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ČECH'S TYPE FUNCTORS AND COMPLETIONS OF SPACES

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Abstract. In this paper Čech's type uniform shape invariant extensions of functors defined on subcategories of uniform homotopy category are constructed and for them continuity theorems are proved. Besides, it is shown that these extensions of functors have equal values on pairs of uniform spaces and their completions.

რეზიუმე. შრომაში აგებულია თანაბარი ჰომოტოპიის კატეგორიის ქვეკატეგორიებზე განსაზღვრული ფუნქტორების თანაბარ შეიპურად ინვარიანტული ჩეხის ტიპის გაგრძელებები და მათთვის დამტკიცებულია უწყვეტობის თეორემები. გარდა ამისა, ნაჩვენებია რომ ფუნქტორების გაგრძელებები ტოლ მნიშვნელობებს ღებულობს თანაბარ სივრცეთა წყვილებზე და მათ გასრულებებზე.

Introduction

The main aim of the present paper is to study the extension problem of functors. The problem of extension of functors from the subcategory of spaces with the homotopy type of "good" spaces to the category of topological spaces is one of the important problems of algebraic topology ([8–12], [15–21]). The achievements in the solution of this problem have interesting applications in different branches of modern topology ([1–10], [17–21]).

The uniform shape theory ([2], [4], [18–20]) is closely connected with the problem of extension of functors from the category of uniform spaces with the uniform homotopy type of uniform polyhedras to the category of uniform topological spaces.

This paper is devoted to the study of this problem. Using the results of uniform shape theory ([4], [18–21]) here is given a general method of construction of uniform shape invariant and continuous extensions of functors with values in the category of abelian groups. In particular, using the inverse

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system approach of pairs of uniform spaces we construct Čech type extensions of covariant (contravariant) functors defined on various subcategories of pairs of uniform spaces and prove that the values of defined extensions on pairs of uniform spaces and their completions are equal.

In this paper we use the following books: J. R. Isbell and I. M. James ([13, 14]) for uniform spaces, S. Mardešic and J. Segal [14] for shape theory, S. Eilenberg and N. Steenrod [11] for algebraic topology.

Let **Unif** denotes the category of uniform spaces and uniform maps. By **fUnif**, **pUnif**, **ANRU** and **UPol** we denote the full subcategories whose objects are finitistic uniform spaces [18], precompact uniform spaces [14], uniform absolute neighborhood retracts [14] and uniform polyhedrons [18], respectively.

We write **HUnif** for the uniform homotopy category of category **Unif**. Similarly, by symbols $\mathbf{H_fUnif}$, $\mathbf{H_pUnif}$, **HANRU** and **HUpol** we denote the uniform homotopy categories of categories $\mathbf{{}_fUnif}$, $\mathbf{{}_pUnif}$, **ANRU** and **Upol**. Besides, by symbols $\mathbf{Unif^2}$, $\mathbf{{}_fUnif^2}$, $\mathbf{Upol^2}$, $\mathbf{ANRU^2}$ and $\mathbf{pUnif^2}$ we denote the uniform categories of pairs of uniform spaces, finitistic uniform spaces, uniform polyhedrons, absolute neighborhood retracts and precompact uniform spaces and by $\mathbf{HUnif^2}$, $\mathbf{H_fUnif^2}$, $\mathbf{HUpol^2}$, $\mathbf{HANRU^2}$ and $\mathbf{HpUnif^2}$ we denote their uniform homotopy categories, respectively.

Let **Ab** be the category of abelian groups and homomorphisms. By symbols $\mathbf{pro-HUnif}$, $\mathbf{pro-H_fUnif}$, $\mathbf{pro-H_pUnif}$, $\mathbf{pro-HANRU}$, $\mathbf{pro-HUpol}$ and $\mathbf{pro-Ab}$ we denote pro-categories of above given categories.

Let $\mathbf{dir-Ab}$ be the category of direct systems of category **Ab**. By $\mathbf{inj-Ab}$ we denote the quotient category of category $\mathbf{dir-Ab}$ [8]. By symbol $\mathbf{Sh_{(K,L)}}$ denoted the shape category [17] for a category **K** and its subcategory **L**.

I. Uniform shape invariant functors. Let **L** be a full subcategory of some subcategory **K** of category $\mathbf{HUnif^2}$. Consider the category $\mathbf{inv-L}$ of inverse systems of category **L**.

Let $T : \mathbf{L} \to \mathbf{Ab}$ be a covariant (contravariant) functor with values in the category of abelian groups. For every object $(\mathbf{X}, \mathbf{X_0}) = \{(X_\lambda, X_{0\lambda}), p_{\lambda\lambda}, \Lambda\}$ of category $\mathbf{inv-L}$ the family $T((\mathbf{X}, \mathbf{X_0})) = \{T(X_\lambda, X_{0\lambda}), T(p_{\lambda\lambda'}), \Lambda\}$ is an object of the category $\mathbf{inv-Ab}$ ($\mathbf{dir-Ab}$). Every morphism $(f_\mu, \varphi) : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0}) = \{(Y_\mu, Y_{0\mu}), q_{\mu\mu}, M\}$ of category $\mathbf{inv-L}$ induces a morphism $(T(f_\mu), \varphi) : T((\mathbf{X}, \mathbf{X_0})) \to T((\mathbf{Y}, \mathbf{Y_0}))(T(f_\mu), \varphi) : T((\mathbf{Y}, \mathbf{Y_0})) \to T((\mathbf{X}, \mathbf{X_0})))$ in the category $\mathbf{inv-Ab}$ ($\mathbf{dir-Ab}$). If (f_μ, φ) and $({f_\mu}^{'}, \varphi^{'})$ are equivalent morphisms of the category $\mathbf{inv-L}$, then $(T(f_\mu), \varphi)$ and $(T(f'_\mu), \varphi')$ are equivalent morphizms of the category $\mathbf{inv-Ab}$ ($\mathbf{dir-Ab}$). Indeed, for each index $\mu \in M$ there exist an index

