies the homogeneous boundary conditions, and an a priori estimate has the form

$$\|\varphi_n\|_{C_\alpha(\overline{G})} \le C_1 \|F\|_{L_p(\overline{G})}, \quad C_1 = \mathrm{const} > 0.$$

The integral operator in the right-hand part of equation (1.10) is denoted by T_G. Note that the operator T_G maps the space $L_p(\overline{G})$ into $C_\beta(\overline{G})$, $\beta = (p-2)/p < \alpha$ [27].

Let consider the following conditions:

(A3) There exists a number $R_1 > 0$, $R_1 \le R$, such that the inequality

$$\|\psi_n\|_{C_\alpha(\overline{G})} + \big(C_1 + \|T_G\|_{L_p(\overline{G}), C_\alpha(\overline{G})}\big)\big(2L|G|^{1/p} R_1\big) \le R_1, \quad |G| = \mathrm{mes}\, G,$$

is fulfilled.

(A4) $2|G|^{1/p} L\big(C_1 + \|T_G\|_{L_p(\overline{G}), C_\alpha(\overline{G})}\big) < 1$.

Assume that conditions **(A1)**–**(A4)** are fulfilled, then there exists a unique solution of problem (1.7)–(1.9) in a ball $\|v_n\|_{C_\alpha(\overline{G})} \le R_1$ [28].

Let us estimate the function $v_n(z)$ from equality (1.10) in the metric of the space $C(\overline{G})$:

$$\|v_n\|_{C(\overline{G})} \le \|\psi_n\|_{C(\overline{G})} + \|\varphi_n\|_{C(\overline{G})} + \big\|T_G[F]\big\|_{C(\overline{G})}. \tag{1.11}$$

Using the previous estimates, from inequality (1.11) we obtain

$$\|v_n\|_{C(\overline{G})} \le \|\psi_n\|_{C(\overline{G})} + \big(C_1 + \|T_G\|_{L_p(\overline{G}),C_\alpha(\overline{G})}\big)\|F\|_{L_p(\overline{G})}. \tag{1.12}$$

By virtue of **(A1)** we have

$$\big|F(z,w_n,\overline{w}_n,w_{n+1},\overline{w}_{n+1})\big| = \big|f(z,w_{n+1},\overline{w}_{n+1}) - f(z,w_n,\overline{w}_n)\big| \le$$
$$\le 2L|w_{n+1}-w_n| = 2L|v_n|.$$

With **(A1)** taken into account, the latter inequality implies that the complex function $F(z,w_n,\overline{w}_n,w_{n+1},\overline{w}_{n+1})$ belongs to the space $L_p(\overline{G})$. Then

$$\|F\|_{L_p(\overline{G})} \le 2L\|v_n\|_{L_p(\overline{G})} \le 2L|G|^{1/p}\|v_n\|_{C(\overline{G})}.$$

Thus, from inequality (1.12) we can write that

$$\|v_n\|_{C(\overline{G})} \le \|\psi_n\|_{C(\overline{G})} + 2L|G|^{1/p}\big(C_1 + \|T_G\|_{C_\alpha(\overline{G})}\big)\|v_n\|_{C(\overline{G})},$$

i.e., taking **(A4)** into account, we finally obtain

$$\|v_n\|_{C(\overline{G})} \le \frac{\|\psi_n\|_{C(\overline{G})}}{1-2L|G|^{1/p}(C_1 + \|T_G\|_{C_\alpha(\overline{G})})}\,. \tag{1.13}$$

Note that the function $\psi_n(z)$ is the solution of the following problem

$$\partial_{\overline{z}}\psi_n(z) = 0, \quad z\in G,$$
$$\mathrm{Re}[\psi_n(z)] = 0, \quad z\in\Gamma\setminus\gamma, \quad \mathrm{Im}\,\big[\psi_n(z^*)\big] = 0,$$
$$\mathrm{Re}\,\big[\psi_n(z)\big] = \sigma\,\mathrm{Re}\,\big[\psi_{n-1}(z_0)\big], \quad z\in\gamma, \quad z_0\in\gamma_0, \quad n=1,2,3,\dots,$$
$$\psi_0(z) = w_1(z) - w_0(z).$$

Since $\mathrm{Re}[\psi_n(z)]$ is a harmonic function, all the conditions of Schwartz' lemma [27] are fulfilled for it and there exists $0<q<1$ which is independent of ψ_n and for which the following inequality [16] is fulfilled:

$$\|\psi_n\|_{C(\overline{G})} \le Mq^n,$$

where the constant $M>0$ depend only on $\phi(z)$.

