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ON YOUNG TYPE INEQUALITIES FOR GENERALIZED CONVOLUTION

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Abstract. A generalized convolution is introduced using which new iterated convolution inequalities have been obtained in a general framework of Lebesgue spaces with different indices and with different weights. In each case, a characterization has been given for the corresponding inequality to hold. These inequalities include several of the known such inequalities.

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1. Introduction

By a weight function or simply a weight, we shall mean a function which is positive, measurable and finite a.e. For $\Omega \subseteq \mathbb{R}^m$, $1 \le p < \infty$ and a weight ρ, we shall denote by $L^p(\Omega, \rho)$, the weighted Lebesgue space which is the space of all measurable functions f for which

$$\|f\|_{L^p(\Omega,\rho)} := \left(\int\limits_{\Omega} |f(\xi)|^p \rho(\xi) d\xi \right)^{1/p} < \infty.$$

When $\rho \equiv 1$, the corresponding non-weighted Lebesgue space will be denoted by $L^p(\Omega)$. According to Young's inequality

$$\|F_1 *_1 F_2\|_{L^r(\mathbb{R})} \le \|F_1\|_{L^p(\mathbb{R})} \|F_2\|_{L^q(\mathbb{R})}, \quad F_1 \in L^p(\mathbb{R}), \quad F_2 \in L^q(\mathbb{R}),$$

where $p, q, r > 0$, $\frac{1}{r} + 1 = \frac{1}{p} + \frac{1}{q}$ and $F_1 *_1 F_2$ is the Fourier convolution defined by

$$(F_1 *_1 F_2)(\eta) = \int\limits_{\mathbb{R}} F_1(\xi) F_2(\eta - \xi) d\xi. \tag{1.1}$$

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Inspired by Young's inequality, several authors have studied more general versions of it: sometimes by considering different type of convolutions and sometimes by introducing weights in the corresponding inequalities. One may refer to [2-10] and references there in.

Castro and Saitoh [2] considered the following three convolutions in addition to (1.1):

$$(F_1 *_2 F_2)(\eta) = \int_{\mathbb{R}} F_1(\xi)\overline{F_2(\xi-\eta)}d\xi, \tag{1.2}$$

$$(F_1 *_3 F_2)(\eta) = \int_{\mathbb{R}} \overline{F_1(\xi)}F_2(\xi+\eta)d\xi, \tag{1.3}$$

$$(F_1 *_4 F_2)(\eta) = \int_{\mathbb{R}} \overline{F_1(\xi)F_2(-\xi-\eta)}d\xi \tag{1.4}$$

and proved the inequality

$$\int_{-\infty}^{\infty} \frac{1}{(\rho_1 * \rho_2)(\eta)} \left\{ |((F_1\rho_1) * (F_2\rho_2))(\eta)|^2 \right\} d\eta \le \left(\int_{-\infty}^{\infty} |F_1(\eta)|^2 \rho_1(\eta) d\eta \right) \left(\int_{-\infty}^{\infty} |F_2(\eta)|^2 \rho_2(\eta) d\eta \right), \tag{1.5}$$

where $*$ denotes any one of the convolutions $*_1, *_2, *_3, *_4$ defined above and ρ_1, ρ_2 are weights. Further, in [7], the authors introduced the so called φ-convolution which is a generalization of the standard convolution. The φ-convolution of F_1 and F_2, denoted by $F_1 *_\varphi F_2$ is defined by

$$(F_1 *_\varphi F_2)(\eta) = \int_{\mathbb{R}^m} F_1(\xi)F_2(\varphi(\xi,\eta))\,|(\varphi_\eta(\xi,\eta))|\,d\xi, \tag{1.6}$$

if the integral on the right exists. Here φ is a mapping from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^m$ and

$$|(\varphi_\eta(\xi,\eta))| = \det \frac{\partial}{\partial \eta}(\varphi(\xi,\eta)).$$

In the framework of the φ-convolution, Nhan, Duc and Tuan [7] proved the following:

Theorem A. *Consider the iterated φ-convolution $\prod_{j=1}^{n} *_\varphi F_j$ defined by*

$$\prod_{j=1}^{n} *_\varphi F_j(\xi) = \left(\prod_{j=1}^{n-1} *_\varphi F_j \right) *_\varphi F_n(\xi), \quad \xi \in \mathbb{R}^m.$$

Let $\rho_j, j = 1, 2, \dots, n$ *be weights such that the iterated* φ*-convolution* $\prod_{j=1}^{n} *_\varphi \rho_j$ *exists. Then, for* $1 < p < \infty$*, the inequality*

$$\left\| \left(\prod_{j=1}^{n} *_\varphi (F_j \rho_j) \right) \left(\prod_{j=1}^{n} *_\varphi \rho_j \right)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)} \leq \prod_{j=1}^{n} \|F_j\|_{L^p(\mathbb{R}^m, \rho_j)}, \tag{1.7}$$

holds for all $F_j \in L^p(\mathbb{R}^m, \rho_j)$*,* $1 \leq j \leq n$*.*

For $\varphi(\xi, \eta) = \eta - \xi$, the convolution $*_\varphi$ is the standard convolution $*_1$ given by (1.1). This case of Theorem A was studied by Andersen [1].

Further, note that the convolutions $*_2$, $*_3$ and $*_4$ can not be obtained as special cases of $*_\varphi$. Thus, it is natural to consider the generalized versions of $*_2$, $*_3$, $*_4$ corresponding to $*_\varphi$. In this direction, we unify all these generalizations and define the following:

Definition 1.1. Define $\circledast_\varphi$-convolution of F_1 and F_2 by

$$(F_1 \circledast_\varphi F_2)(\eta) = \int\limits_{\mathbb{R}^m} F_1(\xi) \circledast F_2(\varphi(\xi, \eta)) \, |\varphi_\eta(\xi, \eta)| \, d\xi,$$

where φ is the mapping as in (1.6) and $\circledast$ is any one of the operations inside the integrals of (1.1)-(1.4). For example, if $\varphi(\xi, \eta) = \xi + \eta$ and $F_1(\xi) \circledast F_2(\tau) = \overline{F_1(\xi)} F_2(\tau)$, then

$$(F_1 \circledast_\varphi F_2)(\eta) = (F_1 *_3 F_2)(\eta).$$

As the first aim of this paper, we shall establish Theorem A in the framework of $\circledast_\varphi$-convolution. Moreover, we shall establish this new result in a more general setting in the sense that different sets of weights ρ_i and β_i are considered on two sides of the corresponding inequality and also different indices p and q are used. A characterization has been obtained for such inequality for the case $1 < p < q < \infty$.

