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ON THE RELATIONSHIP BETWEEN CONDITIONS OF THE DIFFERENTIABILITY AND EXISTENCE OF A GENERALIZED GRADIENT

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Abstract. It is proved that if admissible increments have property of anisotropic density, then there exists a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ which is differentiable almost everywhere but does not has the generalized gradient with respect to the given increments almost nowhere.

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1. Definitions and notation. Below everywhere we will assume that $n \in \mathbb{N}$ and $n \geq 2$.

For $h \in \mathbb{R}^n$ and $i \in \overline{1,n}$, denote by $h(i)$ the point in $\mathbb{R}^n$ such that $h(i)_j = h_j$ for every $j \in \overline{1,n} \backslash \{i\}$ and $h(i)_i = 0$. For $i \in \overline{1,n}$, by L_i will be denoted the hyperplane $\{h \in \mathbb{R}^n : h_i = 0\}$.

By Π_i $(i \in \overline{1,n})$ denote the class of all sets $\Delta \subset \mathbb{R}^n$ with the following properties: $\Delta \cap L_i = \emptyset$ and the origin 0 is a limit point for Δ.

Let $i \in \overline{1,n}$, $\Delta \in \Pi_i$ and f be a function defined in a neighborhood of a point $x \in \mathbb{R}^n$. If there exists the limit

$$\lim_{\Delta \ni h \to 0} \frac{f(x+h) - f(x+h(i))}{h_i},$$

then we call its value the *partial* (i, Δ)*-derivative of* f *at* x and denote it by $D_{i,\Delta} f(x)$.

A basis of gradient generating(briefly, *basis*) will be defined as an n-tuple $\Delta = (\Delta_1, \dots, \Delta_n)$, where $\Delta_i \in \Pi_i$ for every $i \in \overline{1,n}$.

If for a basis $\Delta = (\Delta_1, \dots, \Delta_n)$ a function f has finite partial (i, Δ_i)-derivative for every $i \in \overline{1,n}$ at x then we will say that f has the Δ-*gradient* at x.

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For an interval $I = I_1 \times \cdots \times I_n$ denote

$$r_i(I) = \frac{\max\limits_{j \neq i} |I_j|}{|I_i|} \quad (i \in \overline{1,n}).$$

A set $E \subset \mathbb{R}^n$ let us call *anisotropically dense at point* 0 *with respect to* i*-th variable*, if there exist a number $\alpha > 0$ and a sequence of n-dimensional intervals $(I_k)_{k \in \mathbb{N}}$ such that:

$$\begin{gathered} \operatorname{diam} I_k \to 0 \quad (k \to \infty), \\ 0 \ \text{ is a center of } \ I_k \ \ (k \in \mathbb{N}), \\ r_i(I_k) \to \infty \quad (k \to \infty), \\ \frac{|E \cap I_k|}{|I_k|} \geq \alpha \quad (k \in \mathbb{N}). \end{gathered}$$

Remark 1. In two-dimensional case, any convex set with point 0 on its boundary and axis Ox_2 as a tangent line is anisotropically dense at point 0 with respect to the first variable. In particular, such is the set $E_\varepsilon = \{(t,\tau) : t > 0,\ \ 0 \leq |\tau| \leq t^\varepsilon\}$ for any $\varepsilon \in (0,1)$.

A basis $\Delta = (\Delta_1, \dots, \Delta_n)$ let us call *anisotropically dense* if at least one among its components Δ_i is anisotropically dense at point 0 with respect to a corresponding i-th variable.

2. Result. In the case $\Delta = (\mathbb{R}^n \setminus L_1, \dots, \mathbb{R}^n \setminus L_n)$ the Δ-gradient is called the strong gradient and was introduced by O. Dzagnidze [1]. In [1] it was proved that if a function f has the strong gradient at a point x, then f is differentiable at x and the converse assertion is not true. G. G. Oniani [2] constructed a continuous function $f : \mathbb{R}^n \to \mathbb{R}$ such that f is differentiable almost everywhere but f does not has the strong gradient almost nowhere. The following generalization of this result is true.

