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THE PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR A POLYGONAL DOMAIN WITH A RECTILINEAR CUT

R. BANTSURI AND G. KAPANADZE

Abstract. The plane problem of elasticity for a polygonal domain with a rectilinear cut is considered under the condition that uniformly distributed stretching forces or normal displacements (i.e., under the conditions of the third modified problem of elasticity) are prescribed on the external boundary of the domain, while the cut edges are free from external forces. For solving the problem, the methods of conformal mappings and those of the boundary value problems of analytic functions are used; solutions are given effectively (analytically). Estimates of solution behavior in the vicinity of angular points are presented.

რეზიუმე. განხილულია დრეკადობის თეორიის ბრტყელი ამოცანა სწორხაზოვანი ჭრილის მქონე მრავალკუთხა არისათვის იმ პირობით, რომ არის გარე საზღვარზე ცნობილია თანაბრადგანაწილებული ნორმალური გამჭიმავი ძაბვები ან ნორმალური გადაადგილებები (ე. ი. მესამე სახეშეცვლილი ამოცანის პირობები), ხოლო ჭრილის საზღვარი თავისუფალია გარეგანი დატვირთებისაგან. ამოცანის ამოსახსნელად გამოყენებულია კონფორმულ ასახვათა და ანალიზურ ფუნქციათა სასაზღვრო ამოცანების მეთოდები და ამონახსნი წარმოდგენილია ეფექტური (ანალიზური) ფორმით. მოყვანილია ამონახსნების შეფასებები კუთხეების წვეროთა მახლობლობაში.

In the present paper we consider the plane problem of the theory of elasticity for a finite doubly-connected domain which is bounded by a convex polygon and by a rectilinear cut. It is assumed that the polygon sides are under the action of uniformly distributed normal stretching forces, or normal displacements (i.e., the conditions of the third modified problem of the theory of elasticity), while the cut edges are free from external forces.

To solve the problem under consideration, we use the methods of conformal mappings and those of boundary value problems of analytic functions, and solutions are given in effective (analytic) form.

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Analogous problems of the plane theory of elasticity and plate bending for finite doubly-connected domains bounded by broken lines have been considered in [1] and [2].

Statement of the Problem. Let S be a doubly-connected domain on the plane z of a complex variable bounded by a convex polygon (A) with the boundary L_0 and by a rectilinear cut B_1B_2 with the boundary (cut edges) L_1. By A_l $(j=1,\dots,n)$ we denote vertices (and their affices) of the polygon (A) and direct the ox-axis along the segment B_1B_2, perpendicular to the side A_1A_n (this latter is of no importance). The point $z=0$ lies on the segment B_1B_2. By $\pi\alpha_j^0$ $(j=1,\dots,n)$ we denote sizes of the polygon inner angles and the positive sense on $L=L_0\cup L_1$ ($L_0=\bigcup_1^n L_0^{(k)}, L_0^{(k)}=A_kA_{k+1}$, $k=1,\dots,n$, $A_{n+1}=A_1$; $L_1=\bigcup_1^2 L_1^{(j)}$, $L_1^{(1)}=B_1B_2$, $L_1^{(2)}=B_2B_1$) is assumed that which leaves the domain S on the left. Let $\alpha(t)$ and $\beta(t)$ be the angles lying between the ox-axis and the outer normals to the contours L_0 and L_1 at the point $t\in L$, where $\beta(t)=-\frac{\pi}{2}$, $t\in L_1^{(1)}$ and $\beta(t)=-\frac{3}{2}\pi$, $t\in L_1^{(2)}$, $\alpha(t)=\alpha_k$, $t\in L_0^{(k)}$ (k=1,..., n).

Our problem is to find an elastic equilibrium of the domain if on the polygon boundary the values of the principal vector of external forces (which in our case has the form $C(t)=\mathrm{Re}[e^{-i\nu(t)}i\int_0^s N(t_0)e^{i\nu(t_0)}ds_0]$, $N(t)$ is a normal stress, $\nu(t)=\alpha(t)$, $t\in L_0$; $\nu(t)=\beta(t)$, $t\in L_1$), or normal displacements $v_n(t)$ are known.