Further, in a different paper [8], Nhan, Duc and Tuan studied the reverse of the inequality (1.7). Precisely, they proved the following:

Theorem B. *Let* $\rho_j, j = 1, 2, \dots, n$ *be weights such that the* φ*-convolution* $\prod_{j=1}^{n} *_\varphi \rho_j$ *exists and the function* F_j*'s satisfy* $0 < m_j^{1/p} \leq F_j \leq M_j^{1/p} < \infty$*. Then, for* $1 < p < \infty$*, the inequality*

$$\left\| \left(\prod_{j=1}^{n} *_\varphi (F_j \rho_j) \right) \left(\prod_{j=1}^{n} *_\varphi \rho_j \right)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)} \geq$$
$$\geq \prod_{i=2}^{n} \left\{ A_{p,p'}^{-n} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \prod_{j=1}^{n} \|F_j\|_{L^p(\mathbb{R}^m, \rho_j)}, \tag{1.8}$$

holds for all $F_j \in L^p(\mathbb{R}^m, \rho_j), 1 \leq j \leq n$*.*

In the present paper, we also extend Theorem B by taking different sets of weights and different indices on two sides of the inequality (1.8) and obtain a characterization for the corresponding inequality to hold for $1 < q < p < \infty$.

2. $\circledast_\varphi$-Convolution Inequality: The Case $p = q$

We begin with the following lemma which will be used in the subsequent result:

Lemma 2.1. *Let $1 < p < \infty$ and $\rho_j, j = 1, 2$ be weights on $\mathbb{R}^m$. Then the inequality*

$$\left\| ((F_1\rho_1) \circledast_\varphi (F_2\rho_2))\,(\rho_1 \circledast_\varphi \rho_2)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)} \leq \|F_1\|_{L^p(\mathbb{R}^m,\rho_1)} \|F_2\|_{L^p(\mathbb{R}^m,\rho_2)}$$

holds for all functions $F_j \in L^p(\mathbb{R}^m, \rho_j)$.

Proof. By applying Hölder's inequality, Fubini's theorem and change of variables, we have

$$\begin{aligned}
&\left\| \big((F_1\rho_1) \circledast_\varphi (F_2\rho_2)\big)\,(\rho_1 \circledast_\varphi \rho_2)^{1/p-1} \right\|^p_{L^p(\mathbb{R}^m)} \leq \\
&\leq \int\limits_{\mathbb{R}^m} |((F_1\rho_1) \circledast_\varphi (F_2\rho_2))(\eta)|^p \,|(\rho_1 \circledast_\varphi \rho_2)(\eta)|^{1-p}\, d\eta = \\
&= \int\limits_{\mathbb{R}^m} \left| \int\limits_{\mathbb{R}^m} (F_1\rho_1)(\xi) \circledast (F_2\rho_2)(\varphi(\xi,\eta))\, |\varphi_\eta(\xi,\eta)|\, d\xi \right|^p \times \\
&\quad \times |(\rho_1 \circledast_\varphi \rho_2)(\eta)|^{1-p}\, d\eta = \\
&= \int\limits_{\mathbb{R}^m} \frac{\left|\int_{\mathbb{R}^m} (F_1(\xi) \circledast F_2(\varphi(\xi,\eta)))\,(\rho_1(\xi) \circledast \rho_2(\varphi(\xi,\eta)))\, |\varphi_\eta(\xi,\eta)|\, d\xi\right|^p}{|(\rho_1 \circledast_\varphi \rho_2)(\eta)|^{p-1}} d\eta \leq \\
&\leq \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} |F_1(\xi) \circledast (F_2(\varphi(\xi,\eta))|^p\, |\rho_1(\xi) \circledast \rho_2(\varphi(\xi,\eta))|\, |\varphi_\eta(\xi,\eta)|\, d\xi d\eta \leq \\
&\leq \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} |F_1(\xi)|^p\, |F_2(\varphi(\xi,\eta))|^p\, \rho_1(\xi)\rho_2(\varphi(\xi,\eta))\, |\varphi_\eta(\xi,\eta)|\, d\xi d\eta = \\
&= \int\limits_{\mathbb{R}^m} |F_1(\xi)|^p\, \rho_1(\xi) \Bigg(\int\limits_{\mathbb{R}^m} |F_2(\varphi(\xi,\eta))|^p\, \rho_2(\varphi(\xi,\eta))\, |\varphi_\eta(\xi,\eta)|\, d\eta \Bigg) d\xi = \\
&= \Bigg(\int\limits_{\mathbb{R}^m} |F_1(\xi)|^p\, \rho_1(\xi) d\xi \Bigg) \Bigg(\int\limits_{\mathbb{R}^m} |F_1(\tau)|^p\, \rho_1(\tau) d\tau \Bigg) = \\
&= \|F_1\|^p_{L^p(\mathbb{R}^m,\rho_1)} \|F_2\|^p_{L^p(\mathbb{R}^m,\rho_2)}
\end{aligned}$$

and we are done. $\square$

Now, we prove the following:

Theorem 2.2. *Let $1 < p < \infty$ and $\rho_j, j = 1, 2, \ldots, n$ be weights such that the convolution $\prod_{j=1}^{n} \circledast_{\varphi} \rho_j$ exists. Then the inequality*

$$\left\| \left(\prod_{j=1}^{n} \circledast_{\varphi} (F_j \rho_j) \right) \left(\prod_{j=1}^{n} \circledast_{\varphi} \rho_j \right)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)} \leq \prod_{j=1}^{n} \|F_j\|_{L^p(\mathbb{R}^m, \rho_j)} \tag{2.1}$$

holds for all functions $F_j \in L^p(\mathbb{R}^m, \rho_j)$, $j = 1, 2, \ldots, n$.

Proof. We will prove it by the induction. By Lemma 2.1, the result holds for $n = 2$. For $n = k+1$, by applying Hölder's inequality, Fubini's theorem, change of variables and induction hypothesis, we have

$$\left\| \left(\prod_{j=1}^{k+1} \circledast_{\varphi} (F_j \rho_j) \right) \left(\prod_{j=1}^{k+1} \circledast_{\varphi} \rho_j \right)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)}^p \leq$$

$$\leq \int_{\mathbb{R}^m} \left| \left(\prod_{j=1}^{k+1} \circledast_{\varphi} (F_j \rho_j) \right)(\eta) \right|^p \left| \prod_{j=1}^{k+1} \circledast_{\varphi} \rho_j(\eta) \right|^{1-p} d\eta \leq$$

$$\leq \int_{\mathbb{R}^m} \left[\int_{\mathbb{R}^m} \left| \left(\prod_{j=1}^{k} \circledast_{\varphi} (F_j \rho_j) \right)(\xi) \circledast (F_{k+1}\rho_{k+1}) \left(\varphi(\xi,\eta)\right) \right| \left|\varphi_\eta(\xi,\eta)\right| d\xi \right]^p \times$$

$$\times \left[\int_{\mathbb{R}^m} \left| \left(\prod_{j=1}^{k} \circledast_{\varphi} \rho_j \right)(\xi) \circledast \rho_{k+1} \left(\varphi(\xi,\eta)\right) \right| \left|\varphi_\eta(\xi,\eta)\right| d\xi \right]^{1-p} \leq$$