Using this estimate, from (1.13) we can write

$$\|v_n\|_{C(\overline{G})} \le \frac{M}{1-2L|G|^{1/p}(C_1 + \|T_G\|_{C_\alpha(\overline{G})})}\,q^n. \tag{1.14}$$

Now from (1.14) we can conclude that the series $\sum\limits_{k=1}^{\infty} v_k$ converges uniformly to zero in the domain $\overline{G}$. Hence it follows that the sequence $\{w_n(z)\}$ is fundamental in $C(\overline{G})$ and has the limit $w(z)\in C(\overline{G})$.

Let us consider the integral representation for the function $w_n(z)$:

$$w_n(z) = \psi_n'(z) + \varphi_n'(z) - \frac{1}{\pi}\iint\limits_G \frac{f(\zeta,w_n,\overline{w}_n)}{\zeta - z}\,d\xi\,d\eta, \tag{1.15}$$

where $\psi_n'(z)$ is a holomorphic function that satisfies conditions (1.5)–(1.6), and $\varphi_n'(z)$ is a holomorphic function such that the difference

$$\varphi_n'(z) - \frac{1}{\pi}\iint\limits_G \frac{f(\zeta, w_n, \overline{w}_n)}{\zeta - z}\, d\xi\, d\eta$$

satisfies the homogeneous boundary conditions.

From representation (1.15) we can conclude that $w(z)$ is the solution of problem (1.1)–(1.3) and $w(z) \in C_\alpha(\overline{G})$. By the uniqueness of the holomorphic solution and the integral representation (1.15) we conclude that this solution is unique in the class $C_\alpha(\overline{G})$.

We have thereby proved

Theorem 1. *Let conditions* **(A1)**–**(A4)** *be fulfilled, then the solution of problem* (1.1)–(1.3) *exists in the space* $C_\alpha(\overline{G})$ *and is unique.*

2. Linear Problem

Let us consider, in the domain $\overline{G}$, the Bitsadze–Samarski boundary value problem for a linear differential equation of first order

$$\begin{gathered}\partial_{\overline{z}} w = A(z)w + B(z)\overline{w} + d(z), \quad z \in G, \\ \operatorname{Re}[w(z)] = 0, \quad z \in \Gamma \setminus \gamma, \quad \operatorname{Im}[w(z^*)] = 0, \quad z^* \in \Gamma \setminus \gamma, \\ \operatorname{Re}[w(z(s))] = \sigma \operatorname{Re}\big[w(z_0(s))\big], \quad z(s) \in \gamma, \quad z_0(s) \in \gamma_0. \end{gathered} \tag{2.1}$$

Assume that $A(z), B(z), d(z) \in L_p(\overline{G})$, $p > 2$, $|A|, |B| \le N$.

Denote by $C_\alpha^p(\overline{G})$ the set of functions $w(z) \in C_\alpha(\overline{G})$ such that

$$\begin{gathered}\operatorname{Re}[w(z)] = 0, \quad z \in \Gamma \setminus \gamma, \quad \operatorname{Im}[w(z^*)] = 0, \quad z^* \in \Gamma \setminus \gamma, \\ \operatorname{Re}[w(z(s))] = \sigma \operatorname{Re}\big[w(z_0(s))\big], \quad z(s) \in \gamma, \quad z_0(s) \in \gamma_0 \end{gathered} \tag{2.2}$$

and possessing the norm

$$\|w\|_{C_\alpha^p(\overline{G})} = \|w\|_{C_\alpha(\overline{G})} + \|\partial_{\overline{z}} w\|_{L_p(\overline{G})} < +\infty. \tag{2.3}$$

It is easy to verify that the set $C_\alpha^p(\overline{G})$ is a linear normalized space over the real field with the norm defined by means of equality (2.3). If $p > q > 2$, then $C_\alpha^p(\overline{G}) \subset C_\alpha^q(\overline{G})$ and $\|w\|_{C_\alpha^q(\overline{G})} \le \ell \|w\|_{C_\alpha^p(\overline{G})}$, where ℓ is a positive constant and w is any element from $C_\alpha^p(\overline{G})$.