$\lambda \geq \varphi(\mu), \varphi'(\mu)$ such that

$$T(f_\mu) \cdot T(p_{\varphi(\mu)\lambda}) = T(q_{\mu\mu'}) \cdot T(f_{\mu'}) \cdot T(p_{\varphi'(\mu')\lambda})$$
$$\big(T(p_{\varphi(\mu)\lambda}) \cdot T(f_\mu) = T(p_{\varphi'(\mu)\lambda}) \cdot T(f_{\mu'}) \cdot T(q_{\mu\mu'})\big).$$

This means that the morphisms $(T(f_\mu), \varphi)$ and $(T(f_{\mu'}), \varphi^{'})$ are equivalent in the category $\mathbf{inv} - \mathbf{Ab}$ ($\mathbf{dir} - \mathbf{Ab}$). Thus, every morphism $\mathbf{f} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$ of $\mathbf{pro}-\mathbf{L}$ determines a morphism $T(\mathbf{f}) : T((\mathbf{X}, \mathbf{X_0})) \to T((\mathbf{Y}, \mathbf{Y_0}))(T(\mathbf{f}) : T((\mathbf{Y}, \mathbf{Y_0})) \to T((\mathbf{X}, \mathbf{X_0})))$ of the category $\mathbf{pro} - \mathbf{Ab}$ ($\mathbf{inj} - \mathbf{Ab}$). Thus, if (f_μ, φ) is a representative of morphism $\mathbf{f}$, then $T(\mathbf{f}) = [(T(f_\mu), \varphi)]$. Note that $T(\mathbf{1}_{(\mathbf{X}, \mathbf{X_0})}) = 1_{T((\mathbf{X}, \mathbf{X_0}))}$ and $T(\mathbf{g} \cdot \mathbf{f}) = T(\mathbf{f}) \cdot T(\mathbf{g})$ $(T(\mathbf{g} \cdot \mathbf{f}) = T(\mathbf{f}) \cdot T(\mathbf{g}))$. Thus, every covariant (contravariant) functor $T : \mathbf{L} \to \mathbf{Ab}$ induces the covariant (contravariant) functor

$$\mathrm{pro} - T : \mathbf{pro} - \mathbf{L} \to \mathbf{pro} - \mathbf{Ab} \ \ \big(\mathrm{inj} - T : \mathbf{pro} - \mathbf{L} \to \mathbf{inj} - \mathbf{Ab}\big).$$

Now assume that $\mathbf{L}$ is a dense subcategory of category $\mathbf{K}$ [16]. For each object $(X, X_0) \in \mathbf{K}$ consider a $\mathbf{L}$-expansion $\mathbf{p} : (X, X_0) \to (\mathbf{X}, \mathbf{X_0})$ [14] and the inverse (direct) system $T((\mathbf{X}, \mathbf{X_0}))$. If $\mathbf{p'} : (X, X_0) \to (\mathbf{X'}, \mathbf{X'_0})$ is another $\mathbf{L}$-expansion of (X, X_0), then there is an unique isomorphism $\mathbf{i} : \mathbf{X}(X, X_0) \to (\mathbf{X}^{'}, \mathbf{X}^{'}{}_\mathbf{0})$ of the category $\mathbf{pro} - \mathbf{L}$ such that $\mathbf{i} \cdot \mathbf{p} = \mathbf{p}^{'}$.

It is clear that $\mathbf{i}$ induces an isomorphism $T(\mathbf{i}) : T((\mathbf{X}, \mathbf{X_0})) \to T((\mathbf{X'}, \mathbf{X'_0}))$ $(T(\mathbf{i}) : T((\mathbf{X}^{'}, \mathbf{X}^{'}{}_\mathbf{0})) \to T((\mathbf{X}, \mathbf{X_0})))$ of inverse (direct) systems. Therefore, we can assign to every object $(X, X_0) \in \mathbf{K}$ the equivalence class of $T((\mathbf{X}, \mathbf{X_0}))$. We denote this class by $\mathrm{pro} - T(X, X_0)$ $(\mathrm{inj} - T(X, X_0)))$ and call pro-group (inj-group). Let $F : (X, X_0) \to (Y, Y_0)$ be a uniform shape morphism with representative $\mathbf{f} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{Y}, \mathbf{Y_0})$. The morphism $\mathbf{f}$ determines a morphism

$$T(\mathbf{f}) : T((\mathbf{X}, \mathbf{X_0})) \to T((\mathbf{Y}, \mathbf{Y_0})) \ \ \big(T(\mathbf{f}) : T((\mathbf{Y}, \mathbf{Y_0})) \to T((\mathbf{X}, \mathbf{X_0}))\big)$$

of category $\mathbf{pro}-\mathbf{Ab}$ ($\mathbf{inj}-\mathbf{Ab}$). If $\mathbf{f}^{'} : \mathbf{X}^{'} \to \mathbf{Y}^{'}$ is another representative of F, then $\mathbf{f}^{'} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{f}$ (here $\mathbf{i} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{X}^{'}, \mathbf{X}^{'}{}_\mathbf{0})$ and $\mathbf{j} : (\mathbf{Y}, \mathbf{Y_0}) \to (\mathbf{Y}^{'}, \mathbf{Y}^{'}{}_\mathbf{0})$ are the isomorphisms of the category **pro-L**). Hence,

$$T(\mathbf{f}^{'}) \cdot T(\mathbf{i}) = T(\mathbf{j}) \cdot T(\mathbf{f}) \ \ \big(T(\mathbf{i}) \cdot T(\mathbf{f}^{'}) = T(\mathbf{f}) \cdot T(\mathbf{j})\big).$$