$$\leq \int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| \left(\prod_{j=1}^{k} \circledast_{\varphi} (F_j \rho_j) \right)(\xi) \right|^p \circledast \left| (F_{k+1}\rho_{k+1}) \left(\varphi(\xi,\eta)\right) \right|^p \times$$

$$\times \left| \left(\prod_{j=1}^{k} \circledast_{\varphi} \rho_j \right)(\xi) \right|^{1-p} \circledast \left| \rho_{k+1} \left(\varphi(\xi,\eta)\right) \right|^{1-p} \left|\varphi_\eta(\xi,\eta)\right| d\xi d\eta =$$

$$= \int_{\mathbb{R}^m} \left| \left(\prod_{j=1}^{k} \circledast_{\varphi} (F_j \rho_j) \right)(\xi) \right|^p \left| \left(\prod_{j=1}^{k} \circledast_{\varphi} \rho_j \right)(\xi) \right|^{1-p} \times$$

$$\times \left[\int_{\mathbb{R}^m} \left| (F_{k+1}\rho_{k+1}) \left(\varphi(\xi,\eta)\right) \right|^p \left(\rho_{k+1} \left(\varphi(\xi,\eta)\right) \right)^{1-p} \left| \left(\varphi_\eta(\xi,\eta)\right) \right| d\eta \right] d\xi =$$

$$= \left\| \left(\prod_{j=1}^{k} \circledast_{\varphi} (F_j \rho_j) \right) \left(\prod_{j=1}^{k} \circledast_{\varphi} \rho_j \right)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)}^p \|F_{k+1}\|_{L^p(\mathbb{R}^m, \rho_{k+1})}^p \leq$$

$$\leq \prod_{j=1}^{k+1} \|F_j\|_{L^p(\mathbb{R}^m, \rho_j)}^p$$

and the assertion follows. $\square$

Remark 2.3. If the functions F_j are real valued then $\overline{F_j} = F_j$ and therefore inequality (2.1) reduces to (1.7). Consequently, Theorem 2.2 extends Theorem A. At the same time, the convolution $\circledast_\varphi$ in inequality (2.1) is more general than $*_2$, $*_3$ and $*_4$ as well, and so (2.1) also extends (1.5) given by Castro and Saitoh [2].

3. $\circledast_\varphi$-Convolution Inequality: The Case $p < q$

Throughout, for $r > 0$, we shall denote

$$I_r = \underbrace{(-r, r) \times (-r, r) \times \cdots \times (-r, r)}_{m-\text{times}}.$$

In this section, we shall study inequality (2.1) for different sets of weights and for different indices on both sides of it. A weight characterization has been obtained for the corresponding inequality. We prove a lemma first:

Lemma 3.1. *Let $1 < p < q < \infty$ and $\rho_j, \beta_j, j = 1, 2$ be weights on $\mathbb{R}^m$. If*

$$C := \sup_{r>0} \left\| \frac{\rho_1}{\beta_1^{p/q}} \right\|_{L^{\frac{q}{q-p}}(I_r)} \left\| \frac{\rho_2}{\beta_2^{p/q}} \right\|_{L^{\frac{q}{q-p}}(I_r)} < \infty, \tag{3.1}$$

then the inequality

$$\begin{aligned} &\left\| \left((F_1\rho_1) \circledast_\varphi (F_2\rho_2) \right) (\rho_1 \circledast_\varphi \rho_2)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)} \leq \\ &\leq C \|F_1\|_{L^q(\mathbb{R}^m, \beta_1)} \|F_2\|_{L^q(\mathbb{R}^m, \beta_2)} \end{aligned} \tag{3.2}$$

holds for all functions $F_j \in L^q(\mathbb{R}^m, \beta_j)$, $j = 1, 2$.

Proof. By applying Hölder's inequality, Fubini's theorem, change of variables and (3.1), we have

$$\begin{aligned}
&\left\| \left((F_1\rho_1) \circledast_\varphi (F_2\rho_2) \right) (\rho_1 \circledast_\varphi \rho_2)^{1/p-1} \right\|^p_{L^p(\mathbb{R}^m)} \leq \\
&\leq \int_{\mathbb{R}^m} \left| \left((F_1\rho_1) \circledast_\varphi (F_2\rho_2) \right) (\eta) \right|^p \left| (\rho_1 \circledast_\varphi \rho_2)(\eta) \right|^{1-p} d\eta = \\
&= \int_{\mathbb{R}^m} \left| \int_{\mathbb{R}^m} (F_1\rho_1)(\xi) \circledast (F_2\rho_2)(\varphi(\xi,\eta)) \left| \varphi_\eta(\xi,\eta) \right| d\xi \right|^p \times \\
&\quad \times \left| (\rho_1 \circledast_\varphi \rho_2)(\eta) \right|^{1-p} d\eta = \\
&= \int_{\mathbb{R}^m} \frac{\left| \int_{\mathbb{R}^m} \left(F_1(\xi) \circledast F_2(\varphi(\xi,\eta)) \right) \left(\rho_1(\xi) \circledast \rho_2(\varphi(\xi,\eta) \right) \left| \varphi_\eta(\xi,\eta) \right| d\xi \right|^p}{\left| (\rho_1 \circledast_\varphi \rho_2)(\eta) \right|^{p-1}} d\eta \leq
\end{aligned}$$

$$\le \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} |F_1(\xi) \circledast F_2(\varphi(\xi,\eta))|^p \, |\rho_1(\xi) \circledast \rho_2(\varphi(\xi,\eta))| \, |\varphi_\eta(\xi,\eta)| \, d\xi d\eta \le$$

$$\le \int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} |F_1(\xi)|^p \, |F_2(\varphi(\xi,\eta))|^p \, \rho_1(\xi)\rho_2(\varphi(\xi,\eta)) \, (\beta_1(\xi)\beta_2(\varphi(\xi,\eta)))^{p/q} \times$$

$$\times \, (\beta_1(\xi)\beta_2(\varphi(\xi,\eta)))^{-p/q} \, |\varphi_\eta(\xi,\eta)| \, d\xi d\eta \le$$

$$\le \left(\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} |F_1(\xi)|^q \, |F_2(\varphi(\xi,\eta))|^q \, \beta_1(\xi)\beta_2(\varphi(\xi,\eta)) \, |\varphi_\eta(\xi,\eta)| \, d\xi d\eta \right)^{p/q} \times$$

$$\times \left(\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} \left(\frac{\rho_1(\xi)\rho_2(\varphi(\xi,\eta))}{(\beta_1(\xi)\beta_2(\varphi(\xi,\eta)))^{p/q}} \right)^{q/(q-p)} |\varphi_\eta(\xi,\eta)| \, d\xi d\eta \right)^{1-p/q} \le$$

$$\le C \left(\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} |F_1(\xi)|^q \, |F_2(\varphi(\xi,\eta))|^q \, \beta_1(\xi)\beta_2(\varphi(\xi,\eta)) \, |\varphi_\eta(\xi,\eta)| \, d\xi d\eta \right)^{p/q} =$$

$$= C \left(\int\limits_{\mathbb{R}^m} |F_1(\xi)|^q \, \beta_1(\xi) \left(\int\limits_{\mathbb{R}^m} |F_2(\varphi(\xi,\eta))|^q \, \beta_2(\varphi(\xi,\eta)) \, |\varphi_\eta(\xi,\eta)| \, d\eta \right) d\xi \right)^{p/q} =$$

$$= C \, \|F_1\|^p_{L^q(\mathbb{R}^m,\beta_1)} \, \|F_2\|^p_{L^q(\mathbb{R}^m,\beta_2)}$$

and the inequality (3.2) is obtained. □

Now, we prove the following:

Theorem 3.2. *Let $1 < p < q < \infty$ and $\rho_j, \beta_j, j = 1, 2, \dots, n$ be weights such that the convolution $\prod_{j=1}^n \circledast_\varphi \rho_j$ exists. Then the inequality*

$$\left\| \left(\prod_{j=1}^n \circledast_\varphi (F_j \rho_j) \right) \left(\prod_{j=1}^n \circledast_\varphi \rho_j \right)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)} \le C_1 \prod_{j=1}^n \|F_j\|_{L^q(\mathbb{R}^m,\beta_j)} \quad (3.3)$$

holds for some constant $C_1 > 0$ and for all functions $F_j \in L^q(\mathbb{R}^m, \beta_j)$, $j = 1, 2, \dots, n$, if and only if

$$\sup_{r>0} \prod_{j=1}^n \left\| \frac{\rho_j}{\beta_j^{p/q}} \right\|_{L^{\frac{q}{q-p}}(I_r)} < \infty. \quad (3.4)$$

Proof. Assume first that (3.4) holds. We shall apply induction procedure. For $n = 2$, the assertion follows from Lemma 3.1. For $n = k + 1$, we apply Hölder's inequality for $q/p > 1$, Fubini's theorem, change of variables and induction hypothesis, and obtain

$$\left\| \left(\prod_{j=1}^{k+1} \circledast_\varphi (F_j \rho_j) \right) \left(\prod_{j=1}^{k+1} \circledast_\varphi \rho_j \right)^{1/p-1} \right\|^p_{L^p(\mathbb{R}^m)} \le$$

$$\leq \int\limits_{\mathbb{R}^m} \Big[\int\limits_{\mathbb{R}^m} \Big| \Big(\prod_{j=1}^{k} \circledast_{\varphi} (F_j \rho_j) \Big)(\xi) \Big| \left| (F_{k+1}\rho_{k+1})(\varphi(\xi,\eta)) \right| \left| \varphi_\eta(\xi,\eta) \right| d\xi \Big]^p \times$$

$$\times \Big[\int\limits_{\mathbb{R}^m} \Big| \Big(\prod_{j=1}^{k} \circledast_{\varphi} \rho_j \Big)(\xi) \Big| \left| \rho_{k+1} \left(\varphi(\xi,\eta) \right) \right| \left| \varphi_\eta(\xi,\eta) \right| d\xi \Big]^{1-p} d\eta \leq$$

$$\leq \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^m} \Big| \Big(\prod_{j=1}^{k} \circledast_{\varphi} (F_j \rho_j) \Big)(\xi) \Big|^p \left| (F_{k+1}\rho_{k+1}) \left(\varphi(\xi,\eta) \right) \right|^p \Big| \Big(\prod_{j=1}^{k} \circledast_{\varphi} \rho_j \Big)(\xi) \Big|^{1-p} \times$$

$$\times \left(\rho_{k+1} \left(\varphi(\xi,\eta) \right) \right)^{1-p} \left| \varphi_\eta(\xi,\eta) \right| d\xi d\eta =$$

$$= \int\limits_{\mathbb{R}^m} \Big| \Big(\prod_{j=1}^{k} \circledast_{\varphi} (F_j \rho_j) \Big)(\xi) \Big|^p \Big| \Big(\prod_{j=1}^{k} \circledast_{\varphi} \rho_j \Big)(\xi) \Big|^{1-p} \Big[\int\limits_{\mathbb{R}^m} \left| (F_{k+1}\rho_{k+1})(\varphi(\xi,\eta)) \right|^p \times$$

$$\times \left(\rho_{k+1} \left(\varphi(\xi,\eta) \right) \right)^{1-p} \left| \varphi_\eta(\xi,\eta) \right| d\eta \Big] d\xi =$$

$$= \Big\| \Big(\prod_{j=1}^{k} \circledast_{\varphi} (F_j \rho_j) \Big) \Big(\prod_{j=1}^{k} \circledast_{\varphi} \rho_j \Big)^{1/p-1} \Big\|_{L^p(\mathbb{R}^m)}^p \times$$

$$\times \int\limits_{\mathbb{R}^m} \left| F_{k+1}(\sigma) \right|^p \rho_{k+1}(\sigma) \beta_{k+1}^{-p/q}(\sigma) \beta_{k+1}^{p/q}(\sigma) d\sigma \leq$$

$$\leq \prod_{j=1}^{k} \Big\| \frac{\rho_j}{\beta_j^{p/q}} \Big\|_{L^{q/(q-p)}(\mathbb{R}^m)} \prod_{j=1}^{k} \| F_j \|_{L^q(\mathbb{R}^m,\beta_j)}^p \Big(\int\limits_{\mathbb{R}^m} \left| F_{k+1}(\sigma) \right|^q \beta_{k+1}(\sigma) d\sigma \Big)^{p/q} \times$$

$$\times \Big(\int\limits_{\mathbb{R}^m} \Big(\frac{\rho_{k+1}(\sigma)}{\beta_{k+1}^{p/q}(\sigma)} \Big)^{q/(q-p)} d\sigma \Big)^{(q-p)/q} =$$

$$= \prod_{j=1}^{k+1} \Big\| \frac{\rho_j}{\beta_j^{p/q}} \Big\|_{L^{q/(q-p)}(\mathbb{R}^m)} \prod_{j=1}^{k+1} \| F_j \|_{L^q(\mathbb{R}^m,\beta_j)}^p \leq C_1^p \prod_{j=1}^{k+1} \| F_j \|_{L^q(\mathbb{R}^m,\beta_j)}^p$$

with $C_1^p = \sup\limits_{r>0} \prod\limits_{j=1}^{n} \Big\| \dfrac{\rho_j}{\beta_j^{p/q}} \Big\|_{L^{\frac{q}{q-p}}(I_r)}$ and (3.3) follows.