Theorem 1. *If a basis* Δ *is anisotropically dense, then there exists a continuous function* $f : \mathbb{R}^n \to \mathbb{R}$ *such that:*
1) f *is differentiable almost everywhere;*
2) f *has no* Δ*-gradient almost nowhere.*

Remark 2. A function f will be constructed so as the condition 2) will be satisfied in the following context: $\overline{\lim\limits_{\Delta_i \ni h \to 0}} \frac{f(x+h) - f(x+h(i))}{h_i} = \infty$ almost everywhere, where i is an index for which the component Δ_i of the basis Δ is anisotropically dense at point 0 with respect to i-th variable.

3. Auxiliary statements. We shall note that the construction of desirable function is carried out by modification of G. G. Oniani's [2] scheme.

For simplicity, we consider the two-dimensional case.

Afterwards, without restriction of generality, we shall suppose that Δ_1 is anisotropically dense at zero with a parameter $0 < \alpha \leq 1$.

For simplicity, we shall make one more agreement. Suppose that (I_k) is a sequence of intervals from the property of anisotropic density of the set Δ_1. Denote by $I_k^{(p)}$ $(p \in \overline{1,4})$ the intersection of the interval $I_k^{(p)}$ with a corresponding coordinate quarter. It is obvious that there exists such a $p \in \overline{1,4}$ that

$$\frac{|I_k^{(p)} \cap \Delta_1|}{|I_k^{(p)}|} \geq \alpha$$

for infinite number of k. We shall assume that this condition is satisfied when $p = 1$, i.e. for the first coordinate quarter.

Let us introduce the following three maximal operators: for a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, a point $x \in \mathbb{R}^2$ and numbers $0 < \delta < \eta$ set:

$$M(f)(x) = \sup_{h \neq 0} \frac{|f(x+h) - f(x)|}{\|h\|},$$

$$S_\eta(f)(x) = \sup_{h \in \Delta_1,\, \|h\| < \eta} \frac{|f(x+h) - f(x+h(1))|}{h_1},$$

$$S_{\delta,\eta}(f)(x) = \sup_{h \in \Delta_1,\, \|h\| < \eta,\, |h_1| > \delta} \frac{|f(x+h) - f(x+h(1))|}{h_1}.$$

Basic construction. Suppose that $k \in \mathbb{N}$. Let us consider numbers $0 < t_1(k) < t_2(k)$ such that:

$$t_1(k) \text{ and } t_2(k) \text{ are numbers of the form } \frac{1}{2^m} \quad (m \in \mathbb{N}),$$

$$\frac{t_2(k)}{t_1(k)} \geq 2^{5k+2},$$

$$t_2(k) \leq \frac{1}{2^k},$$

$$\frac{|\Delta_1 \cap I_k|}{|I_k|} \geq \alpha, \quad \text{where } I_k = [0, t_1(k)] \times [0, t_2(k)].$$

The existence of such numbers $t_1(k)$ and $t_2(k)$ follows from the assumption of anisotropic density of Δ_1.

Let us introduce the following notation:

$$t(k) = (t_1(k), t_2(k));$$

$$B_k = B\left[t(k), \frac{\alpha t_1(k)}{2}\right],$$

i.e. B_k is a closed disc with the center at $t(k)$ and with the radius $\frac{\alpha t_1(k)}{2}$;

$$\widetilde{B}_k = B\left[t(k), 2^{2k+1} t_1(k)\right].$$

Let $f_k : \mathbb{R}^2 \to \mathbb{R}$ be a continuously differentiable function with the properties:

$$\operatorname{supp} f_k \subset B_k,$$
$$f_k\,(t(k)) = 2^k t_1(k),$$
$$0 \le f_k(x) \le 2^k t_1(k) \quad (x \in B_k).$$

It is easy to check that the function f_k has the following properties:

$$1) \quad \left\{ M(f_k) > \frac{1}{2^k} \right\} \subset \widetilde{B}_k;$$
$$2) \quad \left| \left\{ S_{2t_1(k)}(f_k) > 2^k \right\} \cap I_k \right| \ge \frac{\alpha}{2}\, |I_k|.$$