As the isomorphisms $\mathbf{i} : (\mathbf{X}, \mathbf{X_0}) \to (\mathbf{X}^{'}, \mathbf{X}^{'}{}_\mathbf{0})$ and $\mathbf{j} : (\mathbf{Y}, \mathbf{Y_0}) \to (\mathbf{Y}^{'}, \mathbf{Y}^{'}{}_\mathbf{0})$ induce the isomorphisms $T(\mathbf{i})$ and $T(\mathbf{j})$ in the category $\mathbf{pro} - \mathbf{Ab}$ ($\mathbf{inj} - \mathbf{Ab}$) the morphisms $T(\mathbf{f})$ and $T(\mathbf{f}^{'})$ are coincided in the category $\mathbf{pro} - \mathbf{Ab}$ ($\mathbf{inj} - \mathbf{Ab}$). Thus, we can associate to every uniform shape morphism $F : (X, X_0) \to (Y, Y_0)$ well defined morphism

$$\mathrm{pro} - T(F) : \mathrm{pro} - T(X, X_0) \to \mathrm{pro} - T(Y, Y_0)$$
$$\big(\mathrm{inj} - T(F) : \mathrm{inj} - T(Y, Y_0) \to \mathrm{inj} - T(X, X_0)\big).$$

Consequently, we have the following

Theorem 1. *For each covariant (contravariant) functor* $T : \mathbf{L} \to \mathbf{Ab}$ *there exists a covariant (contravariant) functor*

$$\text{pro} - T : \mathbf{uSH}_{(\mathbf{K},\mathbf{L})} \to \mathbf{pro} - \mathbf{Ab} \ \big(\text{inj} - T : \mathbf{uSH}_{(\mathbf{K},\mathbf{L})} \to \mathbf{inj} - \mathbf{Ab}\big)$$

such that for each objects $(X, X_0) \in \mathbf{uSH}_{(\mathbf{K},\mathbf{L})}$

$$(\text{pro}{-}T)((X, X_0)) = \text{pro}{-}T((X, X_0)) \ \big(\text{inj}{-}T)((X, X_0)) = \text{inj}{-}T((X, X_0))\big)$$

and for each uniform shape morphism $F : (X, X_0) \to (Y, Y_0)$ *of* $\mathbf{uSH}_{(\mathbf{K},\mathbf{L})}$

$$(\text{pro} - T)(F) = \text{pro} - T(F) \ \big((\text{inj} - T)(F) = \text{inj} - T(F)\big).$$

For every object $(X, X_0) \in \mathbf{uSH}_{(\mathbf{K},\mathbf{L})}$ define a group

$$\check{T}((X, X_0)) = \lim_{\leftarrow} \text{pro} - T((X, X_0)) \ \big(\hat{T}((X, X_0)) = \lim_{\rightarrow} \text{inj} - T((X, X_0))\big)$$

and for each morphism $F : (X, X_0) \to (Y, Y_0)$ of $\mathbf{uSH}_{(\mathbf{K},\mathbf{L})}$ define a homomorphism of groups

$$\check{F} = \check{T}(F) = \lim_{\leftarrow} \text{pro} - T(F) : \check{T}((X, X_0)) \to \check{T}((Y, Y_0))$$

$$\big(\hat{F} = \hat{T}(F) = \lim_{\leftarrow} \text{dir} - T(F : \hat{T}((Y, Y_0)) \to \hat{T}((X, X_0))\big).$$

Thus, we have obtained

Theorem 2. *For every covariant (contravariant) functor* $T : \mathbf{L} \to \mathbf{Ab}$ *there exist covariant (contravariant) functor*

$$\check{T} : \mathbf{uSH}_{(\mathbf{K},\mathbf{L})} \to \mathbf{Ab}(\widehat{\mathbf{T}} : \mathbf{uSH}_{(\mathbf{K},\mathbf{L})} \to \mathbf{Ab})$$

which assigns to every object $(X, X_0) \in \mathbf{uSH}_{(\mathbf{K},\mathbf{L})}$ *the group* $\check{T}((X, X_0))$ $(\hat{T}((X, X_0)))$ *and to every uniform shape morphism* F *the homomorhism*

$$\check{F} : \check{T}((X, X_0)) \to \check{T}((Y, Y_0)) \ \big(\hat{F} : \hat{T}((Y, Y_0)) \to \hat{T}((X, X_0))\big).$$

Corollary 3. *If* $\text{ush}_{(K,L)}((X, X_0)) = \text{ush}_{(K,L)}((Y, Y_0))$*, then*

$$\check{T}((X, X_0)) = \check{T}((Y, Y_0)) \ \big(\hat{T}((X, X_0)) = \hat{T}((Y, Y_0))\big).$$

Using the above obtained results and the results of ([4], [6–8], [18–20]) we can construct uniform shape invariant extensions of functors defined on the categories $\mathbf{HANRU^2}$, $\mathbf{HUPol^2}$ and $\mathbf{H}(_{\mathbf{p}}\mathbf{Unif^2} \cap \mathbf{ANRU^2})$. Let $\mathbf{K} = \mathbf{HUnif^2}$ and $\mathbf{L} = \mathbf{HANRU^2}$**.** We have the following results.