Solution of the Problem. The problem is solved by using the methods of conformal mappings and of the theory of boundary value problems of analytic functions.

Here we present some results dealt with the conformally mapping function of a polygonal doubly-connected domain onto a circular ring [3]. In particular, the finding of a conformally mapping function $z=\omega(\zeta)$ of the domain under consideration onto a circular ring $D=\{1<|\sigma|<R\}$ is reduced to the Riemann-Hilbert problem

$$\mathrm{Re}\left[ite^{-i\nu(t)}\omega'(t)\right]=0,\quad t\in l,\tag{1}$$

where $l=l_0\cup l_1$, $l_0=\{|\zeta|=R\}$, $l_1=\{|\zeta|=1\}$.

To solve the problem (1) (with respect to the function $\omega'(\zeta)$) of the class $h(b_1,b_2)$ [4] (the index of the given class problem (1) is equal to zero), it is necessary and sufficient that the condition

$$\prod_{k=1}^n\left(\frac{a_k}{R}\right)^{\alpha_k^0-1}\cdot b_1\cdot b_2=1,\quad (A_k=\omega a_k,\quad B_k=\omega(b_k)),\tag{2}$$

be fulfilled, and a solution itself is given by the formula

$$\omega'(\zeta)=k_0\cdot\prod_{k=-\infty}^{\infty}G(R^{2k}\zeta)g(R^{2k}\zeta)R^2\zeta^{-2}R^{2\delta_k},\tag{3}$$

where

$$G(\zeta)=\prod_{j=1}^{n}(\zeta-a_j)^{\alpha_j^0-1};\quad g(\zeta)=\prod_{j=1}^{2}(\zeta-b_j);\quad \sigma_k\begin{cases}0, & k\geq 0\\ 1, & k\leq -1\end{cases},$$

K_0 is an arbitrary real constant.

When solving mixed problems of the plane theory of elasticity by the methods of Kolosov-Muskhelishvili [5], in the case of doubly-connected domains bounded by broken lines it becomes possible to reduce these problems to two Riemann-Hilbert problems for a circular ring with respect to the complex potentials $\varphi(z)$ and $\psi(z)$,

$$\operatorname{Re}\left[e^{-i\nu(t)}\varphi(t)\right]=f_{1j}(t),\quad t\in L_j\quad (j=0,1),\tag{4}$$

$$\operatorname{Re}\left[e^{-i\nu(t)}\left(\varphi(t)+\overline{\varphi'(t)}+\overline{\psi(t)}\right)\right]=f_{2j}(t),\quad t\in L_j,\tag{5}$$

where

$$f_{1j}(t)=\frac{1}{\varkappa+1}\left[C_j(t)+2\mu v_n(t)+\operatorname{Re}(C_j^0+iC_j^1)\right],$$
$$f_2(t)=C(t)+\operatorname{Re}(C_j^0+iC_j^1),$$
$$t\in L_j,\quad j=0,1$$

(the constant C_j^0 and C_j^1 $(j=0,1)$ are unknown beforehand (two of them may be fixed arbitrarily).

$$C_j(t)=\begin{cases}C(t), & j=0\\ 0, & j=1.\end{cases}$$

Taking into account the above results concerning the problem (1), we can conclude that the function $e^{2i\nu(t)}$ can be represented in the form

$$e^{2i\nu(t)}=t\omega'(t)\cdot\left[\overline{t\omega'(t)}\right]^{-1},\quad t\in l_j,\quad j=0,1,$$

and, hence, the boundary value problem (4) with respect to the function $\Omega(\zeta)=[\zeta\omega'(\zeta)]^{-1}\varphi_0(\zeta)$ $\varphi_0(\zeta)=\varphi[\omega(\zeta)]$ reduces to the Riemann–Hilbert boundary value problem