Lemma A (see [2]). *Suppose that for continuous functions $f_k : \mathbb{R}^2 \to \mathbb{R}$ $(k \in \mathbb{N})$, a point $x \in \mathbb{R}^2$ and a number $\lambda > 0$ there are satisfied the following conditions:*

$$\operatorname{supp} f_i \cap \operatorname{supp} f_j = \varnothing \quad (i \ne j),$$
$$x \notin \bigcup_{k=1}^{\infty} \operatorname{supp} f_k,$$
$$M(f_k)(x) \le \lambda \ \ (k \in \mathbb{N}).$$

Then

$$M\left(\sum_{k=1}^{\infty} f_k \right)(x) \le \lambda.$$

Lemma 1. *For each $m \in \mathbb{N}$ there exists a function $\nu_m : \mathbb{R}^2 \to \mathbb{R}$ such that:*

1) *ν_m is a continuously differentiable and 1-periodic,*

2) $0 \le \nu_m(x) \le \frac{1}{2^m}$ $(x \in \mathbb{R}^2)$,

3) $\left| \left\{ M(\nu_m) > \frac{1}{2^m} \right\} \cap [0,1]^2 \right| < \frac{1}{2^m}$,

4) $\left| \left\{ S_{1/2^m}(\nu_m) > 2^m \right\} \cap [0,1]^2 \right| > \frac{\alpha}{2}$.

Proof. For each number $k \in \mathbb{N}$ we shall consider parameters $t_1(k)$, $t_2(k)$, I_k, B_k, $\widetilde{B}_k$ and f_k existing due to the basic construction.

Let us divide the unit square $[0,1]^2$ into the intervals $I_{k,p,q}$ $(1 \le p \le 1/t_1(k),\ 1 \le q \le 1/t_2(k))$ congruent to the interval I_k. Let for indexes

$$1 \le p \le 1/t_1(k) - 1 \quad \text{and} \quad 1 \le q \le 1/t_2(k) - 1$$

$T_{p,q}$ be the shift which moves I_k into $I_{k,p,q}$. Let us introduce the following notation:

$$f_{k,p,q} = f_k \circ T_{p,q}, \quad B_{k,p,q} = T_{p,q}(B_k), \quad \widetilde{B}_{k,p,q} = T_{p,q}(\widetilde{B}_k).$$

Let ν_m be the 1-periodic function such that

$$\nu_m(x) = \sum_{p=1}^{1/t_1(k)-1} \sum_{q=1}^{1/t_2(k)-1} f_{k,p,q}(x) \quad \text{for each } x \in [0,1]^2.$$

If we take a sufficiently large k, the function ν_m defined in such a way will satisfy all four required conditions. Actually, 1) and 2) are obvious. Property 4) directly follows from the invariance of the operator $S_{1/2^m}$ with respect to shifts, as well as on the basis of property 2) of a function f_k ensured by a basic construction. In order to check property 3) we have to take into account: i) invariance of the operator M with respect to shifts; and ii) property 1) of f_k ensured by a basic construction. Due to this properties and Lemma A the set

$$\left\{ M(\nu_m) > \frac{1}{2^m} \right\} \cap [0,1]^2$$

will be contained in the union of strips Γ_q $(1 \le q \le 1/t_2(k) - 1)$, where Γ_q denotes the minimal horizontal strip containing the discs $\widetilde{B}_{k,p,q}$ $(1 \le p \le 1/t_1(k) - 1)$. Further noting that the union of the strips, for a sufficiently large k, cuts off from the square $[0,1]^2$ a subset of the measure smaller than $1/2^m$, we conclude the validity of the property 3). □

Remark 3. Taking into account that $\lim_{\delta \to 0} S_{\delta,\eta}(f)(x) = S_\eta(f)(x)$ for each $f \in C(\mathbb{R}^2)$, $\eta > 0$ and $x \in \mathbb{R}^2$, by applying continuity of measure we shall see that for any sufficiently small number $\delta_m \in (0, \frac{1}{2^m})$ there is satisfied the following condition that is stronger than 4) one

4′) $\left|\left\{ S_{\delta_m,1/2^m}(\nu_m) > 2^m \right\} \cap [0,1]^2 \right| > \frac{\alpha}{2}$.