Corollary 4. *For each covariant (contravariant) functor* $T : \mathbf{HANRU^2} \to \mathbf{Ab}$ *there exists a covariant (contravariant) functor*

$$(\text{pro} - T) : \mathbf{uSH}^{\mathbf{2}}_{(\mathbf{HUnif^2},\mathbf{HANRU^2})} \to \mathbf{pro} - \mathbf{Ab}$$

$$\big((\text{inj} - T) : \mathbf{uSH}^{\mathbf{2}}_{(\mathbf{HUnif^2},\mathbf{HANRU^2})} \to \mathbf{inj} - \mathbf{Ab}\big)$$

such that for each $(X, X_0) \in \text{ob}(\mathbf{uSH}^{\mathbf{2}}_{(\mathbf{HUnif^2},\mathbf{HANRU^2})})$

$$(\text{pro} - T)(X, X_0) = \text{pro} - T(X, X_0) \ \big((\text{inj} - T)(X, X_0) = \text{inj} - T(X, X_0)\big)$$

and for each $F \in \mathbf{uS}^{\mathbf{2}}_{(\mathbf{HUnif^2},\mathbf{HANRU^2})}((X,X_0),(Y,Y_0))$

$$(\mathrm{pro}-T)(F) = \mathrm{pro}-T(F) \ \ \big((\mathrm{inj}-T)(F) = \mathrm{inj}-T(F)\big).$$

Corollary 5. *Let* $(X,X_0),(Y,Y_0) \in \mathbf{HUnif^2}$.
If $\mathrm{ush}^{\mathbf{2}}_{(\mathrm{HUnif^2},\mathrm{HANRU^2})}(X,X_0) = \mathrm{ush}^{\mathbf{2}}_{(\mathrm{HUnif^2},\mathrm{HANRU^2})}(Y,Y_0)$*, then*

$$\mathrm{pro}-T(X,X_0) = \mathrm{pro}-T(Y,Y_0) \ \ \big(\mathrm{inj}-T(X,X_0) = \mathrm{inj}-T(Y,Y_0)\big).$$

In this case for each pair (X,X_0) of uniform spaces also are defined the groups

$$\check{T}(X,X_0) = \lim_{\leftarrow} \mathrm{pro}-T(X,X_0) \ \ \big(\hat{T}(X,X_0) = \lim_{\rightarrow} \mathrm{inj}-T(X,X_0)\big)$$

and for each uniform shape morphism $F:(X,X_0)\to(Y,Y_0)$ are defined the morphism of abelian groups

$$\check{T}(F) = \lim_{\leftarrow} \mathrm{pro}-T(F) : \check{T}(X,X_0) \to \check{T}(Y,Y_0)$$
$$\big(\hat{T}(F) = \lim_{\leftarrow} \mathrm{inj}-T(F) : \hat{T}(X,X_0) \to \hat{T}(Y,Y_0)\big).$$

Corollary 6. *Let* (X,X_0) *and* (Y,Y_0) *be pairs of uniform spaces. If* $\mathrm{ush}^{2}_{(\mathrm{HUnif^2},\mathrm{HANRU^2})}(X,X_0) = \mathrm{ush}^{2}_{(\mathrm{HUnif^2},\mathrm{HANRU^2})}(Y,Y_0)$*, then*

$$\check{T}(X,X_0) = \check{T}(Y,Y_0) \ \ \big(\hat{T}(X,X_0) = \hat{T}(Y,Y_0)\big).$$

Now consider the pair of categories $\mathbf{K} = \mathbf{HfUnif^2}$ and $\mathbf{L} = \mathbf{HUpol^2}$. We have the following results.

Corollary 7. *For each covariant* (*contravariant*) *functor* $T:\mathbf{HUpol^2}\to \mathbf{Ab}$ *there exists a covariant* (*contravariant*) *functor*

$$\mathrm{pro}-T : \mathbf{uSH}^{\mathbf{2}}_{(\mathbf{HfUnif^2},\mathbf{HUpol^2})} \to \mathbf{pro}-\mathbf{Ab}$$
$$\big(\mathrm{inj}-T : \mathbf{uSH}^{\mathbf{2}}_{(\mathbf{HfUnif^2},\mathbf{HUpol^2})} \to \mathbf{pro}-\mathbf{Ab}\big)$$

such that for each object $(X,X_0) \in \mathbf{uSH^2}_{(\mathbf{HfUnif^2},\mathbf{HUpol^2})}$

$$(\mathrm{pro}-T)(X,X_0) = \mathrm{pro}-T(X,X_0) \ \ \big((\mathrm{inj}-T)(X,X_0) = \mathrm{inj}-T(X,X_0)\big)$$

and for each morphism $F \in \mathbf{uSH^2}_{(\mathbf{HfUnif^2},\mathbf{HUpol^2})}((X,X_0),(Y,Y_0))$

$$(\mathrm{pro}-T)(F) = \mathrm{pro}-T(F) \ \ \big((\mathrm{inj}-T)(F) = \mathrm{inj}-T(F)\big).$$

In this case for every pair of finitistic uniform spaces (X,X_0) and for each uniform shape morphism $F:(X,X_0)\to(Y,Y_0)$ of $\mathbf{uSH^2}_{(\mathbf{HfUnif^2},\mathbf{HUpol^2})}$ also are defined the groups

$$\check{T}(X,X_0) = \lim_{\leftarrow} \mathrm{pro}-T(X,X_0) \ \ \big(\hat{T}(X,X_0) = \lim_{\rightarrow} \mathrm{inj}-T(X,X_0)\big)$$

and the homomorphisms of groups

$$\check{F} = \lim_{\leftarrow} \mathrm{pro} - T(F) : \check{T}(X, X_0) \to \check{T}(Y, Y_0)$$
$$\big(\hat{F} = \lim_{\rightarrow} \mathrm{inj} - T(F) : \hat{T}(Y, Y_0) \to \hat{T}(X, X_0)\big).$$

Corollary 8. *Let* $(X, X_0), (Y, Y_0) \in {}_{\mathbf{f}}\mathbf{Unif^2}$. *If* $\mathrm{ush}^2_{(\mathrm{HfUnif},\mathrm{HUpol})}(X, X_0) = \mathrm{ush}^2_{(\mathrm{HfUnif},\mathrm{HUpol})}(Y, Y_0)$, *then*

$$\mathrm{pro} - T(X, X_0){=}\mathrm{pro} - T(Y, Y_0) \ \ \big(\mathrm{inj} - T(X, X_0) = \mathrm{inj} - T(Y, Y_0)\big).$$

Corollary 9. *Let* $\mathrm{ush}^2_{(\mathrm{HfUnif},\mathrm{HUpol})}(X, X_0) = \mathrm{ush}^2_{(\mathrm{HfUnif},\mathrm{HUpol})}(Y, Y_0)$ *for the pairs* $(X, X_0), (Y, Y_0) \in \mathbf{fUnif^2}$. *Then*

$$\check{T}(X, X_0) = \check{T}(Y, Y_0) \ \ \big(\hat{T}(X, X_0) = \hat{T}(Y, Y_0)\big).$$

Let $(\mathbf{K}, \mathbf{L})$ be the pair of categories $\mathbf{K} = \mathbf{HpUnif}$ and $\mathbf{L} = \mathbf{H}(\mathbf{ANRU^2} \cap \mathbf{pUnif^2})$.