Conversely, assume that (3.3) holds. For $j = 1, 2, \ldots, n$, we use the a.e. finiteness of the weights β_j. There exist sets A_j with $m(A_j) = 0$ such that β_j are finite on $\mathbb{R}^m \setminus A_j$. Similarly, there exist sets B_j with $m(B_j) = 0$ such that ρ_j are finite on $\mathbb{R}^m \setminus B_j$. If we write $E_j = A_j \cup B_j$, then clearly measure of E_j is zero, i.e., $m(E_j) = 0$. Now, define the function

$$F_j = \chi_{I_r - E_j} \left(\frac{\beta_j}{\rho_j} \right)^{1/(p-q)}. \tag{3.5}$$

In view of the above arguments, we find that F_j is a positive, measurable, finite and bounded function. Then there exist m_j, M_j such that

$$0 < m_j^{1/p} \leq F_j \leq M_j^{1/p} < \infty.$$

It was proved in ([8], p. 82) that the following inequality holds:

$$\left(\left(\prod_{j=1}^{n} *_{\varphi} (F_j\rho_j)\right)(\eta)\right)^p \left(\left(\prod_{j=1}^{n} *_{\varphi}\rho_j\right)(\eta)\right)^{1-p} \geq$$
$$\geq \prod_{i=2}^{n} \left\{ A_{p,p'}^{-np} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \left(\prod_{j=1}^{n} *_{\varphi}(F_j^p\rho_j) \right)(\eta), \quad \eta \in \mathbb{R}^m. \tag{3.6}$$

Note that if F_j's in (3.6) are taken as defined by (3.5), then (3.6) also holds with the operation $*_{\varphi}$ replaced by $\circledast_{\varphi}$ since these functions are real valued. Thus, the following holds:

$$\left(\left(\prod_{j=1}^{n} \circledast_{\varphi} (F_j\rho_j)\right)(\eta)\right)^p \left(\left(\prod_{j=1}^{n} \circledast_{\varphi}\rho_j\right)(\eta)\right)^{1-p} \geq$$
$$\geq \prod_{i=2}^{n} \left\{ A_{p,p'}^{-np} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \left(\prod_{j=1}^{n} \circledast_{\varphi}(F_j^p\rho_j) \right)(\eta), \quad \eta \in \mathbb{R}^m. \tag{3.7}$$

By using (3.7), the inequality (3.3) reduces to

$$\prod_{i=2}^{n} \left\{ A_{p,p'}^{-np} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \int\limits_{\mathbb{R}^m} \left| \prod_{j=1}^{n} \circledast_{\varphi}(F_j^p\rho_j) \right| (\eta) d\eta \leq C_1^p \prod_{j=1}^{n} \|F_j\|_{L_q(\mathbb{R}^m,\beta_j)}^p,$$

i.e.,

$$\prod_{i=2}^{n} \left\{ A_{p,p'}^{-np} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \int\limits_{\mathbb{R}^m} \left| \int\limits_{\mathbb{R}^m} \left(\prod_{j=1}^{n-1} \circledast_{\varphi}(F_j^p\rho_j) \right)(\xi) \times \right.$$
$$\left. \times \left(F_n^p\rho_n\right)\left(\varphi(\xi,\eta)\right) \left|\varphi_\eta(\xi,\eta)\right| d\xi \right| d\eta \leq C_1^p \prod_{j=1}^{n} \|F_j\|_{L_q(\mathbb{R}^m,\beta_j)}^p,$$

which by Fubini's theorem and change of variables gives

$$\prod_{i=2}^{n} \left\{ A_{p,p'}^{-np} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \int\limits_{\mathbb{R}^m} \left(\prod_{j=1}^{n-1} \circledast_{\varphi}(F_j^p\rho_j) \right)(\xi) d\xi \int\limits_{\mathbb{R}^m} (F_n^p\rho_n)(\sigma) d\sigma \leq$$
$$\leq C_1^p \prod_{j=1}^{n} \|F_j\|_{L_q(\mathbb{R}^m,\beta_j)}^p.$$

By using the definition of convolution, then Fubini's theorem and change of variables $(n-2)$ times on the last inequality, we get

$$\prod_{i=2}^{n}\left\{A_{p,p'}^{-np}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\prod_{j=1}^{n}\int_{\mathbb{R}^m}\left(F_j^p\rho_j\right)(x)dx\le C_1^p\prod_{j=1}^{n}\|F_j\|_{L_q(\beta_j)}^p .$$

In view of the fact that $m(E_j)=0$, the last inequality, for F_j's given by (3.5), becomes

$$\prod_{j=1}^{n}\left(\int_{I_r}\left(\frac{\rho_j(x)}{\beta_j^{p/q}(x)}\right)^{q/(q-p)}dx\right)^{(q-p)/q}\le C_1^p\prod_{i=2}^{n}\left\{A_{p,p'}^{np}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}<\infty$$

or

$$\sup_{r>0}\prod_{j=1}^{n}\left\|\frac{\rho_j}{\beta_j^{p/q}}\right\|_{L^{q/(q-p)}(I_r)}<\infty$$

and the result is proved. $\square$

Remark 3.3. If $\rho_j=\beta_j$ in Theorem 3.2, then we can allow $p=q$ as well and the corresponding inequality extends Theorem A as well as inequality (1.5). Thus Theorem 3.2 is much more general than the similar existing results.

Remark 3.4. It is of interest to investigate the case $q\le p$ when $\beta_j\ne\rho_j$.

4. Reverse Inequality

In this section, we shall prove some reverse convolution inequalities. We shall be using the following version of reverse Hölder's inequality [8] (see also [8-11]):

Theorem C. *Let $p>1$, $\frac{1}{p}+\frac{1}{p'}=1$, p be a weight function and f be a positive function defined on $\Omega\subseteq\mathbb{R}^m$ satisfying*

$$0<m^{1/p}\le f(x)\le M^{1/p}<\infty.$$

Then the inequality

$$\left(\int_{\Omega}|f(x)|^p\rho(x)dx\right)^{1/p}\left(\int_{\Omega}\rho(x)dx\right)^{1/p'}\le A_{p,p'}^n\left(\frac{m}{M}\right)\int_{\Omega}f(x)\rho(x)dx$$

holds with

$$A_{p,p'}(t)=p^{-\frac{1}{p}}p'^{-\frac{1}{p'}}\frac{t^{-\frac{1}{pp'}}(1-t)}{\left(1-t^{\frac{1}{p}}\right)^{\frac{1}{p}}\left(1-t^{\frac{1}{p'}}\right)^{\frac{1}{p'}}}.$$

We begin with the following lemma:

Lemma 4.1. *Let $1 < q < p < \infty$, $\rho_j, \beta_j, j = 1, 2$ be weights on $\mathbb{R}^m$ and F_1, F_2 be measurable functions satisfying*

$$0 < m_1^{1/p} \le F_1 \le M_1^{1/p} < \infty, \quad 0 < m_2^{1/p} \le F_2 \le M_2^{1/p} < \infty.$$

If

$$D := \sup_{r>0} \left\| \frac{\beta_1^{p/q}}{\rho_1} \right\|_{L^{\frac{q}{p-q}}(I_r)} \left\| \frac{\beta_2^{p/q}}{\rho_2} \right\|_{L^{\frac{q}{p-q}}(I_r)} < \infty \tag{4.1}$$

then, the inequality

$$\begin{aligned} D_1 \Big\| \big((F_1\rho_1) *_\varphi (F_2\rho_2)\big) \left(\rho_1 *_\varphi \rho_2\right)^{1/p-1} \Big\|_{L^p(\mathbb{R}^m)} \ge \\ \ge \|F_1\|_{L^q(\mathbb{R}^m,\beta_1)} \|F_2\|_{L^q(\mathbb{R}^m,\beta_2)} \end{aligned} \tag{4.2}$$

holds for all functions $F_j \in L^q(\mathbb{R}^m, \beta_j)$, $j = 1, 2$ with $D_1 = A^n_{p,p'} \left(\frac{m_1 m_2}{M_1 M_2}\right) D^{1/p}$.