$$\Omega(t)+\overline{\Omega(t)}=F_j(t),\quad t\in l_j,\quad j=0,1,\tag{6}$$

where

$$F_j(t)=2e^{i\nu(t)}[t\omega'(t)]^{-1}f_j(t).$$

The necessary and sufficient condition for the solvability of the problem (6) is the form

$$\int_0^{2\pi}F_1(e^{i\theta})d\theta=\int_0^{2\pi}F_0(\operatorname{Re}i^{i\theta})d\theta,\tag{7}$$

and a solution itself is given by the formula

$$\Omega(\zeta) = \frac{1}{2\pi i}\sum_{j=0}^{1}\int_{l_j} K(\zeta;t)F_j(t)dt + iC_0^{**}, \quad K(\zeta,t) = \sum_{k=-\infty}^{\infty}\frac{1}{t-R^{2k}\zeta}, \tag{8}$$

where C_0^{**} is the real constant.

Thus, the solution of the problem (4) has the form

$$\varphi_0(\zeta) = \zeta(\Omega)(\zeta)\omega'(\zeta),$$

where $\Omega(\zeta)$ is defined by formula (8).

Taking into account the type of the function $\omega'(\zeta)$ in the neighborhood of the point $(k = 1,\dots,n)$, we can conclude that for the function $\varphi_0(\zeta)$ to be continuously extendable in the domain $D + l$, it is necessary and sufficient that the condition

$$\Omega(a_k) = 0, \quad (k = 1,\dots,n) \tag{9}$$

be fulfilled.

Bearing in mind both the behavior of the Cauchy type integral in the neighborhood of the points of density discontinuity and that of the conformally mapping function in the neighborhood of angular points, we can prove that the function $\varphi(z) = \varphi_0'(\zeta)[\omega'(\zeta)]^{-1}$ in the neighborhood of the points B_j $(j = 1,2)$ satisfies the condition

$$|\varphi'(z)| < M|z - B_j|^{-1/2}, \quad M = \text{const},$$

but in the neighborhood of the points A_k $(k = 1,2,\dots,n)$ it is bounded. Analogously, for $\varphi''(z)$, we have the estimates

$$|\varphi''(z)| < M|z - B_j|^{-3/2}, \ \ j=1,2, \ \ |\varphi''(z)| < M|z - A_k|^{\frac{1}{\alpha_k^0}-2}, \ \ k=1,\dots,n.$$

After the function $\varphi(z)$ is at hand, the finding of the function $\psi(z)$ on the basis of (5) is reduced (with a simple modification) to the above-studied problem, i.e., to the Riemann-Hilbert boundary value problem with a bounded right-hand side with respect to the function $\Psi(z) = \psi(z) + P(z)\varphi'(z)$.

$$\mathrm{Re}[e^{i\nu(t)}\Psi(t)] = \Theta_j(t), \quad t \in L_j, \quad j = 1,2, \tag{10}$$

where

$$\Theta_j(t) = f_2(t) - \mathrm{Re}\left[e^{i\nu(t)}(\varphi(t) + (\bar{t} - P(t))\varphi'(t))\right];$$

$P(t)$ is the so-called interpolation polynomial defined by the condition $P(B_k) = \overline{B}_k$ $(k = 1,2)$.

A solution of the problem (10) (with the requirement for the function $\Psi(z)$ to be continuously extendable up to the boundary) can be constructed analogously to the foregoing one, and the conditions for solvability of that problem will have the form, similar to the conditions (7) and (9). All these conditions can be represented in the form of an inhomogeneous system (all together $n+4$ equations) with real coefficients with respect to the unknown

$n + 4$ real constants. It is proved that the obtained system is uniquely solvable and, hence, the problem under consideration has a unique solution.

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Authors' addresses:

R. Bantsuri

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6, Tamarashvili St., Tbilisi 0177, Georgia

G. Kapanadze

A. Razmadze Mathematical Institute
I. Javakhishvili Tbilisi State University
6, Tamarashvili St., Tbilisi 0177, Georgia

I. Vekua Institute Applied Mathematics
of I. Javakhishvili Tbilisi State University
2, University Str., Tbilisi 0186, Georgia