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THE INVESTIGATION OF A NONLINEAR CHARACTERISTIC PROBLEM BY USING ANALOGUES OF RIEMANN INVARIANTS

M. MENTESHASHVILI

Abstract. The nonlocal characteristic problem for a second order quasi-linear equation with rectilinear characteristics and admissible parabolic degeneration is investigated.

რეზიუმე. ნაშრომში განხილულია არალოკალური მახასიათებელი ამოცანა მეორე რიგის კვაზიწრფივი პარაბოლურად გადაგვარებადი განტოლებისათვის. შესწავლილია ამონახსნის არსებობისა და ერთადერთობის საკითხი განტოლების ზოგადი ამონახსნის გამოყენების გარეშე.

1. Introduction

In the present work we investigate a modified characteristic problem for a second order quasi-linear equation with real characteristics. As is known, characteristics of quasi-linear equations may depend on values of an unknown solution and its derivatives. In such a case they are unknown and should be defined simultaneously with a solution (see [1]). Characteristics of such equations may form families of any geometry. However, there exist equations for which these characteristic families may have a quite definite configuration. For the equation

$$u_{xx} + (1 + u_x + u_y) \cdot u_{xy} + (u_x + u_y) \cdot u_{yy} = 0, \tag{1}$$

the family defined by the characteristic root $\lambda_1 = 1$ is represented by straight lines $x - y = c$, but as regards to the family of the root $\lambda_2 = u_x + u_y$, one has to take into account the following fact: the equation itself along the characteristics of that family is written simply as $d(u_x) + d(u_y) = 0$. This characteristic differential relation results in the equality $u_x + u_y = \text{const}$ along any characteristic of the given family. Thus, relying on the analysis of joint characteristic differential relations, we have obtained explicitly the

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so-called characteristic invariants, the analogues of the well-known Riemann invariants:

$$\begin{cases} \xi = y - x, \\ \xi_1 = e^{u_y}(u_x + u_y - 1), \end{cases} \tag{2}$$

for the family of the root λ_1, and

$$\begin{cases} \eta = y - (u_x + u_y)x, \\ \eta_1 = u_x + u_y, \end{cases} \tag{3}$$

for the family of the root λ_2.

The invariants (2) and (3) are constant along every characteristic of the corresponding family. It should be noted that equation (1) has no another characteristic invariants, independent of (2) and (3) (see [2]).

Judging by its characteristic roots λ_1, λ_2, equation (1) is hyperbolic. However, the case is not excluded in which the values of these roots coincide and hence equation (1) degenerates parabolically. This occurs for $u_x + u_y = 1$. Therefore, a class of hyperbolic solutions of the equation under consideration should be defined by the condition

$$u_x + u_y - 1 \neq 0. \tag{4}$$

Analyzing structures of characteristic invariants η and η_1, we can conclude that inclination of every separately taken characteristic is defined by the value of the invariant η_1, and hence is constant. This implies that all these characteristics form a family of straight lines.

Of the works based on the idea to apply the method of characteristics to nonlinear hyperbolic problems the works [3–8] are worth mentioning. They investigate structures of domains of definition of solutions and of domains of influence of initial and characteristic perturbations in singular cases.

A Nonlocal Characteristic Problem.

In this section we formulate a nonlocal problem in the statement of which all the above-mentioned properties and singularities of equation (1) will be taken into account. The attention earn the works [5–7] referring to the investigation of analogous problems for non-strictly hyperbolic second order equations with quasi-linear principal part.

If on some set of points A the value of the sum

$$u_x(x_0, y_0) + u_y(x_0, y_0) = \alpha(x_0, y_0), \quad (x_0, y_0) \in A$$

of the first order derivatives u_x, u_y of an unknown solution $u(x, y)$ is known, then the family characteristics of the root λ_2 are representable in the form

$$y - y_0 = \alpha(x_0, y_0)(x - x_0).$$

In the case if $\alpha(x_0, y_0) = 1$ at an arbitrary point $(x_0, y_0) \in A$, then the equation degenerates parabolically along the whole line $y - y_0 = x - x_0$. If the set A is a segment $J = \{y = 0, x \in [0, a]\}$ of the line $y = 0$, then all the family characteristics of the root λ_2 will be defined and, hence, the points of their intersection with characteristions of another family, including with the charactersitic $y = x$, will be known. Thus, under the condition

$$u_x(x,0) + u_y(x,0) = \alpha(x), \quad x \in J \tag{5}$$

the characteristics of the family λ_2 passing through an arbitrary point $(x_0, 0)$, $x_0 \in [0,a]$ have the form

$$y = \alpha(x_0)(x - x_0)$$

and intersect with the line $y = x$ at the point $Q(\mu(x_0), \mu(x_0))$, where

$$\mu(x) = \frac{x\,\alpha(x)}{\alpha(x) - 1}\,.$$

The Nonlocal Problem.

