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ON SOLVING THE DIRICHLET GENERALIZED PROBLEM FOR A HARMONIC FUNCTION IN THE CASE OF INFINITE PLANE WITH HOLES

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Abstract. An algorithm for an approximate solution of definite type Dirichlet generalized problem is given. It consists of the following stages: 1) reduction of the Dirichlet generalized problem to an ordinary new (auxiliary) problem for harmonic function; 2) approximate solution of the new problem by the modified version of MFS (the method of fundamental solutions); 3) definition of the approximate solution of the posed generalized problem by the solution of the new problem. Examples of application of the proposed algorithm and the results of numerical experiments are given.

რეზიუმე. ნაშრომში მოცემულია გარკვეული სახის დირიხლეს განზოგადებული ამოცანის მიახლოებითი ამოხსნის ალგორითმი, რომელიც შედგება შემდეგი ეტაპებისგან: 1) დირიხლეს განზოგადებული ამოცანის დაყვანა ჩვეულებრივ ახალ ამოცანაზე ჰარმონიული ფუნქციისათვის; 2) ახალი ამოცანის მიახლოებითი ამოხსნა ფუნდამენტურ ამოხსნათა მეთოდის მოდიფიცირებული ვერსიის საშუალებით; 3) დასმული განზოგადებული ამოცანის ამონახსნის განსაზღვრა ახალი ამოცანის ამონახსნის საშუალებით. მოცემულია შემოთავაზებული ალგორითმის გამოყენების მაგალითები და რიცხვითი ექსპერიმენტების შედეგები.

1. Introduction

Let a domain D be the infinite plane $z = x + iy \equiv (x, y)$ with the holes B_i $(i = 1, 2, \dots, m)$, which are bounded by closed piecewise smooth contours S_i $\big(S_i = \bigcup\limits_{j=1}^{l} S_i^j,\ i = 1, 2, \dots, m\big)$, respectively, having no multiple points. It is evident that the whole boundary of domain D will be $S = \bigcup\limits_{i=1}^{m} S_i$. Moreover, we assume that parametric equations of the smooth curves S_i^j are given and $S_k \bigcap S_j = \varnothing$ for $k = j$.

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It is known that the classical statement of the Dirichlet ordinary boundary problem for harmonic function requires continuity of the boundary function. However, in practical problems (for example, during determination of the temperature of the thermal field or of the potential of the electric field and so on) there are cases when the boundary function is piecewise continuous and therefore it is necessary to consider the Dirichlet generalized problem (see [1,2,3,4]).

A. *On the boundary S of the domain D a function $g(\tau)$ is given which is continuous everywhere, except a finite number of points $\tau_1, \tau_2, \dots, \tau_n$ at which it has first kind break points. It is required to find a function $u(z) \equiv u(x,y) \in C^2(D) \bigcap C(\overline{D} \setminus \{\tau_1, \tau_2, \dots, \tau_n\})$ satisfying the conditions*

$$\Delta u(z) = 0, \quad z \in D, \tag{1.1}$$

$$u(\tau) = g(\tau), \quad \tau \in S, \quad \tau \neq \tau_k \quad (k = 1, 2, \dots, n), \tag{1.2}$$

$$u(z) = c + O\left(\frac{1}{|z|}\right) \quad for \quad |z| \to \infty, \tag{1.3}$$

where Δ is the Laplace operator and c is a real constant provided $|c| < \infty$.

It is known (see [1,2]) that problem (1.1)–(1.3) is correct, i. e., the solution exists, is unique, depends continuously on the data, and for the generalized solution $u(z)$ the generalized extremum principle is valid:

$$\min_{z \in S} u(z) < \underset{z \in D}{u(z)} < \max_{z \in S} u(z), \tag{1.4}$$

where for $z \in S$ it is assumed that $z \neq \tau_k$ $(k = \overline{1, n})$.