Proof. We have, by applying Hölder's inequality for $p/q > 1$, Fubini's theorem and change of variables,

$$\begin{aligned} &\|F_1\|^p_{L^q(\mathbb{R}^m,\beta_1)} \|F_2\|^p_{L^q(\mathbb{R}^m,\beta_2)} = \\ &= \left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |F_1(\xi)|^q |F_2(\varphi(\xi,\eta))|^q \beta_1(\xi)\beta_2(\varphi(\xi,\eta)) \left|\varphi_\eta(\xi,\eta)\right| d\xi d\eta \right)^{p/q} \le \\ &\le \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |F_1(\xi)|^p |F_2(\varphi(\xi,\eta))|^p \rho_1(\xi)\rho_2(\varphi(\xi,\eta)) \left|\varphi_\eta(\xi,\eta)\right| d\xi d\eta \times \\ &\quad \times \left(\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \left(\frac{\beta_1(\xi)\beta_2(\varphi(\xi,\eta))}{\rho_1^{q/p}(\xi)\rho_2^{q/p}(\varphi(\xi,\eta))} \right)^{p/(p-q)} \left|\varphi_\eta(\xi,\eta)\right| d\xi d\eta \right)^{(p-q)/q} = \\ &= \prod_{j=1}^{2} \left(\int_{\mathbb{R}^m} \left(\frac{\beta_j^{p/q}(\eta)}{\rho_j(\eta)} \right)^{q/(p-q)} d\eta \right)^{(p-q)/q} \times \\ &\quad \times \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} |F_1(\xi)|^p |F_2(\varphi(\xi,\eta))|^p \rho_1(\xi)\rho_2(\varphi(\xi,\eta)) \left|\varphi_\eta(\xi,\eta)\right| d\xi d\eta. \end{aligned} \tag{4.3}$$

Let us consider

$$f(\xi) = F_1^p(\xi) F_2^p(\varphi(\xi,\eta)) \rho_1(\xi)\rho_2(\varphi(\xi,\eta)) \left|\varphi_\eta(\xi,\eta)\right|$$

and

$$g(\xi) = \rho_1(\xi)\rho_2(\varphi(\xi,\eta)) \left|\varphi_\eta(\xi,\eta)\right|.$$

Then for all $\xi \in \mathbb{R}^m$, we find, by using the hypothesis, that

$$0 < m_1 m_2 \le \frac{f(\xi)}{g(\xi)} \le M_1 M_2 < \infty.$$

Now, in view of Theorem C, we obtain

$$A^n_{p,p'}\left(\frac{m_1m_2}{M_1M_2}\right)\int\limits_{\mathbb{R}^m} F_1(\xi)\rho_1(\xi)F_2(\varphi(\xi,\eta))\rho_2(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,d\xi \geq$$

$$\geq \left(\int\limits_{\mathbb{R}^m} |F_1(\xi)|^p\rho_1(\xi)|F_2(\varphi(\xi,\eta))|^p\rho_2(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,d\eta\right)^{1/p}\times$$

$$\times\left(\int\limits_{\mathbb{R}^m} \rho_1(\xi)\rho_2(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,d\xi\right)^{1-1/p}$$

or

$$\left\|\,((F_1\rho_1)*_\varphi(F_2\rho_2))\,(\rho_1*_\varphi\rho_2)^{1/p-1}\,\right\|^p_{L^p(\mathbb{R}^m)} \geq A^{-np}_{p,p'}\left(\frac{m_1m_2}{M_1M_2}\right)\times$$

$$\times\int\limits_{\mathbb{R}^m}\int\limits_{\mathbb{R}^m} |F_1(\xi)|^p\rho_1(\xi)|F_2(\varphi(\xi,\eta))|^p\rho_2(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,d\xi d\eta$$

which on using (4.3) gives

$$\|F_1\|_{L^q(\mathbb{R}^m,\beta_1)}\|F_2\|_{L^q(\mathbb{R}^m,\beta_2)} \leq \prod_{j=1}^{2}\left\|\frac{\beta_j^{p/q}}{\rho_j}\right\|^{1/p}_{L^{\frac{q}{(p-q)}}(\mathbb{R}^m)} A^n_{p,p'}\left(\frac{m_1m_2}{M_1M_2}\right)\times$$

$$\times\left\|((F_1\rho_1)*_\varphi(F_2\rho_2))\,(\rho_1*_\varphi\rho_2)^{1/p-1}\right\|_{L^p(\mathbb{R}^m)} \leq$$

$$\leq D_1\left\|((F_1\rho_1)*_\varphi(F_2\rho_2))\,(\rho_1*_\varphi\rho_2)^{1/p-1}\right\|_{L^p(\mathbb{R}^m)}$$

and the assertion is proved. □

Now, we prove the following characterization:

Theorem 4.2. *Let $1<q<p<\infty$ and $\rho_j,\beta_j, j=1,2,\dots,n$ be weights such that the convolution $\prod_{j=1}^n *_\varphi\rho_j$ exists. Let F_j be functions satisfying*

$$0<m_j^{1/p}\leq F_j\leq M_j^{1/p}<\infty,\quad j=1,2,\dots,n.$$

Then the inequality

$$D_2\left\|\left(\prod_{j=1}^n *_\varphi(F_j\rho_j)\right)\left(\prod_{j=1}^n *_\varphi\rho_j\right)^{1/p-1}\right\|_{L^p(\mathbb{R}^m)} \geq \prod_{j=1}^n\|F_j\|_{L^q(\mathbb{R}^m,\beta_j)} \tag{4.4}$$

holds for some constant $D_2>0$ and for all functions $F_j\in L^q(\mathbb{R}^m,\beta_j)$, $j=1,2,\dots,n$, if and only if

$$\mathfrak{D} := \sup_{r>0}\prod_{j=1}^n\left\|\frac{\beta_j^{p/q}}{\rho_j}\right\|_{L^{\frac{q}{p-q}}(I_r)} <\infty,\quad j=1,2,\dots,n. \tag{4.5}$$