Let $\mathbf{puSH^2}$ be the precompact shape category constructed in [4] for given pair of categories. We have the following results.

Corollary 10. *For every covariant* (*contravariant*) *functor* $T : \mathbf{H}(\mathbf{ANRU^2} \cap \mathbf{pUnif^2}) \to \mathbf{Ab}$ *there exists a covariant* (*contravariant*) *functor*

$$\mathrm{pro} - T{:}\mathbf{puSH^2} \to \mathrm{pro} - \mathbf{Ab} \ \ \big(\mathrm{inj} - T : \mathbf{puSH^2} \to \mathbf{inj} - \mathbf{Ab}\big)$$

which assigns to every precompact space (X, X_0) *the* pro-*group* (inj-*group*)

$$(\mathrm{pro}{-}T)((X, X_0)) = \mathrm{pro}{-}T((X, X_0)) \ \ \big((\mathrm{inj}{-}T)((X, X_0)) = \mathrm{inj}{-}T((X, X_0))\big)$$

and to each precompact uniform shape morphism $F : (X, X_0) \to (Y, Y_0)$ *the morphism*

$$(\mathrm{pro} - T)(F) = \mathrm{pro} - T(F) \ \ \big((\mathrm{inj} - T)(F) = \mathrm{inj} - T(F)\big).$$

Corollary 11. *If* $\mathrm{push}((X, X_0)) = \mathrm{push}((Y, Y_0))$ *for* $(X, X_0), (Y, Y_0) \in \mathbf{pUnif^2}$, *then*

$$\mathrm{pro} - T((X, X_0)) = \mathrm{pro} - T((Y, Y_0)) \ \ \big(\mathrm{inj} - T((X, X_0)) = \mathrm{inj} - T((Y, Y_0))\big).$$

For every precompact uniform space X *define the group*

$$\check{T}((X, X_0)) = \lim_{\leftarrow} \ \mathrm{pro} - \check{T}((X, X_0)) \ \ \big(\hat{T}((X, X_0)) = \lim_{\rightarrow} \ \mathrm{inj} - \hat{T}((X, X_0))\big)$$

and for each uniform shape morphism $F{:}(X, X_0) \to (Y, Y_0)$ *define a mophism*

$$\check{T}(F) = \lim_{\leftarrow} \ \mathrm{pro} - \check{T}(F) \ \ \big(\hat{T}(F) = \lim_{\rightarrow} \mathrm{inj} - \hat{T}(F)\big).$$

Corollary 12. *If for the pair of precompact uniform spaces* (X, X_0) *and* (Y, Y_0) $\mathrm{push}((X, X_0)) = \mathrm{push}((Y, Y_0))$, *then*

$$\check{T}((X, X_0)) = \check{T}((Y, Y_0)) \ \ \big(\hat{T}((X, X_0)) = \hat{T}((Y, Y_0))\big).$$

II. Uniform Continuous functors. Now we prove the following continuity theorem for the functor $\check{T}$ $(\hat{T})$.

Theorem 13. *Let* $\mathbf{p}:(X,X_0) \to (\mathbf{X},\mathbf{X_0})$ *be a* $\mathbf{K}$*-expansion of object* $(X,X_0) \in \mathbf{K}$ *and let*

$$(\mathbf{p}) : \check{T}((X,X_0)) \to \check{T}((\mathbf{X},\mathbf{X_0})) \ \big(\hat{T}(\mathbf{p}) : \hat{T}((\mathbf{X},\mathbf{X_0})) \to \hat{T}((X,X_0))\big)$$

be the induced by $\mathbf{p}$ *morphism of the category* $\mathbf{pro}-\mathbf{Ab}$ $(\mathbf{inj}-\mathbf{Ab})$. *Then the homomorphism*

$$\check{p} : \check{T}((X,X_0)) \to \lim_{\leftarrow} \check{T}((\mathbf{X},\mathbf{X_0})) \ \big(\hat{p} : \lim_{\leftarrow} \hat{T}((\mathbf{X},\mathbf{X_0})) \to \hat{T}((X,X_0))\big)$$

induced by $\check{T}(\mathbf{p}$ $(\hat{T}(\mathbf{p}))$ *is an isomorphism of groups.*

Proof. Consider only covariant case. The proof for contravariant case is dual.

By π_λ denote the protection

$$\pi_\lambda : \lim_{\leftarrow} \check{T}((X,X_0)) \to \check{T}((X_\lambda, X_{0\ \lambda})), \ \ \lambda \in \Lambda \ .$$

