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ON MEASURABILITY PROPERTIES OF BERNSTEIN SETS

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Abstract. We envisage Bernstein subsets of the real line $\mathbf{R}$ from the point of view of their measurability with respect to certain classes of measures on $\mathbf{R}$. In particular, it is shown that there exists a Bernstein set absolutely nonmeasurable with respect to the class of all nonzero σ-finite translation quasi-invariant measures on $\mathbf{R}$, and that there exist countably many Bernstein sets which collectively cover $\mathbf{R}$ and are absolutely negligible with respect to the same class of measures.

რეზიუმე. ნაშრომში ბერნშტეინის სიმრავლეები შესწავლილია მათი ზომადობის თვალსაზრისით ზომათა გარკვეული კლასის მიმართ. კერძოდ, დამტკიცებულია ბერნშტეინის ისეთი სიმრავლის არსებობა, რომელიც აბსოლუტურად არაზომადია ნამდვილ ღერძზე მოცემულ ყველა არანულოვან სიგმა-სასრულ კვაზი-ინვარიანტულ ზომათა კლასის მიმართ. აგრეთვე დადგენილია, რომ არსებობს ნამდვილი ღერძის თვლადი დაფარვა, რომლის ყველა წევრი ბერნშტეინის სიმრავლეებია, თანაც ეს სიმრავლეები აბსოლუტურად უგულვებელყოფადია ზომათა ზემოთ აღნიშნული კლასის მიმართ.

According to the standard definition, a subset B of the real line $\mathbf{R}$ is a Bernstein set if, for any nonempty perfect set $P \subset \mathbf{R}$, the relations $P \cap B \neq \emptyset$ and $P \cap (\mathbf{R} \setminus B) \neq \emptyset$ are fulfilled.

Recall that such a set B was first constructed by Bernstein [1] in 1908. In his argument Bernstein essentially relies on an uncountable form of the Axiom of Choice and uses the method of transfinite recursion. Later it was recognized that this technique is necessary for obtaining B.

The importance of Bernstein sets in various questions of general topology, measure theory, and the theory of Boolean algebras is well known (see, e.g., [6], [11], [13], [14]). In classical measure theory, the significance of these sets is primarily caused by providing delicate counterexamples for seemingly valid statements in real analysis and by constructions of measures lacking nice regularity properties (see [6], [9]). For instance, one interesting application of Bernstein sets may be found in [12] where it is shown that some

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Lebesgue nonmeasurable real-valued functions turn out to be integrable in the Boks sense.

In this paper we study measurability properties of Bernstein sets with respect to the class of all nonzero σ-finite translation invariant or translation quasi-invariant measures on $\mathbf{R}$.

For our further purposes, we need several auxiliary notions.

Let E be a base (ground) set and let μ be a measure defined on some σ-algebra of subsets of E.

Recall that μ is said to be diffused (or continuous) if all singletons in E belong to the domain of μ and μ vanishes at all of them.

A set $Z \subset E$ is said to be μ-thick in E if the equality $\mu_*(E \setminus Z) = 0$ holds true, where μ_* denotes the inner measure associated with μ.

In view of the above definitions, the following remark is relevant.

Remark 1. Let $\mathcal{M}$ denote the class of the completions of all nonzero σ-finite diffused Borel measures on $\mathbf{R}$. It is not difficult to see that if B is any Bernstein set in $\mathbf{R}$ and μ is any measure from the class $\mathcal{M}$, then both B and $\mathbf{R} \setminus B$ are μ-thick subsets of $\mathbf{R}$ and, consequently, they are nonmeasurable with respect to μ. Actually, this property completely characterizes Bernstein sets in $\mathbf{R}$. More precisely, for a subset T of $\mathbf{R}$, the following two assertions are equivalent:

(a) T is a Bernstein set;

(b) for every measure $\mu \in \mathcal{M}$, the set T is nonmeasurable with respect to μ.

In particular, assertion (b) indicates that all Bernstein sets have extremely bad properties from the point of view of topological measure theory.

Now, we are going to discuss some measurability properties of Bernstein sets with respect to invariant (quasi-invariant) measures.

Let $(G, +)$ be a commutative group and let μ be a nonzero σ-finite measure defined on some σ-algebra of subsets of G. Let H be a subgroup of G.

Recall that μ is an H-quasi-invariant measure if the domain of μ ($=$ $\mathrm{dom}(\mu)$) and the σ-ideal generated by all μ-measure zero sets ($= \mathcal{I}(\mu)$) are H-invariant classes of subsets of G.