Theorem 2. *For any function* $d(z) \in L_p(\overline{G})$, $p > 2$, *the solution* $w(z)$ *of problem* (2.1) *exists, belongs to the space* $C_\alpha^p(\overline{G})$ *and the following a priori estimate holds for it*

$$\|w\|_{C_\alpha^p(\overline{G})} \le \lambda \|d\|_{L_p(\overline{G})}, \tag{2.4}$$

where λ *is the positive constant depending only on* p, N *and* $|G| = \operatorname{mes} G$.

Proof. The existence and uniqueness of the solution of problem (2.1) immediately follows from Theorem 1. It remains to prove the validity of the a priori estimate (2.4).

We have to reduce problem (2.1) to an integral equation. For this we introduce the operator

$$T_G[z,f] = -\frac{1}{\pi}\iint\limits_G \frac{f(t)}{t-z}\, d\xi\, d\eta, \quad t = \xi + i\eta$$

and the operator $S_G[z,f]$ from $L_p(\overline{G})$ into a subset of analytic functions, which satisfies the conditions

$$\begin{gathered}
\operatorname{Re}\Big\{T_G[z,f] + S_G[z,f]\Big\} = 0, \quad z \in \Gamma \setminus \gamma, \\
\operatorname{Re}\Big\{T_G[z,f] + S_G[z,f]\Big\} = \sigma \operatorname{Re}\Big\{T_G[z_0,f] + S_G[z_0,f]\Big\}, \\
z_0 \in \gamma_0, \quad z \in \gamma, \\
\operatorname{Im}\Big\{T_G[z^*,f] + S_G[z^*,f]\Big\} = 0,
\end{gathered} \tag{2.5}$$

where $z^* \in \Gamma \setminus \gamma$ is a fixed point.

Due to conditions (2.5) we define the operator $S_G[z,f]$ uniquely. Let us define the operators

$$\begin{gathered}
P(f) = T_G[z,f] + S_G[z,f], \\
P_{AB}(f) = P(Af) + P(B\overline{f}),
\end{gathered} \tag{2.6}$$

where the functions $A(z)$ and $B(z)$ are from the right-hand part of equation (2.1).

Taking now into account that $\partial_{\overline{z}} P(f) = f(z)$, it can be easily proved that the solution of problem (2.1) satisfies the following integral equation

$$w(z) = P_{AB}(w) + P(d). \tag{2.7}$$

It is likewise easy to show that problems (2.1) and (2.7) are equivalent. Using the properties of the operators $T_G[z,f]$ and $P(f)$ [27], it can be shown that these operators are completely continuous over the field of real numbers. It is obvious that the operator $P_{AB}(f)$, too, is completely continuous.

Since for $d(z) = 0$ equation (2.1) has only the trivial solution, the equation $w(z) = P_{AB}(w)$ will also have only the trivial solution. Hence, because the operator $P_{AB}(f)$ is completely continuous, we obtain the existence and boundedness of the operator $(I - P_{AB})^{-1}$, where I is the identity operator.

We introduce the notation

$$\|I - P_A\|^{-1}_{C_\alpha(\overline{G}), L_p(\overline{G})} = M, \quad \|P\|_{L_p(\overline{G}), C_\alpha(\overline{G})} = M_P,$$

where M and M_p are positive constants. From equation (2.7) we immediately obtain

$$\|w(z)\|_{C_\alpha(\overline{G})} \le M M_p \|d\|_{L_p(\overline{G})}. \tag{2.8}$$

Equation (2.1) immediately implies

$$\|\partial_{\overline{z}} w\|_{L_p(\overline{G})} \leq 2N\|w\|_{C_\alpha(\overline{G})} + \|d\|_{L_p(\overline{G})}. \tag{2.9}$$

From inequalities (2.8), (2.9) we obtain the estimate

$$\|w\|_{C^p_\alpha(\overline{G})} = \|w\|_{C_\alpha(\overline{G})} + \|\partial_{\overline{z}} w\|_{L_p(\overline{G})} \leq \lambda\|d\|_{L_p(\overline{G})},$$

where $\lambda = MM_p(2N+1)+1$. □

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