The set $E \subset \mathbb{R}^n$ is called *l-periodic* $(l > 0)$, if its characteristic function χ_E is l-periodic with respect to each variable.

The following lemma belongs to A. Calderón (see e.g. [3, Ch. XIII, Section 1]).

Lemma B. *Let $(E_k)_{k\in\mathbb{N}}$ be a sequence of measurable and l-periodic subsets of $\mathbb{R}^n$ such that*

$$\sum_{k=1}^{\infty} |E_k \cap [0,l]^n| = \infty.$$

Then there exist points $x_k \in \mathbb{R}^n$ such that $\overline{\lim}_{k\to\infty} (x_k + E_k)$ is a set of full measure in $\mathbb{R}^n$.

The following theorem belongs to V. Stepanov (see e.g. [4, Ch. IX, Section 14]).

Theorem A. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a measurable function and*

$$\overline{\lim_{h \to 0}} \frac{|f(x+h) - f(x)|}{\|h\|} < \infty$$

at every point x of a set E. Then f is differentiable at almost every point of the set E.

4. Proof of Theorem 1. Assume that the functions ν_m and the numbers δ_m are chosen in accordance with Lemma 1.

We choose the sequence of the indexes $m(1) < m(2) < \cdots$ so that for each $i \geq 2$ the following conditions are satisfied:

$$2^{m(i)-1} > i + \sum_{j=1}^{i-1} \max_{x \in \mathbb{R}^2} \left| D_1 \nu_{m(j)}(x) \right|, \tag{1}$$

where D_1 denotes the partial derivative with respect to the first variable;

$$\omega \left(D_1 \nu_{m(j)}, \frac{1}{2^{m(i)}} \right) < 1 \quad \text{for every} \quad j \in \overline{1, i-1}, \tag{2}$$

where $\omega(F, t)$ denotes the modulus of continuity of a function F;

$$\frac{1}{\min\{\delta_{m(1)}, \ldots, \delta_{m(i)}\} 2^{m(i+1)}} < \frac{1}{2^i}\,. \tag{3}$$

Let us use Lemma B and find values of shifts x_i such that the upper limit of the sequence of the sets

$$x_i + \left\{ S_{\delta_{m(i)}, 1/2^{m(j)}}(\nu_{m(i)}) > 2^{m(i)} \right\} \tag{4}$$

to be of full measure in $\mathbb{R}^2$.

Let $\widetilde{\nu}_{m(i)}$ be the x_i shift of the function $\nu_{m(i)}$, that is

$$\widetilde{\nu}_{m(i)}(\cdot) = \nu_{m(i)}(\cdot - x_i) \quad (i \in \mathbb{N}).$$

Using invariance of the operators M and $S_{\delta,\eta}$ with respect to shifts, we have

$$\left\{ M(\widetilde{\nu}_{m(i)}) > \frac{1}{2^{m(i)}} \right\} = x_i + \left\{ M(\nu_{m(i)}) > \frac{1}{2^{m(i)}} \right\}, \tag{5}$$

$$\left\{ S_{\delta_{m(i)}, 1/2^{m(i)}}(\widetilde{\nu}_{m(i)}) > 2^{m(i)} \right\} = x_i + \left\{ S_{\delta_{m(i)}, 1/2^{m(i)}}(\nu_{m(i)}) > 2^{m(i)} \right\}. \tag{6}$$

Note that by the invariance of D_1 with respect to shifts, the conditions analogous to (1) and (2) will be satisfied for functions $\widetilde{\nu}_{m(i)}$.

The function f let us define in the following way

$$f = \sum_{i=1}^{\infty} \widetilde{\nu}_{m(i)}.$$

It is obvious that the function f is continuous and 1-periodic.