It should be noted that condition (1.3) plays an important role in the extremum principle (1.4) and, consequently, in the theorem on uniqueness of the solution to Problem A (see [1,2]). To see that this is so, it is sufficient, e. g., to consider an exterior of the disk with the center at the origin and of radius r, as the domain D, i. e. $S : |z| = r$. If the function $u_1(z)$ is a solution of problem A without condition (1.3), then the functions of the type $u_2(z) = u_1(z) + k \ln \dfrac{|z|}{r}$ are the solutions of problem A, where $k \neq 0$ is a real constant.

It can be easily shown that if we fix in advance the value of the constant c, this will be a rather strong restriction. Really, since under conditions (1.1), (1.2), (1.3) for the function $u(x, y)$ the minimax principle is fulfilled (see [1,2]), hence problem A with hitherto fixed c may turn out generally unsolvable. To avoid this fact the constant c should be defined from the condition (1.2) while solving problem A.

If $g^-(\tau_k)$ and $g^+(\tau_k)$ are the limit values of the boundary function $g(\tau)$, when τ tends to the point τ_k along S, respectively, in the positive and negative directions (under the positive direction the counter-clockwise movement along the boundary direction is meant), then the following theorem explains the behavior of the generalized solution in the neighbourhood of the point τ_k (see [1,5]).

Theorem 1. *The limit values of the solution $u(z)$ of the Dirichlet generalized problem, when the point $z \in D$ approaches the point τ_k lie between $g^-(\tau_k)$ and $g^+(\tau_k)$.*

Remark 1. If the domain D is the exterior of the circle $S: \ x = a\cos t, \ y = a\sin t \ (0 \le t \le 2\pi)$, then the solution of the problem A is represented by Piosson's integral (see [1,2]):

$$u(z) = \frac{1}{2\pi}\int\limits_0^{2\pi} g(ae^{it})\frac{r^2 - a^2}{r^2 - 2ar\cos(t-\varphi) + a^2}\,dt, \tag{1.5}$$

where $r > a$ and $z = re^{i\varphi}$ $(0 \le \varphi \le 2\pi)$. When $r = a$ representation (1.5) loses sense. However, it is proved (see [1,2]) that

$$\lim_{z\to\tau} u(z) = g(\tau), \quad \tau = ae^{it}, \quad \tau \ne \tau_k, \quad z \in D.$$

Remark 2. On the basis of the formula (1.5) the problem A for simply connected domains can be solved by the method of conformal mapping (see [6]). In particular, for this it is necessary to know the function $z = \omega(\zeta)$ which conformally maps the unit disk $G(|\zeta| < 1)$ onto a simply connected domain D, and for calculation of the solution to the problem A at an arbitrary point of the initial domain D (also for determination the pre-images t_k of the points $\tau_k (k = 1, 2, \ldots, n)$ in conformal mapping $z = \omega(\zeta)$) it is necessary to know the function $\zeta = f(z)$ which is inverse to the function $z = \omega(\zeta)$.

2. On Application of the MFS for Generalized Problem

In general, it is known (see [3,7]) that the methods used for approximate solution of the ordinary boundary problems are less suitable (or not suitable at all) for solving problems with singularities. In particular, the convergence is very slow and, consequently, the accuracy is very low in the neighbourhood of singularity of the boundary function. Similar case takes place in solving the generalized Dirichlet boundary problem by the MFS. Therefore researchers try to perform preliminary improvment of the posed boundary problem. More precisely, they try to reduce, if possible, the posed problem by smoothing a boundary function to solving the ordinary problem (see e.g., [3,5,7]). For example, in the case of finite domains, the question about application of the MFS to harmonic and biharmonic problems with certain singularities is considered in (see e.g., [8,9,10,11]). In these papers it is noted that from the view-point of the accuracy in the neighbourhood of boundary singularities the MFS is ineffective for solution of harmonic and biharmonic problems with boundary singularities. Therefore, for solution of the considered problems authors have used so-called modified versions of the MFS, which are based on the direct subtraction of the leading terms of the singular local solution (which must be determined) from the original mathematical problem.