Also, recall that μ is an H-invariant measure if $\mathrm{dom}(\mu)$ is an H-invariant class of subsets of G and the equality $\mu(h + X) = \mu(X)$ is satisfied for any element $h \in H$ and any set $X \in \mathrm{dom}(\mu)$.

Clearly, every H-invariant measure is simultaneously H-quasi-invariant. The converse assertion is not true in general.

Throughout this paper the class of all nonzero σ-finite H-quasi-invariant measures on G will be denoted by the symbol $\mathcal{M}(G, H)$.

A set $Y \subset G$ is called H-absolutely nonmeasurable if $Y \notin \mathrm{dom}(\mu)$ for any measure $\mu \in \mathcal{M}(G, H)$.

It is clear that if H_1 and H_2 are two subgroups of G and $H_1 \subset H_2$, then any H_1-absolutely nonmeasurable subset of G is also H_2-absolutely nonmeasurable.

A set $Z \subset G$ is called H-absolutely negligible, if for every measure $\mu \in \mathcal{M}(G,H)$, there exists a measure $\mu' \in \mathcal{M}(G,H)$ extending μ and such that $\mu'(Z)=0$.

Various properties of H-absolutely nonmeasurable sets and H-absolutely negligible sets are discussed, e.g., in [2], [8], [9], [17].

Let G coincide with the additive group $\mathbf{R}$ and let $H \subset \mathbf{R}$ be an uncountable vector space over the field $\mathbf{Q}$ of all rational numbers. We are going to demonstrate that:

(*) there exists a Bernstein set B on $\mathbf{R}$ which is absolutely nonmeasurable with respect to the class $\mathcal{M}(\mathbf{R},H)$;

(**) there exists a countable family $\{B_i : i \in I\}$ of Bernstein subsets of $\mathbf{R}$ such that $\cup\{B_i : i \in I\} = \mathbf{R}$ and each B_i $(i \in I)$ is an H-absolutely negligible set.

Statement (*) shows that there are Bernstein sets extremely bad from the point of view of measurability with respect to all measures belonging to $\mathcal{M}(\mathbf{R},H)$.

Statement (**) implies that, given any measure $\mu \in \mathcal{M}(\mathbf{R},H)$, there exists a Bernstein set B such that $\mu'(B)=0$ for some measure $\mu' \in \mathcal{M}(\mathbf{R},H)$ extending μ. Moreover, if μ is an H-invariant measure, then we may suppose that its extension μ' is H-invariant too.

In what follows, the symbol ω denotes the least infinite cardinal number, ω_1 denotes the least uncountable cardinal number, and the symbol $\mathbf{c}$ stands for the cardinality continuum.

For establishing (*), we need one auxiliary proposition.

Lemma 1. *Let H be an uncountable subgroup of $\mathbf{R}$. Then there exist two groups $H' \subset H$ and $H'' \subset \mathbf{R}$ satisfying these three relations:*

(1) *H' is uncountable;*

(2) *H'' is a Bernstein subset of $\mathbf{R}$;*

(3) *$H' \cap H'' = \{0\}$.*

Proof. We identify $\mathbf{c}$ with the least ordinal number of cardinality $\mathbf{c}$. Let $\{P_\xi : \xi < \mathbf{c}\}$ denote the family of all nonempty perfect subsets of $\mathbf{R}$. Consider two possible cases.

(a) $\omega < \mathrm{card}(H) < \mathbf{c}$.

In this case, it is not difficult to construct by transfinite recursion a strictly increasing (by inclusion) family $\{H''_\xi : \xi < \mathbf{c}\}$ of subgroups of $\mathbf{R}$ such that, for each ordinal $\xi < \mathbf{c}$, the following conditions are fulfilled:

(i) $\mathrm{card}(H''_\xi) \le \mathrm{card}(\xi) + \omega$;

(ii) $H''_\xi \cap H = \{0\}$;

(iii) $H''_\xi \cap P_\xi \neq \emptyset$.

Now, putting $H' = H$ and $H'' = \cup\{H''_\xi : \xi < \mathbf{c}\}$, we obtain the required groups H' and H''.

(b) $\mathrm{card}(H) = \mathbf{c}$.