Denote

$$E_1 = \varlimsup_{i\to\infty} \left\{ S_{\delta_{m(i)},1/2^{m(i)}}(\widetilde{\nu}_{m(i)}) > 2^{m(i)} \right\} \cap (0,1)^2,$$
$$E_2 = \varlimsup_{i\to\infty} \left\{ M(\widetilde{\nu}_{m(i)}) > \frac{1}{2^{m(i)}} \right\} \cap (0,1)^2.$$

We have that (by Lemma 1, (4),(5) and (6))

$$|E_1| = 1 \quad \text{and} \quad |E_2| = 0. \tag{7}$$

Suppose $i \geq 2$ and

$$x \in \left\{ S_{\delta_{m(i)},1/2^{m(i)}}(\widetilde{\nu}_{m(i)}) > 2^{m(i)} \right\} \cap (0,1)^2.$$

Let $h \in \Delta_1$ be such that $|h_1| > \delta_{m(i)}$, $\|h\| < 1/2^{m(i)}$ and

$$\left| \frac{\widetilde{\nu}_{m(i)}(x+h) - \widetilde{\nu}_{m(i)}(x+h(1))}{h_1} \right| > 2^{m(i)}. \tag{8}$$

From (2) and (3), by using Lagrange formula, we conclude that for every $j \in \overline{1, i-1}$

$$\left| \frac{\widetilde{\nu}_{m(i)}(x+h) - \widetilde{\nu}_{m(i)}(x+h(1))}{h_1} \right| \leq |D_1 \widetilde{\nu}_{m(i)}(x)| + 1. \tag{9}$$

By (8), (9), (1), (2) and (3) we write

$$\begin{aligned}
\left| \frac{f(x+h) - f(x+h(1))}{h_1} \right| \geq \left| \frac{\widetilde{\nu}_{m(i)}(x+h) - \widetilde{\nu}_{m(i)}(x+h(1))}{h_1} \right| - \\
- \sum_{j=1}^{i-1} \left| \frac{\widetilde{\nu}_{m(j)}(x+h) - \widetilde{\nu}_{m(j)}(x+h(1))}{h_1} \right| - \\
- \sum_{j=i+1}^{\infty} \left| \frac{\widetilde{\nu}_{m(j)}(x+h) - \widetilde{\nu}_{m(j)}(x+h(1))}{h_1} \right| > \\
> 2^{m(i)} - \sum_{j=1}^{i-1} \big(D_1(\widetilde{\nu}_{m(j)}(x) + 1) \big) - \sum_{j=i+1}^{\infty} \frac{1}{\delta_{m(i)}} \cdot \frac{2}{2^{m(j)}} > 2^{m(i)-1}.
\end{aligned}$$

Consequently

$$\varlimsup_{\Delta_1 \ni h \to 0} \frac{f(x+h) - f(x+h(1))}{h_1} = \infty \tag{10}$$

for every $x \in E_1$. Now, if we take into account (7) and 1-periodicity of the function f we can conclude that the condition (10) is satisfied almost everywhere on $\mathbb{R}^2$.

Suppose $x \in (0,1)^2$ and $x \notin E_2$. It is obvious that there exists $i \geq 2$ such that

$$x \notin \left\{ M(\widetilde{\nu}_{m(i)}) > \frac{1}{2^{m(i)}} \right\} \cap (0,1)^2, \quad \text{when} \;\; j \geq i.$$

Therefore

$$M\Big(\sum_{j=i}^{\infty}\widetilde{\nu}_{m(j)}\Big)(x)\le\sum_{j=i}^{\infty}M(\widetilde{\nu}_{m(j)})(x)\le\sum_{j=i}^{\infty}\frac{1}{2^{m(j)}}<1. \tag{11}$$

On the other hand, by using Lagrange formula we obtain that

$$M\Big(\sum_{j=1}^{i-1}\widetilde{\nu}_{m(j)}\Big)(x)\le$$

$$\le\sum_{j=1}^{i-1}\Big[\max_{y\in\mathbb{R}^2}|D_1\widetilde{\nu}_{m(j)}(y)|+\max_{y\in\mathbb{R}^2}|D_2\widetilde{\nu}_{m(j)}(y)|\Big]<\infty. \tag{12}$$

(11) and (12) imply that $M(f)(x)<\infty$. Now, taking into account (7) and 1-periodicity of the function f we have

$$M(f)(x)<\infty \ \text{ almost everywhere on } \ \mathbb{R}^2.$$

Therefore, by virtue of Theorem A, f is differentiable almost everywhere on $\mathbb{R}^2$. This proves the theorem. □

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