Proof. Assume first that (4.5) holds. We shall apply induction procedure. For $n = 2$, the assertion follows from Lemma 4.1. For $n = k + 1$, we apply induction hypothesis, Hölder's inequality for $p/q > 1$ and change of variables, and obtain

$$\prod_{j=1}^{k+1} \|F_j\|_{L^q(\mathbb{R}^m,\beta_j)} = \prod_{j=1}^{k} \|F_j\|_{L^q(\mathbb{R}^m,\beta_j)} \|F_{k+1}\|_{L^q(\mathbb{R}^m,\beta_j)} \leq$$
$$\leq \prod_{j=1}^{k} \left\| \frac{\beta_j^{p/q}}{\rho_j} \right\|_{L^{q/(p-q)}(\mathbb{R}^m)}^{1/p} \prod_{i=2}^{k} \left\{ A_{p,p'}^{n} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \times$$
$$\times \left\| \left(\prod_{j=1}^{k} *_{\varphi} (F_j \rho_j) \right) \left(\prod_{j=1}^{k} *_{\varphi} \rho_j \right)^{1/p-1} \right\|_{L^p(\mathbb{R}^m)} \|F_{k+1}\|_{L^q(\mathbb{R}^m,\beta_j)} =$$
$$= \prod_{j=1}^{k} \left\| \frac{\beta_j^{p/q}}{\rho_j} \right\|_{L^{q/(p-q)}(\mathbb{R}^m)}^{1/p} \prod_{i=2}^{k} \left\{ A_{p,p'}^{n} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \times$$
$$\times \left(\int_{\mathbb{R}^m} \left| \left(\prod_{j=1}^{k} *_{\varphi} (F_j \rho_j) \right)(\xi) \right|^p \left| \left(\prod_{j=1}^{k} *_{\varphi} \rho_j \right)(\xi) \right|^{1-p} d\xi \right)^{1/p} \times$$
$$\times \left(\int_{\mathbb{R}^m} |F_{k+1}|^p (\sigma) \rho_{k+1}(\sigma) d\sigma \right)^{1/p} \times$$
$$\times \left(\int_{\mathbb{R}^m} \left(\frac{\beta_{k+1}^{p/q}(\sigma)}{\rho_{k+1}(\sigma)} d\sigma \right)^{q/(p-q)} d\sigma \right)^{(p-q)/pq} =$$
$$= \prod_{j=1}^{k+1} \left\| \frac{\beta_j^{p/q}}{\rho_j} \right\|_{L^{q/(p-q)}(\mathbb{R}^m)}^{1/p} \prod_{i=2}^{k} \left\{ A_{p,p'}^{n} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \times$$
$$\times \left(\int_{\mathbb{R}^m} \int_{\mathbb{R}^m} \left| \left(\prod_{j=1}^{k} *_{\varphi} (F_j \rho_j) \right)(\xi) \right|^p \left| \left(\prod_{j=1}^{k} *_{\varphi} \rho_j \right)(\xi) \right|^{1-p} \times \right.$$
$$\left. \times |F_{k+1}|^p (\varphi(\xi,\eta)) \rho_{k+1}(\varphi(\xi,\eta)) |\varphi_\eta(\xi,\eta)| \, d\xi d\eta \right)^{1/p}. \tag{4.6}$$

Let us consider

$$f_\eta^p(\xi) = \left(\prod_{j=1}^{k} *_{\varphi} (F_j \rho_j) \right)^p (\xi) \times$$
$$\times \left(\prod_{j=1}^{k} *_{\varphi} \rho_j \right)^{1-p} (\xi) \left(F_{k+1}^p \rho_{k+1} \right) (\varphi(\xi,\eta)) |\varphi_\eta(\xi,\eta)| ,$$

and

$$g_\eta^{p'}(\xi) = \left(\prod_{j=1}^{k} *_\varphi \rho_j\right)(\xi)\rho_{k+1}(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,.$$

Then, for all $\xi \in \mathbb{R}^m$

$$0 < \prod_{j=1}^{k+1} m_j \le \frac{f_\eta^p(\xi)}{g_\eta^{p'}(\xi)} \le \prod_{j=1}^{k+1} M_j < \infty.$$

Now, in view of Theorem C, we obtain

$$A^n_{p,p'}\left(\prod_{j=1}^{k+1}\frac{m_j}{M_j}\right)\int\limits_{\mathbb{R}^m} f_\eta(\xi)g_\eta(\xi)d\xi \ge \left(\int\limits_{\mathbb{R}^m} f_\eta^p(\xi)d\xi\right)^{1/p}\left(\int\limits_{\mathbb{R}^m} g_\eta^{p'}(\xi)d\xi\right)^{1/p'}$$

i.e.,

$$\begin{aligned}
A^n_{p,p'}\left(\prod_{j=1}^{k+1}\frac{m_j}{M_j}\right)&\int\limits_{\mathbb{R}^m}\left(\prod_{j=1}^{k} *_\varphi (F_j\rho_j)\right)(\xi)\left(F_{k+1}\rho_{k+1}\right)(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,d\xi \ge \\
&\ge \left(\int\limits_{\mathbb{R}^m}\left(\prod_{j=1}^{k} *_\varphi (F_j\rho_j)\right)^p(\xi)\left(\prod_{j=1}^{k} *_\varphi \rho_j\right)^{1-p}(\xi)\times\right. \\
&\qquad \left.\times\left(F^p_{k+1}\rho_{k+1}\right)(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,d\xi\right)^{1/p}\times \\
&\qquad \times\left(\int\limits_{\mathbb{R}^m}\left(\prod_{j=1}^{k} *_\varphi \rho_j\right)(\xi)\rho_{k+1}(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,d\xi\right)^{1-1/p}
\end{aligned}$$

or

$$\begin{aligned}
\int\limits_{\mathbb{R}^m}&\left(\prod_{j=1}^{k} *_\varphi (F_j\rho_j)\right)^p(\xi)\left(\prod_{j=1}^{k} *_\varphi \rho_j\right)^{1-p}(\xi)\times \\
&\qquad\times\left(F^p_{k+1}\rho_{k+1}\right)(\varphi(\xi,\eta))\,|\varphi_\eta(\xi,\eta)|\,d\xi \le \\
&\le A^{np}_{p,p'}\left(\prod_{j=1}^{k+1}\frac{m_j}{M_j}\right)\left(\prod_{j=1}^{k+1} *_\varphi (F_j\rho_j)\right)^p(\eta)\left(\prod_{j=1}^{k+1} *_\varphi \rho_j\right)^{1-p}(\eta). \qquad (4.7)
\end{aligned}$$