In this case, we again use the method of transfinite recursion and construct two strictly increasing (by inclusion) families $\{H'_\xi : \xi < \mathbf{c}\}$ and $\{H''_\xi : \xi < \mathbf{c}\}$ of subgroups of $\mathbf{R}$ such that, for any ordinal $\xi < \mathbf{c}$, the following conditions are fulfilled:

(j) $H'_\xi \subset H$;

(jj) $\mathrm{card}(H'_\xi) \le \mathrm{card}(\xi) + \omega$ and $\mathrm{card}(H''_\xi) \le \mathrm{card}(\xi) + \omega$;

(jjj) $H'_\xi \cap H''_\xi = \{0\}$ and $H''_\xi \cap P_\xi \neq \emptyset$.

Now, putting $H' = \cup\{H'_\xi : \xi < \mathbf{c}\}$ and $H'' = \cup\{H''_\xi : \xi < \mathbf{c}\}$, we come to the desired groups $H' \subset H$ and $H'' \subset \mathbf{R}$. □

Theorem 1. *Let $H \subset \mathbf{R}$ be a vector space over the field $\mathbf{Q}$ of all rational numbers. The following two assertions are equivalent:*

(1) *H is uncountable;*

(2) *there exist H-absolutely nonmeasurable Bernstein sets in $\mathbf{R}$.*

Proof. Notice first that if a group $H \subset \mathbf{R}$ is at most countable, then it is easy to define a nonzero σ-finite H-invariant measure whose domain coincides with the power set of $\mathbf{R}$. This circumstance directly implies that in the case $\mathrm{card}(H) \le \omega$ there are no H-absolutely nonmeasurable subsets of $\mathbf{R}$. Consequently, (1) follows from (2).

It remains to demonstrate the validity of the implication (1) $\Rightarrow$ (2).

Suppose (1). By virtue of Lemma 1, we may assume (without loss of generality) that $\mathrm{card}(H) = \omega_1$ and H satisfies the following condition: there exists a Bernstein group $H'' \subset \mathbf{R}$ such that

$$H \cap H'' = \{0\}, \quad H + H'' = \mathbf{R}.$$

Further, we follow the argument presented in [9]. Namely, according to Lemmas 2 and 3 of Chapter 11 from [9], there are a countable group $H_0 \subset H$, an uncountable group $H_1 \subset H$, and a set $X \subset H$ such that:

(a) $H_0 + X = H$;

(b) $\mathrm{card}((g + X) \cap (h + X)) \le \omega$ for any two distinct elements g and h of H_1.

Let us define $Y = X + H''$. Clearly, Y is a Bernstein set in $\mathbf{R}$. Now, the proof of Theorem 1 from Chapter 11 in [9] yields that Y is an H-absolutely nonmeasurable set as well. So we conclude that there are H-absolutely nonmeasurable Bernstein sets in $\mathbf{R}$. □

Putting $H = \mathbf{R}$ we obtain from Theorem 1 that there exist $\mathbf{R}$-absolutely nonmeasurable Bernstein sets.

On the other hand, let us demonstrate that if H is an uncountable subgroup of $\mathbf{R}$, then there are Bernstein sets with relatively good properties with respect to the class $\mathcal{M}(\mathbf{R}, H)$. For this purpose, we need the method developed in the work [8].

Lemma 2. *Let $(E, ||\cdot||)$ be a normed vector space over the field $\mathbf{R}$ and let H be a nonseparable subgroup of the additive group of E. Then any ball in E is an H-absolutely negligible set.*

For a proof, see Proposition 4 from [8] which is much stronger than Lemma 2.

In what follows the symbol $l_2(\omega_1)$ stands for the real Hilbert space having an orthogonal basis equinumerous with ω_1. Notice that in our further consideration it is convenient to treat both $\mathbf{R}$ and $l_2(\omega_1)$ as vector spaces over the field $\mathbf{Q}$ of all rational numbers. In this case, the algebraic dimension of $\mathbf{R}$ and of $l_2(\omega_1)$ is equal to $\mathbf{c}$, so these vector spaces are isomorphic to each other.

Lemma 3. *Let H be an uncountable subgroup of the additive group $\mathbf{R}$. There exists a group isomorphism*

$$\phi : \mathbf{R} \to l_2(\omega_1)$$

satisfying the following conditions:

(1) *the group $\phi(H)$ is everywhere dense in $l_2(\omega_1)$;*

(2) *the set $\phi^{-1}(\{x \in l_2(\omega_1) : ||x|| = 1\})$ is a Bernstein subset of $\mathbf{R}$.*

Proof. As before, we identify $\mathbf{c}$ with the least ordinal number of cardinality $\mathbf{c}$. Denote by $\{P_\xi : \xi < \mathbf{c}\}$ the family of all nonempty perfect subsets of $\mathbf{R}$. Without loss of generality, we can additionally assume that the partial family $\{P_{2\xi+1} : \xi < \mathbf{c}\}$ also consists of all nonempty perfect subsets of $\mathbf{R}$.