In general, the MFS may be used for solving both ordinary problem and generalized problem (see [12,13]). Concerning the rate of the convergence and accuracy in the neighbourhood of singularity of the boundary function, the noted fact was expected. Indeed, the fundamental solutions (functions) have a high degree of smoothness on the contour S, therefore, such smooth functions are less suitable for approximation of discontinuous functions. Taking into account the fact that for very big N computation becomes complicated, the above noted facts make the MFS less suitable (or not suitable at all) for approximate solving the Problem A. An analogous circumstance takes place when D is infinite domain. Thus, in the case of generalised problem the MFS is ineffective from the view-point of accuracy.

3. A Method of Reduction of the Dirichlet Generalized Problem to an Ordinary Problem in the Case of the Infinite Plane with Holes

For reduction of Problem A to an ordinary problem it is sufficient to have a function $u_0(z)$ which would be a solution of equation (1.1), bounded in $\overline{D}$, continuous in $\overline{D}$ everywhere, except the points $\tau = \tau_k$, and would have the same jumps at the points τ_k, as $g(\tau)$. Indeed, if such a function is constructed, then by introduction of a new unknown function

$$v(z) = u(z) - u_0(z) \tag{3.1}$$

for its determination we have already a Dirichlet ordinary problem.

B.

$$\Delta v(z) = 0, \quad z \in D, \tag{3.2}$$

$$v(\tau) = f(\tau), \quad \tau \in S, \tag{3.3}$$

where $f(\tau) = g(\tau) - u_0(\tau)$ is a continuous function on the contour S (since the function $f(\tau)$ has removable break points at τ_k, i.e., $f(\tau_k) = f^-(\tau_k) = f^+(\tau_k)$).

Since the domain D is infinite, for the uniqueness of the solution of Problems B and A (see [1,2]) we require additionally that

$$\lim v(z) = c_1 \quad for \quad z \to \infty, \tag{3.4}$$

$$\lim u_0(z) = c_2 \quad for \quad z \to \infty. \tag{3.5}$$

It is evident that in this case, since $c = c_1 + c_2$, c_2 must be given in advance, and c_1 should be found while solving Problem (3.2), (3.3). Conditions (1.4), (3.4) and (3.5) are essential, respectively, for the uniqueness of the solution of problems A and B in the case of an infinite domain.

After construction of function $v(z) = v(x, y)$ from (3.1) we have

$$u(z) = v(z) + u_0(z), \quad z \in \overline{D}, \quad z \neq \tau_k. \tag{3.6}$$

For simplicity of presentation, the case, when D is the infinite plane with the hole B_1 (i.e. $m = 1, S \equiv S_1$) we consider separately. In this case in the role of $u_0(z)$ we can take the function (see [5])

$$u_0(z) = \sum_{k=1}^{n} u_k(z), \tag{3.7}$$
$$u_k(z) = \frac{h_k}{\delta_k} w_k(z),$$
$$w_k(z) = \arg\left(\overline{\frac{z-\tau_k}{(z-z_0)(z_0-\tau_k)}}\right),$$

where h_k and δ_k are the jumps of the functions $g(\tau)$ and $w_k(\tau)$ at the point τ_k along S, respectively; in particular

$$h_k = g^+(\tau_k) - g^-(\tau_k), \quad \delta_k = \varphi_k^+ - \varphi_k^-,$$
$$\varphi_k^+ = \lim_{\tau \to \tau_k+} w_k(\tau), \quad \varphi_k^- = \lim_{\tau \to \tau_k-} w_k(\tau), \quad \tau \in S;$$

in the expression of the function $w_k(z)$ the sign "—-" denotes complex conjugate, z_0 is the inner point of the finite domain B_1 (to avoid difficulties in calculations, it is better to take the "center" of B_1 as z_0), *arg* denotes the properly chosen branch of the argument.