By using (4.7) in (4.6), we obtain

$$\begin{aligned}
\prod_{j=1}^{k+1}\|F_j\|_{L^q(\mathbb{R}^m,\beta_j)} \le \prod_{j=1}^{k+1}&\left\|\frac{\beta_j^{p/q}}{\rho_j}\right\|^{1/p}_{L^{q/(p-q)}(\mathbb{R}^m)}\prod_{i=2}^{k+1}\left\{A^n_{p,p'}\left(\prod_{j=1}^{i}\frac{m_j}{M_j}\right)\right\}\times \\
&\times\left(\int\limits_{\mathbb{R}^m}\left|\left(\prod_{j=1}^{k+1} *_\varphi (F_j\rho_j)\right)(\eta)\right|^p\left|\left(\prod_{j=1}^{k+1} *_\varphi \rho_j\right)(\eta)\right|^{1-p}d\eta\right)^{1/p}.
\end{aligned}$$

Taking supremum over $r > 0$, we find that the inequality (4.4) holds with

$$D_2 := \prod_{i=2}^{n} \left\{ A_{p,p'}^{n} \left(\prod_{j=1}^{i} \frac{m_j}{M_j} \right) \right\} \mathfrak{D}^{1/p}.$$

Conversely, assume that (4.4) holds. We shall be using the following inequality proved in ([7], Lemma 2.7):

For $\eta \in \mathbb{R}^m$, we have

$$\left| \left(\prod_{j=1}^{n} *_{\varphi} (F_j \rho_j) \right)(\eta) \right|^p \leq \left| \left(\prod_{j=1}^{n} *_{\varphi} \rho_j \right)(\eta) \right|^{p-1} \left(\left(\prod_{j=1}^{n} *_{\varphi} (|F_j|^p \rho_j) \right)(\eta) \right)$$

using which (4.4) gives

$$\begin{aligned} \prod_{j=1}^{n} \|F_j\|^p_{L^q(\mathbb{R}^m, \beta_j)} &\leq D_2^p \int\limits_{\mathbb{R}^m} \left(\left(\prod_{j=1}^{n} *_{\varphi} (|F_j|^p \rho_j) \right)(\eta) \right) d\eta = \\ &= D_2^p \int\limits_{\mathbb{R}^m} \int\limits_{\mathbb{R}^m} \left(\left(\prod_{j=1}^{n-1} *_{\varphi} (|F_j|^p \rho_j) \right)(\xi) \right) \times \\ &\quad \times \left(|F_n|^p \rho_n \right) \left(\varphi\left(\xi, \eta\right) \right) \left| \left(\varphi_\eta \left(\xi, \eta\right) \right) \right| d\xi d\eta = \\ &= D_2^p \int\limits_{\mathbb{R}^m} \left(\left(\prod_{j=1}^{n-1} *_{\varphi} (|F_j|^p \rho_j) \right)(\xi) \right) d\xi \int\limits_{\mathbb{R}^m} \left(|F_n|^p \rho_n \right)(\sigma) d\sigma. \end{aligned}$$

By applying the definition of convolution, then Fubini's theorem and change of variables $(n-2)$ times on the last inequality, we get

$$\prod_{j=1}^{n} \|F_j\|^p_{L^q(\mathbb{R}^m, \beta_j)} \leq D_2^p \prod_{j=1}^{n} \int\limits_{\mathbb{R}^m} \left(|F_j|^p \rho_j \right)(x) dx. \tag{4.8}$$

For $j = 1, 2, \ldots, n$, consider the functions as defined in (3.5), i.e.,

$$F_j = \chi_{I_r - E_j} \left(\frac{\beta_j}{\rho_j} \right)^{1/(p-q)}.$$

Then, for these functions, (4.8) gives

$$\prod_{j=1}^{n} \left(\int\limits_{I_r} \left(\frac{\beta^{p/q}(x)}{\rho(x)} \right)^{q/(p-q)} dx \right)^{(p-q)/q} \leq D_2^p.$$

Taking supremum over $r > 0$, the necessity follows. □

Remark 4.3. In Theorem 4.2, $q < p$. However, if $\beta_j = \rho_j$, then we can allow $p = q$ as well. In that case, 4.5 is automatically satisfied. The corresponding inequality (4.4) is the one proved in [8].

Remark 4.4. It is of interest to investigate the remaining case $p \leq q$ when $\beta_j \neq \rho_j$.

5. Special Cases

The inequalities proved in this paper include several known convolutions such as Fourier convolution, Mellin convolution, Laplace convolution. As an example, we derive Mellin convolution inequality, as a special case, from Theorem 2.2 that has very recently been proved in [6]. Note that the inequalities proved in the previous sections in the framework of $\mathbb{R}^m$ are also valid for $\mathbb{R}^m_+$.

Example 5.1. Let $n = 2$, $F_1 \circledast_\varphi F_2 = F_1 *_\varphi F_2$, $\varphi(\xi, \eta) = \xi\eta$ and replace ρ_j by $\rho_j^{q/p}$, $j = 1, 2$ in the inequality (2.1). Here $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, we get the following inequality

$$\| (F_1 *_\varphi F_2) \|_{L^p(\mathbb{R}^m_+, \rho^{-1})} \leq \|F_1\|_{L^p(\mathbb{R}^m_+, \rho_1^{-1})} \|F_2\|_{L^p(\mathbb{R}^m_+, \rho_2^{-1})}, \tag{5.1}$$

which holds for all functions $F_j \in L^p(\mathbb{R}^m_+, \rho_j^{-1}), j = 1, 2$, where

$$\rho(\eta) = \left[\left(\rho_1^{q/p} *_\varphi \rho_2^{q/p} \right) (\eta) \right]^{p/q}, \quad \eta \in \mathbb{R}^m_+.$$

The inequality (5.1) is exactly the one proved in ([6], (1.4)), $*_\varphi$ being the Mellin convolution used there.

The reverse of the inequality (5.1) can also be derived easily by Theorem 4.2 (in view of Remark 4.3) or directly by using Theorem B. Precisely, we have the following:

Example 5.2. Let $n = 2$, $F_1 \circledast_\varphi F_2 = F_1 *_\varphi F_2$, $\varphi(\xi, \eta) = \xi\eta$ and ρ_j replaced by $\rho_j^{q/p}$, $j = 1, 2$ in the inequality (1.8). Here $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, we get the following inequality

$$\begin{aligned} &\left\| (F_1 *_\varphi F_2) \right\|_{L^p(\mathbb{R}^m_+, \rho^{-1})} \geq \\ &\qquad \geq A_{p,p'}^{-n} \left(\frac{m_1 m_2}{M_1 M_2} \right) \|F_1\|_{L^p(\mathbb{R}^m_+, \rho_1^{-1})} \|F_2\|_{L^p(\mathbb{R}^m_+, \rho_2^{-1})}, \end{aligned} \tag{5.2}$$

which holds for all functions $F_j \in L^p(\mathbb{R}^m_+, \rho_j^{-1}), j = 1, 2$, where

$$\rho(\eta) = \left[\left(\rho_1^{q/p} *_\varphi \rho_2^{q/p} \right) (\eta) \right]^{p/q}, \quad \eta \in \mathbb{R}^m_+.$$

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