Since $\omega_1 \leq \mathbf{c}$ and the topological weight of the space $l_2(\omega_1)$ is equal to ω_1, we may denote by $\{U_\xi : \xi < \mathbf{c}\}$ some base of open sets in $l_2(\omega_1)$. Without loss of generality, we can additionally assume that the partial family $\{U_{2\xi} : \xi < \omega_1\}$ also forms a base of open sets in $l_2(\omega_1)$.

Further, by using the method of transfinite recursion, we define simultaneously two $\mathbf{c}$-sequences

$$\{t_\xi : \xi < \mathbf{c}\} \subset \mathbf{R}, \quad \{e_\xi : \xi < \mathbf{c}\} \subset l_2(\omega_1)$$

such that:

(a) the family $\{t_\xi : \xi < \mathbf{c}\}$ is linearly independent over $\mathbf{Q}$ in $\mathbf{R}$ and the vector space $\mathrm{span}_{\mathbf{Q}}(\{t_\xi : \xi < \mathbf{c}\})$ has co-dimension $\mathbf{c}$ in $\mathbf{R}$;

(b) the family $\{e_\xi : \xi < \mathbf{c}\}$ is linearly independent over $\mathbf{Q}$ in $l_2(\omega_1)$, and the vector space $\mathrm{span}_{\mathbf{Q}}(\{e_\xi : \xi < \mathbf{c}\})$ has co-dimension $\mathbf{c}$ in $l_2(\omega_1)$;

(c) $t_\xi \in P_\xi$ and $||e_\xi|| = 1$ if $\xi < \mathbf{c}$ is an odd ordinal number;

(d) $e_\xi \in U_\xi$ if $\xi < \omega_1$ is an even ordinal number;

(e) $t_\xi \in H$ if $\xi < \omega_1$ is an even ordinal number.

Transfinite construction of the two above-mentioned $\mathbf{c}$-sequences can be carried out in a standard manner, so we omit its details here. Afterwards, we put

$$\phi(t_\xi) = e_\xi \qquad (\xi < \mathbf{c}).$$

In view of (a) and (b), the function ϕ can be extended to a group isomorphism between $\mathbf{R}$ and $l_2(\omega_1)$, for which we preserve the same notation ϕ.

According to (d) and (e), the group $\phi(H)$ is everywhere dense in the Hilbert space $l_2(\omega_1)$, i.e., condition (1) is fulfilled.

According to (c), the set $B = \phi^{-1}(\{x \in l_2(\omega_1) : ||x|| = 1\})$ intersects every nonempty perfect subset of $\mathbf{R}$. It can easily be seen that there exists $t \in \mathbf{R}$ for which $B \cap (B + t) = \emptyset$, whence it immediately follows that the set $\mathbf{R} \setminus B$ also intersects every nonempty perfect subset of $\mathbf{R}$. We thus conclude that B is a Bernstein set in $\mathbf{R}$, i.e., condition (2) is fulfilled, too. □

Theorem 2. *Let H be a subgroup of the additive group $\mathbf{R}$. The following two assertions are equivalent:*

(1) *H is uncountable;*

(2) *there exists a countable family $\{B_n : n < \omega\}$ of Bernstein subsets of $\mathbf{R}$ such that $\cup\{B_n : n < \omega\} = \mathbf{R}$ and each set B_n $(n < \omega)$ is H-absolutely negligible in $\mathbf{R}$.*

Proof. It can easily be verified that if a group $H \subset \mathbf{R}$ is at most countable, then no nonempty subset of $\mathbf{R}$ is H-absolutely negligible. This circumstance directly indicates that (1) follows from (2).

It remains to demonstrate the validity of the implication (1) $\Rightarrow$ (2).

Suppose (1). By virtue of Lemma 3, there exists a group isomorphism

$$\phi : \mathbf{R} \to l_2(\omega_1)$$

such that these two relations are fulfilled:

(a) the group $\phi(H)$ is everywhere dense in $l_2(\omega_1)$;

(b) the set $\phi^{-1}(\{x \in l_2(\omega_1) : ||x|| = 1\})$ is a Bernstein subset of $\mathbf{R}$.