From (3.7) for the value of the constant c_2 (see (3.5)) we have

$$c_2 = \lim_{z \to \infty} u_0(z) = \sum_{k=1}^{n} \frac{h_k}{\delta_k} \arg(z_0 - \tau_k). \tag{3.8}$$

Now we consider the general case, when D is the infinite plane with the holes B_k $(k = 1, 2, \ldots, m)$. It should be noted that in problem A it is not necessary for points of discontinuity to be placed on all contours S_k $(k = 1, 2, \ldots, m)$. For simplicity we introduce the following notations. We denote by $\Gamma_1, \Gamma_2, \ldots, \Gamma_l$ $(1 \leq l \leq m)$ those of the contours S_k $(k = 1, 2, \ldots, m)$ on which the points of discontinuity are situated, and suppose that the number of the points of discontinuity on the contour Γ_i is k_i. It is clear that for the natural numbers k_i we have $1 \leq k_i \leq n$ and $k_1 + k_2 + \cdots + k_l = n$. Further, we denote by τ_{ik} $(k = 1, 2, \ldots, k_i)$ the points of discontinuity which are situated on the contour Γ_i, and we introduce the notation $h_{ik} = g^+(\tau_{ik}) - g^-(\tau_{ik})$, where h_{ik} is a jump of the function $g(\tau)$ at the point τ_{ik}.

In paper [5] it is shown that for smoothing of the function $g(\tau)$ on the contour Γ_i we can take the function

$$u_i(z) = \sum_{k=1}^{k_i} u_{ik}(z), \tag{3.9}$$
$$u_{ik}(z) = \frac{h_{ik}}{\delta_{ik}} w_{ik}(z),$$
$$w_{ik}(z) = \arg\left(\overline{\frac{z-\tau_{ik}}{(z-z_{i0})(z_{i0}-\tau_{ik})}}\right),$$

where z_{i0} is the "center" of the finite domain G_i with the boundary $\Gamma_i (z_{i0} \in G_i)$, while

$$\delta_{ik} = \varphi_{ik}^{+} - \varphi_{ik}^{-},\ \varphi_{ik}^{+} = \lim_{\tau \to \tau_{ik}+} w_{ik}(\tau),\ \ \varphi_{ik}^{-} = \lim_{\tau \to \tau_{ik}-} w_{ik}(\tau), \quad \tau \in \Gamma_i.$$

In the considered case we can take the function (see [5])

$$u_0(z) = \sum_{i=1}^{l} \sum_{k=1}^{k_i} u_{ik}(z) \tag{3.10}$$

in the role of $u_0(z)$, then the function $v(z) = u(z) - u_0(z)$ is a solution of problem B with the continuous boundary function

$$f(\tau) = g(\tau) - u_0(\tau), \quad \tau \in S \tag{3.11}$$

Thus in the general case the solution $u(z)$ of problem A can be represented in the form

$$u(z) = v(z) + \sum_{i=1}^{l} \sum_{k=1}^{k_i} u_{ik}(z). \tag{3.12}$$

From (3.10) for the velue of the constant c_2 we have

$$c_2 = \lim_{z \to \infty} u_0(z) = \sum_{i=1}^{l} \sum_{k=1}^{k_i} \frac{h_{ik}}{\delta_{ik}} \arg(z_{i0} - \tau_{ik}). \tag{3.13}$$

4. On Application of the Modified Version of MFS to Solution of the Dirichlet Ordinary Problem in the Case of Infinite Plane with Holes

It is known (see [12,13,14]) that the method of fundamental solutions can be used in the general case to solve approximately both internal and external boundary value problems (besides, also the number of connectivity of domain and dimension of space do not matter). On the basis of theory [12,13] the system

$$\left\{\varphi_{i,k}(z)\right\}_{k=1}^{\infty} = \left\{\ln|z - \widetilde{z}_{i,k}|\right\}_{k=1}^{\infty}, \quad z \in S \quad (i = 1, 2, \ldots, m), \tag{4.1}$$

plays the role of the system of fundamental solutions of the Laplace operator. In (4.1) $\{\widetilde{z}_{i,k}\}_{k=1}^{\infty}$ is a countable set of points lying everywhere densely on the auxiliary closed Liapunov contours $\widetilde{S}_i$ ($\widetilde{S} = \bigcup\limits_{i=1}^{m} \widetilde{S}_i$), where contours $\widetilde{S}_i$ lie respectively inside the finite domains B_i and $\min \rho(S_i, \widetilde{S}_i) > 0$, where ρ is the distance between S_i and $\widetilde{S}_i$. It is known that the system (4.1) is linearly independent and complete not only in the space $L_2(S)$, but also in $C(S)$. Theoretically, with the aid of system (4.1), the boundary function $g(z)$ can be approximated to within any accuracy, but it is inconvenient for an approximate solution of problem (3.2), (3.3), (3.4). Indeed, when using the MFS, the approximate solution is sought in the form