Note that for each index $\lambda \in \Lambda$

$$\pi_\lambda \cdot \check{p} = \check{T}(p_\lambda) \tag{1}$$

and for each pair $\lambda \le \lambda^{'}$

$$\check{T}(p_{\lambda\lambda'}) \cdot \pi_{\lambda'} = \pi_\lambda. \tag{2}$$

Now consider a $\mathbf{K}$-expansion $\mathbf{q}:(X,X_0) \to (\mathbf{Y},\mathbf{Y_0}) = \{(Y_\mu, Y_{0\mu}), q_{\mu\mu'}, M\}$ of $(X,X_0) \in \mathbf{K}$. By definition of $\check{T}((X,X_0))$, $\check{T}(\mathbf{q}) = \check{\mathbf{q}} : \check{T}((X,X_0)) \to \check{T}((\mathbf{Y},\mathbf{Y_0}))$ is an inverse limit of $\check{T}((\mathbf{Y},\mathbf{Y_0}))$. Note that there is a morphism $\mathbf{f} : (\mathbf{X},\mathbf{X_0}) \to (\mathbf{Y},\mathbf{Y_0})$ of the category $\mathbf{pro}-\mathbf{K}$ such that

$$\mathbf{f} \cdot \mathbf{p} = \mathbf{q}. \tag{3}$$

It is clear that the morphism $\mathbf{f}$ induces the morphism $\check{T}(\mathbf{f}) : \check{T}((X,X_0)) \to \check{T}((Y,Y_0))$ of pro-groups such that

$$\check{T}(\mathbf{f}) \cdot \check{T}(\mathbf{p}) = \check{T}(\mathbf{q}). \tag{4}$$

The morphism $\check{T}(\mathbf{f})$ defines a homomorphism

$$\check{f} : \lim_{\leftarrow} \check{T}((\mathbf{X},\mathbf{X_0})) \to \check{T}((X,X_0))$$

of abelian groups such that

$$\check{T}(\mathbf{f}) \cdot \pi = \check{T}(\mathbf{q}) \cdot \check{f}, \tag{5}$$

where $\pi = (\pi_\lambda)$. If we assume that (f_μ, φ) is a representative of $\mathbf{f}$, then (5) yields

$$\check{T}(f_\mu) \cdot \pi_{\varphi(\mu)} = \check{T}(q_\mu) \cdot \check{f} \tag{6}$$

for each $\mu \in M$. Now we show that $\check{f}$ is inverse of $\check{p}$. Indeed, using formulas (6), (1) and (4) we obtain:

$$\check{T}(q_\mu) \cdot \check{f} \cdot \check{p} = \check{T}(f_\mu) \cdot \pi_{\varphi(\mu)} \cdot \check{p} = \check{T}(f_\mu) \cdot \check{T}_{p_{\varphi(\mu)}} = \check{T}(q_\mu), \mu \in M.$$

This means that

$$\check{T}(\mathbf{q}) \cdot \check{f} \cdot \check{p} = \check{T}(\mathbf{q}). \tag{7}$$

In (7) $\check{T}(\mathbf{q}) : \check{T}((X, X_0)) \to \check{T}((\mathbf{Y}, \mathbf{Y_0}))$ is an inverse limit of $\check{T}((X, X_0))$, i.e.,

$$\check{f} \cdot \check{p} = 1_{\check{T}((X,X_0))}.$$

Now we establish that $\check{f} \cdot \check{p} = 1_{\underleftarrow{\lim} \check{T}((X,X_0))}$. For this aim we prove that $\pi_\lambda \cdot \check{f} \cdot \check{p} = \pi_\lambda$ for every $\lambda \in \Lambda$. By formula (1) this is equivalent to

$$\check{T}(p_\lambda) \cdot \check{f} = \pi_\lambda, \lambda \in \Lambda \ . \tag{8}$$

Let $\mathbf{r} = (r_\nu) : (X_\lambda, X_{0\lambda}) \to (\mathbf{Z}, \mathbf{Z_0}) = \{(Z_\nu, Z_{0\nu}), r_{\nu\nu'}, N\}$ be a $\mathbf{K}$-expansion of $(X_\lambda, X_{0\lambda})$. Since $\mathbf{q} : (X, X_0) \to (\mathbf{Y}, \mathbf{Y_0})$ is a $\mathbf{K}$-expansion of (X, X_0) and $(Z_\nu, Z_{0\nu}) \in \mathbf{K}$, then there exist an index $\mu \in M$ and a uniform map $g : (Y_\lambda, Y_{0\lambda}) \to (Z_\nu, Z_{0\nu})$ such that

$$r_\nu \cdot p_\lambda = g \cdot q_\mu. \tag{9}$$

By (3) we have

$$r_\nu \cdot p_\lambda = g \cdot f_\mu \cdot p_{\varphi(\mu)}.$$

By property (E2) of expansion [17] there is an index $\lambda' \geq \lambda$, $\varphi(\mu)$ such that

$$r_\nu \cdot p_{\lambda\lambda'} = g \cdot f_\mu \cdot p_{\varphi(\mu)\lambda'}. \tag{10}$$

Applying the functor $\check{T}$ to (10) and using (8) and (2) we obtain:

$$\check{T}(r_\nu) \cdot \check{T}(p_\lambda) \cdot \check{f} = \check{T}(g) \cdot \check{T}(f_\mu) \cdot \check{T}_{p_{\varphi(\mu)}} \cdot \check{f} = \check{T}(g) \cdot \check{T}(f_\mu) \cdot \pi_{\varphi(\mu)} =$$

$$= \check{T}(g) \cdot \check{T}(f_\mu) \cdot \check{T}(p_{\varphi(\mu)\lambda'}) \cdot \pi_{\lambda'} = \check{T}(r_\nu) \cdot \check{T}(p_{\lambda\lambda'}) \cdot \pi_{\lambda'} = \check{T}(r_\nu) \cdot \pi_{\lambda'}.$$

This means that

$$\check{T}(p_\lambda) \cdot \check{f} = \pi_\lambda \ , \ \lambda \in \Lambda.$$

Consequently, for each index $\lambda \in \Lambda$

$$\pi_\lambda \cdot \check{p} \cdot \check{f} = \pi_\lambda.$$

Thus,

$$\check{p} \cdot \check{f} = 1_{\underleftarrow{\lim} \check{T}((X,X_0))}. \qquad \square$$

Corollary 14. *Let* $\mathbf{p}:(X, X_0) \to (\mathbf{X}, \mathbf{X_0})$ *be an* $\mathbf{HUnif^2}$*-expansion of pair* $(X, X_0) \in \mathbf{HUnif^2}$. *Then the homomorphism*

$$\check{p} : \check{T}(X, X_0) \to \underleftarrow{\lim} \check{T}(\mathbf{X}, \mathbf{X_0}) \ \left(\hat{p} : \underrightarrow{\lim} \hat{T}(\mathbf{X}, \mathbf{X_0}) \to \hat{T}(X, X_0)\right)$$

induced by the morphism

$$\check{T}(\mathbf{p}){:}\check{T}(X,X_0)\to\check{T}(\mathbf{X},\mathbf{X_0})\ \ \big(\hat{T}(\mathbf{p}){:}\hat{T}(\mathbf{X},\mathbf{X_0})\to\hat{T}(X,X_0)\big)$$

is an isomorphism of groups.