Keeping in mind (a) and (b), let us define

$$B_n = \phi^{-1}(\{x \in l_2(\omega_1) : ||x|| \le n + 1\}) \qquad (n < \omega).$$

We thus obtain the countable family $\{B_n : n < \omega\}$ of subsets of $\mathbf{R}$ and we are going to show that this family is as required.

Obviously, the family of balls

$$\{x \in l_2(\omega_1) : ||x|| \le n + 1\} \qquad (n < \omega)$$

forms a countable covering of the space $l_2(\omega_1)$, whence it follows that

$$\cup\{B_n : n < \omega\} = \mathbf{R}.$$

Now, consider any set B_n, where $n < \omega$. In view of (b), the set B_0 is a Bernstein subset of $\mathbf{R}$. Since $B_0 \subset B_n$, the set B_n meets every nonempty perfect subset of $\mathbf{R}$. Similarly to the proof of Lemma 3, there exists $t \in \mathbf{R}$

for which $B_n \cap (B_n + t) = \emptyset$, whence it immediately follows that the set $\mathbf{R} \setminus B_n$ also meets every nonempty perfect subset of $\mathbf{R}$. We thus conclude that B_n is a Bernstein set in $\mathbf{R}$.

Finally, according to Lemma 2, any ball $\{x \in l_2(\omega_1) : ||x|| \leq n+1\}$, where $n < \omega$, is $\phi(H)$-absolutely negligible in the additive group $l_2(\omega_1)$. Since ϕ^{-1} is a group isomorphism, we readily deduce that each set B_n $(n < \omega)$ is H-absolutely negligible in $\mathbf{R}$. □

Of course, the most interesting case in the above theorem is when H coincides with the whole real line $\mathbf{R}$.

Remark 2. It would be interesting to generalize Theorems 1 and 2 to the case of a multi-dimensional Euclidean space instead of the real line $\mathbf{R}$. More precisely, let $n \geq 2$ and let H be a subgroup of the group of all isometric transformations of the Euclidean space $\mathbf{R}^n$. The natural question arises: under what assumptions on H there exists an H-absolutely nonmeasurable Bernstein subset of $\mathbf{R}^n$? The second question can be formulated as follows: under what assumptions on H there exists a countable family $\{B_n : n < \omega\}$ of Bernstein subsets of $\mathbf{R}^n$ such that $\cup\{B_n : n < \omega\} = \mathbf{R}^n$ and all B_n $(n < \omega)$ are H-absolutely negligible sets? In this connection, the work [2] should be mentioned in which it is proved that if H contains the group of all translations of $\mathbf{R}^n$, then there exists a countable covering of $\mathbf{R}^n$ with H-absolutely negligible sets (extensive information about this and further results may be found in [17]).

Remark 3. Bernstein sets on $\mathbf{R}$ form a family of so-called pathological subsets of $\mathbf{R}$ (cf. [6], [9], [11], [13], [14]). Another family of pathological subsets of $\mathbf{R}$ is constituted by Vitali sets (see [3], [9], [11], [13], [14], [16]). Recall that any Vitali set is a selector of the quotient set $\mathbf{R}/\mathbf{Q}$ and is absolutely nonmeasurable with respect to the class of all those measures on $\mathbf{R}$ which extend the Lebesgue measure λ and are $\mathbf{Q}$-invariant. On the other hand, it was shown in [10] that there exists a Vitali set which is measurable with respect to a certain translation quasi-invariant measure on $\mathbf{R}$ extending λ. Some other measurability properties of Vitali type sets can be found in [3], [9], [17]. By using an argument similar to the proof of Theorem 1, it can be demonstrated that there exists an $\mathbf{R}$-absolutely nonmeasurable subset of $\mathbf{R}$ which is sumultaneously a Vitali set and a Bernstein set.

Remark 4. It was proved in [4] (see also [15]) that the following two assertions are equivalent:

(1) the Continuum Hypothesis (**CH**);

(2) there exists a countable family $\{Z_i : i \in I\}$ of Hamel bases of $\mathbf{R}$ such that $\cup\{Z_i : i \in I\} = \mathbf{R} \setminus \{0\}$.

It was shown in [7] that every Hamel basis of $\mathbf{R}$ is an $\mathbf{R}$-absolutely negligible set. Therefore, the analogue of Theorem 2 in terms of Hamel bases cannot be established within **ZFC** set theory. This analogue is valid if and only if **CH** holds true (see [7]).

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