$$u_N(z) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \ln |z - \widetilde{z}_{i,k}|, \quad z \in \overline{D},$$

where $N = N_1 + N_2 + \cdots + N_m$, the points $\widetilde{z}_{i,k}(k = 1, 2, \dots, N_i)$ are situated "uniformly" on the auxiliary contour $\widetilde{S}_i$, and $a_k^{(N_i)}$ are the coefficients of expansion of the function $g(z)$ into a series with respect to the first N functions of system (4.1). It is obvious that $\lim u_N(z) = \infty$ as $|z| \to \infty$, which means that condition (3.4) for the solution to be unique is not fulfilled.

Remark 3. Further (see [14]) while solving approximately the boundary value problem, under contour $\widetilde{S}_i$ we will mean the Jordan contour which represents the boundary of the plane figure $\widetilde{B}_i(\widetilde{B}_i \subset B_i)$. The figure $\widetilde{B}_i$ is similar to B_i, oriented in the same way and they have one and same the "center" of gravity. As for the values $\rho(S_i, \widetilde{S}_i)$ and N_i, they can be chosen during the numerical realization of the algorithm, taking into account an *a posteriori* estimates of the accuracy of the results.

Remark 4. If we seek the solution to problem B in the form

$$u_N(z) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \ln |z - \widetilde{z}_{i,k}| + c_N,$$

under the condition $\sum\limits_{i=1}^{m} \sum\limits_{k=1}^{N_i} a_k^{(N_i)} = 0$, where c_N is a real constant and $|c_N| < \infty$, then it can be easily proved that $u_N(\infty) = c_N$. However while finding the constants $a_k^{(N_i)}$ $(i = 1, 2, \dots, m;\ k = 1, 2, \dots, N_i)$ and c_N some considerable difficulties arise which are connected with investigation of the questions about solvability of obtained systems and conditionality of its matrix.

To avoid the above noted situations in (see [15]) the modified version of the system of fundamental solutions (4.1) is constructed by the method of conformal mapping, which has the following form

$$\{\psi_{i,k}(z)\}_{k=1}^{\infty} \equiv \{\psi(z,\widetilde{z}_{i,k})\}_{k=1}^{\infty} = \left\{ \ln \left| \frac{\widetilde{z}_{i,k} - z}{(z-z_0)(\widetilde{z}_{i,k} - z_0)} \right| \right\}_{k=1}^{\infty}, \tag{4.2}$$

where z_0 is "the center" of either domain (hole) from the finite domains B_i $(i = 1, 2, \dots, m)$.

For system (4.2) following conditions are satisfied (see [15]):

1_0. $\Delta\psi_{i,k}(z) = 0$, $\forall z \in D$;

2_0. *The system* $\{\psi_{i,k}(z)\}_{k=1}^{\infty}(i = 1, 2, \dots, m)$ *is linearly independent and complete not only in the space* $L_2(S)$*, but also in* $C(S)$.

3_0. $\lim \psi_{i,k}(z)$ *is finite as* $|z| \to \infty$.

Since, in our case $f(z) \in C(S)$, therefore on the basis of property 2_0 for the arbitrary $\varepsilon > 0$ there exist such natural numbers $N_i^0(\varepsilon)$ and system of coefficients $a_k^{(N_i)}$ $(i = 1, 2, \dots, m,\ k = 1, 2, \dots, N_i)$, that if $N_i \geq N_i^0$, then

$$\max_{z\in S} \left| f(z) - \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \psi_{i,k}(z) \right| < \varepsilon.$$

If we introduce notation

$$v_N(z) \equiv v_N(x,y) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \psi_{i,k}(z),$$

where $N = N_1 + N_2 + \dots + N_i$, then on the basis of minimax principle we obtain that $\max\limits_{z\in\overline{D}} |v(z) - v_N(z)| < \varepsilon$, where $v(z)$ is exact solution to problem B, i.e. $v_N(z)$ converges uniformly to $v(z)$ in $\overline{D}$ for $N \to \infty$.