Corollary 15. *Let* $\mathbf{p}{:}(X,X_0)\to(\mathbf{X},\mathbf{X_0})$ *be an* $\mathbf{H_fUnif^2}$*-expansion of* $(X,X_0)\in\mathbf{fUnif^2}$*. Then the homomorphism*

$$\check{p}:\check{T}(X,X_0)\to\lim_{\leftarrow}\check{T}(\mathbf{X},\mathbf{X_0})\ \ \big(\hat{p}:\lim_{\rightarrow}\hat{T}(\mathbf{X},\mathbf{X_0})\to\hat{T}(X,X_0)\big)$$

induced by the morphism

$$\check{T}(\mathbf{p}){:}\check{T}(X,X_0)\to\check{T}(\mathbf{X},\mathbf{X_0})\ \ \big(\hat{T}(\mathbf{p}){:}\hat{T}(\mathbf{X},\mathbf{X_0})\to\hat{T}(X,X_0)\big)$$

is an isomorphism of groups.

Corollary 16. *Let* $\mathbf{p}{:}(X,X_0)\to(\mathbf{X},\mathbf{X_0})$ *be an* $\mathbf{HpUnif^2}$*-expansion of* $(X,X_0)\in\mathbf{HpUnif^2}$*. Then the homomorphism*

$$\check{p}:\check{T}(X,X_0)\to\lim_{\leftarrow}\check{T}(\mathbf{X},\mathbf{X_0})\ \ \big(\hat{p}:\lim_{\rightarrow}\hat{T}(\mathbf{X},\mathbf{X_0})\to\hat{T}(X,X_0)\big)$$

induced by the morphism

$$\check{T}(\mathbf{p}){:}\check{T}(X,X_0)\to\check{T}(\mathbf{X},\mathbf{X_0})\ \ \big(\hat{T}(\mathbf{p}){:}\hat{T}(\mathbf{X},\mathbf{X_0})\to\hat{T}(X,X_0)\big)$$

is an isomorphism of groups.

According to [21] we give formulation of the uniform continuity axiom. Let $T\ :\ \mathbf{Unif^2}\to\mathbf{Ab}$ be a covariant (contravariant) functor. The funtcor T is continuous at pair (X,X_0) of uniform spaces if it satisfies the condition:

UCA). If $\mathbf{p}$=(p_α):$(X,\ X_0)\to(\mathbf{X}$,$\mathbf{X_0})$=$(X_\alpha,p_{\alpha\alpha'},A)$ is an uniform resolution [18] of $(X,\ X_0)\in\mathbf{Unif}$, than $T(\mathbf{p})\ =\ (T(p_\alpha))\ :\ T(X,\ X_0)\to T(\mathbf{X},\ \mathbf{X_0})$=$(T(X_\alpha,X_{\alpha'})$, $T(p_{\alpha\alpha'})$,$A)$ is an inverse limit. We say that T is uniform continuous functor if T is continuous at any pair $(X,\ X_0)$ of uniform spaces.

Let $T:\mathbf{ANRU^2}\to\mathbf{Ab}$ be a covariant (contravariant) functor. We say that T satisfies the uniform homotopy axiom, if $f:(X,\ X_0)\to(Y,\ Y_0)$ is uniform homotopy to $g:(X,X_0)\to(Y,\ Y_0)$, then $T(f)=T(g)$. Using the method of proof of Theorem 13 we can prove the following theorems.

Theorem 17. *Let* $T:\mathbf{ANRU^2}\to\mathbf{Ab}$ *be a covariant* (*contravariant*) *functor satisfying the uniform homotopy axiom. Then there exists the covariant* (*contravariant*) *uniform continuous functor* $\check{T}:\mathbf{fUnif^2}\to\mathbf{Ab}$ ($\hat{T}:\mathbf{fUnif^2}\to\mathbf{Ab}$)*, which is an extension of* T *and satisfies the uniform homotopy axiom.*

Theorem 18. *Let* $T:\mathbf{fUnif^2}\to\mathbf{Ab}$ *be a uniform continuous covariant* (*contravariant*) *functor satisfying the uniform homotopy axiom. Then* T *and* $(\check{T}_{|\mathbf{ANRU^2}})(\hat{T}_{|\mathbf{ANRU^2}})$ *are equivalent functors.*