Thus, the approximate solution $v_N(z)$ of problem B by the modified version of MFS has the form

$$v_N(z) \equiv v_N(x,y) = \sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \ln \left| \frac{\widetilde{z}_{i,k} - z}{(z-z_0)(\widetilde{z}_{i,k} - z_0)} \right|, \tag{4.3}$$

where the auxiliary points (simulation sources) $\widetilde{z}_{i,k}(i = 1, 2, \dots, m,\ k = 1, 2, \dots, N_i)$ are situated "uniformly" on the contours $\widetilde{S}_i$.

As for the coefficients $a_k^{(N_i)}$, they can be found (see [12,13,14]) from the system of linear algebraic equations of the form

$$\sum_{i=1}^{m} \sum_{k=1}^{N_i} a_k^{(N_i)} \psi(z_{r,j}, \widetilde{z}_{i,k}) = f(z_{r,j}), \tag{4.4}$$

where the collocation points $z_{r,j}$ $(r = 1, 2, \dots, m;\ j = 1, 2, \dots, N_i)$ are situated "uniformly" on the contours S_r. The matrix of system (4.4) has the same properties as matrix, which was obtained while solving internal problems by system (4.1) (see [15]).

From (4.3) for the approximate value of constant c_1 we have

$$c_1^N = \lim_{z\to\infty} v_N(z) = v_N(\infty) = -\sum_{i=1}^{m}\sum_{k=1}^{N_i} a_k^{(N_i)} \ln\left|\widetilde{z}_{i,k} - z_0\right|,$$

or $|c_1^N| < \infty$.

5. Numerical Examples

In this section on the basis of considered scheme the results of approximate solution of generalized problems for $m = 1$ and $m = 2$ (see section 1) are given.In examples considered below the coefficients $a_k^{(N_i)}$ of expansion (4.3) are found from system (4.4).

In the Tables, N is the number of auxiliary and collocation points on the contours $\widetilde{S}$ and S, respectively; ε is an *a posteriori* error estimate of the solution of the problem B or the problem A:

$$\varepsilon = \max\left\{|f(z_{i,j}) - v_N(z_{i,j})|\right\}_{i=1}^{m},$$

where $f(z_{i,j}) = g(z_{i,j}) - u_0(z_{i,j})(z_{i,j} \neq \tau_{i,k})$ and $\tau_{i,k}$ is the point of discontinuity; The points $z_{i,j}(j = 1, 2, \ldots, M_i)$ are situated "uniformly" on the contour S_i. If $z_{i,j} = \tau_{i,k}$, then $f(z_{i,j}) = f^+(\tau_{i,k}) \equiv f^-(\tau_{i,k})$.

In numerical realization, provided the parametric equation of contour S_i is $z = p_i(t)$ and $\tau_{i,k} = p_i(t_{i,k})$, in the role of $f(\tau_{i,k})$ we can take values $f(p_i(t_{i,k} + \varepsilon_1)$ or $f(p_i(t_{i,k} - \varepsilon_1)$, where $\varepsilon_1 > 0$ is sufficiently small. In numerical experiments $M = M_1 + M_2 + \cdots + M_m$, and computations were realized by the MATLAB system.

Example 1. The domain D is the exterior of the circle $S : x = 2\cos t,\ y = 2\sin t\ (0 \leq t \leq 2\pi)$. Since $m = 1$, therefore for simplicity: $\tau_{1,k} \equiv \tau_k;\ h_{1,k} \equiv h_k;\ \delta_{1,k} \equiv \delta_k$. In the role of $g(\tau)$ we took a function with four break points $\tau_1 = (2;0),\ \tau_2 = (0;2),\ \tau_3 = (-2;0),\ \tau_4 = (0;-2)$. In particular, we took the function

$$g(\tau) = \begin{cases} x + y, & (x, y) \equiv \tau \in \tau_1\tau_2, \\ x^2 - y^2, & \tau \in \tau_2\tau_3, \\ x + y\,, & \tau \in \tau_3\tau_4, \\ x^2 - y^2, & \tau \in \tau_4\tau_1. \end{cases}$$

where $\tau_1\tau_2,\ \tau_2\tau_3,\ \tau_3\tau_4,\ \tau_4\tau_1$ are open arcs of the contour S ($S \equiv S_1, m = 1$). In the considered case: $g^+(\tau_1) = 2,\ g^-(\tau_1) = 4;\ g^+(\tau_2) = -4;\ g^-(\tau_2) = 2;\ g^+(\tau_3) = -2,\ g^-(\tau_3) = 4;\ g^+(\tau_4) = -4; g^-(\tau_4) = -2$, and $h_1 = -2,\ h_2 = -6,\ h_3 = -6,\ h_4 = -2,\ \delta_k = -\pi\,(k = 1, 2, 3, 4),\ z_0 = (0;0)$ (see ((3.7),(4.3))). In the role of optimal auxiliary contour $\widetilde{S}$ (in the sense of accuracy of approximate solution) for the given N the circle $\widetilde{S} : x = (2-\delta)\cos t,\ \ y = (2-\delta)\sin t\ \ (0 \leq t \leq 2\pi,\ \delta = 0.007)$ is taken in solving problem B by the modified version of MFS.

Auxiliary points, collocation points and points for calculation of the *a posteriori* estimate ε in this and other examples are situated uniformly with respect to the parameter t on the contours $\widetilde{S}$ and S, respectively. In the Table 1, $u_N(z_k)$ is the value of approximate solution to the problem A at the point $z_k \in D$ which is calculated with (3.6); $u(z_k)$ is value of exact solution to the problem A at point $z_k \in D$ which is calculated by Piosson's integral (1.5). It is easy to see that in the considered case for the exact solution to problem A we have $\lim u(z) = 0$ for $z \to \infty$. In the case $N = 1000$, $\varepsilon = 0.2E-02$ and for $N = 2000$, $\varepsilon = 0.5E-03$.

Table 1

$M = 5000$; $\varepsilon_1 = 10^{-6}$				
k	z_k	$u(z_k)$	$u_N(z_k), N = 1000$	$u_N(z_k), N = 2000$
1	$(2.00001; 0)$	3.0000082	2.9999979	2.9999984
2	$(0; 2.00001)$	-0.9999441	-0.9999609	-0.9999581
3	$(-2.00001; 0)$	0.9999441	0.9999611	0.9999266
4	$(0; -2.00001)$	-3.0000082	-2.9999951	-3.0000197
5	$(1000; 0)$	0.0032812	0.0038018	0.0032889
6	∞	0	$0.5E-03$	$0.7E-05$

Example 2. The infinite plane with holes B_i $(i = 1, 2)$ is taken in the role D with boundary $S = S_1 \cup S_2$ where the contour S_1 $(S_1 \equiv \Gamma_1)$ is the ellipse $S_1 : x = 2\cos t,\ y = \sin t$ $(0 \le t \le 2\pi)$ and the contour S_2 $(S_2 \equiv \Gamma_2)$ is the circle $S_2 :\ x = 10 + 2\cos t,\ y = 2\sin t$ $(0 \le t \le 2\pi)$.