III. Completions of pairs of uniform spaces. We have the following

Theorem 19. *Let (cX, cX_0) be the completion of pair (X, X_0) of finitistic uniform spaces. Then*

$$\mathrm{ush}^2(X, X_0) = \mathrm{ush}^2(cX, cX_0).$$

Proof. Consider X and cX as uniform subspaces of some $M \in \mathbf{ANRU}$. Let $A = \{a\}$ be the family of all uniform coverings of M of the form $a = \{U_x; x \in M\}$ (see [18]). Consider the $\mathbf{HANRU^2}$-expansions of pairs (X, X_0) and (cX, cX_0)

$$\mathbf{p} = \{p_\alpha\} : (X, X_0) \to (\mathbf{X}, \mathbf{X_0}) = \big\{(X_\alpha, X_{0\alpha}), i_{\alpha\alpha'}, A\big\},$$

$$\mathbf{r} = \{r_\alpha\} : (cX, cX_0) \to (\mathbf{Z}, \mathbf{Z_0}) = \big\{(Z_\alpha, Z_{0\alpha}), j_{\alpha\alpha'}, A\big\},$$

where $X_\alpha = \mathrm{st}(X, \alpha)$, $X_{0\alpha} = \mathrm{st}(X_0, \alpha)$, $Z_\alpha = \mathrm{st}(cX, \alpha)$, $Z_{0\alpha} = \mathrm{st}(cX_0, \alpha)$ for each index $\alpha \in A$ and $i_{\alpha\alpha'}\colon (X_{\alpha'}, X_{0\alpha'}) \to (X_\alpha, X_{0\alpha})$, $j_{\alpha\alpha'}\colon(Z_{\alpha'}, Z_{0\alpha'}) \to (Z_\alpha, Z_{0\alpha})$, $p_\alpha\colon(X, X_0) \to (X_\alpha, X_{0\alpha})$, $r_\alpha\colon(cX, cX_0) \to (Z_\alpha, Z_{0\alpha})$ are uniform inclusion maps for each pair $\alpha \le \alpha'$ in A. For each $\alpha \in A$ there is an uniform refinement $\widetilde{\alpha}$ of α consisting of open sets. We denote by $\tilde{A} = \{\widetilde{\alpha}\}$ the family of all open uniform coverings of M. Consider the inverse systems

$$(\widetilde{\mathbf{X}}, \widetilde{\mathbf{X_0}}) = \{(X_{\widetilde{\alpha}}, \ X_{0\,\widetilde{\alpha}}), i_{\widetilde{\alpha}\,\widetilde{\alpha}'}, A\}, \ \ (\widetilde{\mathbf{Z}}, \widetilde{\mathbf{Z_0}}) = \{(Z_{\widetilde{\alpha}}, Z_{0\,\widetilde{\alpha}}), j_{\widetilde{\alpha}\,\widetilde{\alpha}'}, A\}$$

and the morphisms

$$\widetilde{\mathbf{p}} = \{p_{\widetilde{\alpha}}\} : (X, X_0) \to (\widetilde{\mathbf{X}}, \widetilde{\mathbf{X_0}}), \ \ \widetilde{\mathbf{r}} = \{r_{\widetilde{\alpha}}\} : (cX, cX_0) \to (\widetilde{\mathbf{Z}}, \widetilde{\mathbf{Z_0}}).$$

The morphisms $\widetilde{\mathbf{p}}\colon(X, X_0) \to (\widetilde{\mathbf{X}}, \widetilde{\mathbf{X_0}})$ and $\widetilde{\mathbf{r}}\colon(cX, cX_0) \to (\widetilde{\mathbf{Z}}, \widetilde{\mathbf{Z_0}})$ also are the $\mathbf{HANRU^2}$-expansions of (X, X_0) and (cX, cX_0), respectively. Note that, for each index $\widetilde{\alpha} \in \tilde{A}$, $X_{\widetilde{\alpha}} = Z_{\widetilde{\alpha}}$, $X_{0\,\widetilde{\alpha}} = Z_{0\,\widetilde{\alpha}}$. Indeed, it is clear that $X_{\widetilde{\alpha}} \subset Z_{\widetilde{\alpha}}$ and $X_{0\,\widetilde{\alpha}} \subset Z_{0\,\widetilde{\alpha}}$. Now show that $Z_{\widetilde{\alpha}} \subset X_{\widetilde{\alpha}}$, $Z_{0\,\widetilde{\alpha}} \subset X_{0\,\widetilde{\alpha}}$. By definition $Z_{\widetilde{\alpha}} = \mathrm{st}(cX, \widetilde{\alpha}) = \cup\{U_{\widetilde{\alpha}}\colon U_{\widetilde{\alpha}} \cap cX \neq \emptyset\}$. The nonempty set $U_{\widetilde{\alpha}} \cap cX$ is open in cX. From $cX = \overline{X}$ it follows that $\emptyset \neq (U_{\widetilde{\alpha}} \cap cX) \cap X = U_{\widetilde{\alpha}} \cap X$. Hence, $U_{\widetilde{\alpha}} \in \mathrm{st}(X, \widetilde{\alpha})$ and, consequently, $\mathrm{st}(cX, \widetilde{\alpha}) \subset \mathrm{st}(X, \widetilde{\alpha})$. Analogously, $\mathrm{st}(cX_0, \widetilde{\alpha}) \subset \mathrm{st}(X_0, \widetilde{\alpha})$, $\widetilde{\alpha} \in \tilde{A}$. Consequently, $(Z_{\widetilde{\alpha}}, Z_{0\,\widetilde{\alpha}}) = (X_{\widetilde{\alpha}}, X_{0\,\widetilde{\alpha}})$, $\widetilde{\alpha} \in \tilde{A}$. Hence, $(\widetilde{\mathbf{X}}, \widetilde{\mathbf{X_0}})$ and $(\widetilde{\mathbf{Z}}, \widetilde{\mathbf{Z_0}})$ are isomorphic objects of the category $\mathbf{pro} - \mathbf{HANRU^2}$. □

Corollary 20. *Every pair of precompact uniform spaces has a uniform shape of pair of compact spaces.*

Corollary 21. *For each pair (X, X_0) of finitistic uniform spaces*

$$\check{T}(X, X_0) = \check{T}(cX, cX_0) \ \ \big(\hat{T}(X, X_0) = \hat{T}(cX, cX_0)\big).$$

Corollary 22. *For each pair (X, X_0) of precompact uniform spaces*

$$\check{T}(X, X_0) = \check{T}(cX, cX_0) \ \ \big(\hat{T}(X, X_0) = \hat{T}(cX, cX_0)\big).$$

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