In the role of a boundary function $g(\tau)$ we took the function

$$g(\tau) = \begin{cases} g_1(\tau), & \tau \in S_1, \\ g_2(\tau), & \tau \in S_2. \end{cases} \tag{5.1}$$

In (5.1) the functions $g_1(\tau)$ and $g_2(\tau)$ have the form

$$g_1(\tau) = \begin{cases} 1, & \tau \in \tau_{1,1}\tau_{1,2}\ , \\ 2, & \tau \in \tau_{1,2}\tau_{1,3}\ , \\ 3, & \tau \in \tau_{1,3}\tau_{1,4}\ , \\ 4, & \tau \in \tau_{1,4}\tau_{1,1}\ ; \end{cases}$$

$$g_2(\tau) = \begin{cases} 1, & \tau \in \tau_{2,1}\tau_{2,2}\ , \\ 3, & \tau \in \tau_{2,2}\tau_{2,3}\ , \\ 5, & \tau \in \tau_{2,3}\tau_{2,4}\ , \\ 7, & \tau \in \tau_{2,4}\tau_{2,1}. \end{cases}$$

on the contours S_1 and S_2, respectively.

It is evident that the jumps of the function $g(\tau)$ at the break points: $\tau_{1,1} = (2; 0)$, $\tau_{1,2} = (0; 1)$, $\tau_{1,3} = (-2; 0)$, $\tau_{1,4} = (-1; 0)$, $\tau_{2,1} = (12; 0)$, $\tau_{2,2} = (10; 2)$, $\tau_{2,3} = (8; 0)$, $\tau_{2,4} = (10; -2)$, respectively are equal to: $h_{1,1} =$

$-3,\ h_{1,2}=1,\ h_{1,3}=1,\ h_{1,4}=1,\ h_{2,1}=-6,\ h_{2,2}=2,\ h_{2,3}=2,\ h_{2,4}=2,$ respectively.

On the basis of the Section 3 we used the functions (3.9) and (3.10) for smoothing of the boundary function (5.1), where $l=2, m=2, \delta_{ik}=-\pi$ $(i=1,2;\ k=1,2,3,4;\ k_i=4)$, and $z_{1,0}=(0;0)$ and $z_{2,0}=(10;0)$.

While solving the problem B by the modified version of MFS, in the role of contours $\widetilde{S}_1$ and $\widetilde{S}_2$ we took the contours $\widetilde{S}_1:\ x=(2-\delta_1)\cos t,\ y=(1-\delta_1)\sin t$ and $\widetilde{S}_2:\ x=10+(2-\delta_2)\cos t,\ y=(2-\delta_2)\sin t,\ (0\le t\le 2\pi)$.

In numerical experiment the number of collocation (auxiliary) points on the contours S_1 and S_2 ($\widetilde{S}_1$ and $\widetilde{S}_2$) were equal $N=N_1+N_2$ $(N_1=N_2)$. Analogously $M=M_1+M_2$, where $M_1=M_2$. $\delta_1=0.01,\ \delta_2=0.03$ and $z_0=(0;0)$ (see (4.3)).

In the Table 2 the values of approximate solution $u_N(z)$ of the problem A calculated by (3.12) at the various points $z_k\in D$ are given. For $N=1600,\ \varepsilon=0.4E-03$ and for $N=2800,\ \varepsilon=0.1E-04$.

Table 2

$M=5000;\quad \varepsilon_1=10^{-6}$			
k	z_k	$u_N(z_k), N=1600$	$u_N(z_k), N=2800$
1	$(2.00001;0)$	2.50000673	2.50000672
2	$(0;1.00001)$	1.50000514	1.50000454
3	$(-2.00001;0)$	2.50000283	2.50000283
4	$(0;-1.00001)$	3.50000022	3.49999939
5	$(12.00001;0)$	3.99999860	3.99999860
6	$(10;2.00001)$	2.00000399	2.00000399
7	$(7.99999;0)$	3.99999654	3.99999654
8	$(10;-2.00001)$	5.99999203	5.99999203
9	∞	3.31068243	3.31066393

6. Concluding Remarks

From Tables 1,2 it is clear that for the approximate solution $u_N(z)$ of the problem A at the considered points of the domain D, the conditions of the generalized extremum principle and Theorem 1 are fulfilled.

The results of numerical experiments indicate the effectivness of the proposed algorithm for approximate solution problem of type A. In particular, the algorithm is sufficiently simple for numerical realization and it is characterized by accuracy, which is practically sufficient for many problems.

The proposed algorithm can be applied for approximate solution such generalized three-dimensional Dirichlet problems, which could be reduced to the problems of type A.

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