Find a regular solution $u(x,y)$ *of equation* (1) *and simultaneously a domain of its propagation if it satisfies the condition* (5) *and the nonlocal condition*

$$u(x,0) = \beta(x)\,u(\mu(x),\mu(x)) + \varphi(x), \quad x \in J, \tag{6}$$

where $\alpha, \beta, \varphi \in C^2[0,a]$ *are the given functions.*

Theorem 1. *Let the function* $\alpha(x)$ *satisfy the conditions*

$$|2\alpha(x) - 1| > 1, \quad -\infty < \alpha'(x) < 0, \quad x \in J, \quad \beta(0) \neq 1, \tag{7}$$

then a regular solution of the problem (1), (5,6) *exists in a characteristic triangle* Δ *bounded by the data supports and characteristic* $y = \alpha(a)(x - a)$.

Proof. First of all, we note that if at some point $(x_0, 0)$, $x_0 \in J$ the equality $\alpha(x_0) = 1$ is fulfilled, then the characteristics of both families passing through the point $(x_0, 0)$ coincide, do not intersect the line $y = x$ and, hence, the condition (6) turns out to be ill-posed.

Characteristics of the root λ_2, being the lines, may intersect with each other. The structure of a set of these points of intersection depends completely on the values and behavior of the function $\alpha(x)$. For $\alpha(x) = \text{const}$, such a point lies at infinity, and all characteristics are parallel.

In the case if

$$0 < \alpha(x) < 1, \quad \alpha'(x) > 0, \quad x \in J$$

the characteristics of the family λ_2 may intersect in a half-plane $y > 0$, and a solution in this case may turn out to be nonregular. But if $0 < \alpha(x) < 1$, $\alpha'(x) < 0$, $x \in J$, then characteristics of the family λ_2 do not intersect in the half-plane $y > 0$ with the line $y = x$.

When

$$\alpha(x) > 1, \quad \alpha'(x) < 0, \quad x \in J,$$

the characteristics of the family λ_2 may intersect only in the half-plane $y < 0$, and each of them may intersect with the characteristic of the family λ_1 not more than once.

Assume that the function $\alpha(x)$ satisfies the conditions (7). Then through every point of the segment $[0, a]$ there passes only one characteristic of every family.

Taking into account the condition (5), at all points $Q(\mu(x), \mu(x))$, $x \in [0, a]$ of the characteristic $y = x$ we have

$$\xi_1\Big|_{y=x} = e^{u_y(\mu(x),\mu(x))} \cdot [\alpha(\mu(x)) - 1] = e^{u_y(0,0)} \cdot [\alpha(0) - 1]. \tag{8}$$

On the other hand, the same point Q appears likewise on the family characteristic of the root λ_2, passing through $(x, 0)$. Therefore, taking into account (3) and (5), we have

$$u_x(\mu(x), \mu(x)) + u_y(\mu(x), \mu(x)) = u_x(x, 0) + u_y(x, 0) = \alpha(x). \tag{9}$$

Consequently, the relation (8) takes at the point Q the form

$$e^{u_x(\mu(x),\mu(x))} \cdot [\alpha(x) - 1] = e^{u_x(0,0)} \cdot [\alpha(0) - 1]. \tag{10}$$

The conditions (7) ensure strict monotonicity of the function $\mu(x)$. In particular, $\mu'(x) \neq 0$ everywhere on J and, naturally, the sign remains unchanged. This function in the neighborhood of any point $x_0 \in [0, a]$ is locally invertible. But we assume that the stronger condition

$$\alpha(x) > 0, \quad -\infty < \alpha'(x) < 0, \quad x \in J, \quad \beta(0) \neq 1 \tag{11}$$

is fulfilled.

Note that the case $\alpha(x) < 0$, $-\infty < \alpha'(x) < 0$, $x \in J$ is considered analogously.

From the relation (10), we can easily find a value of the derivative of an unknown function with respect to the variable y along the characteristic $y = \ x$. Taking into account (9) and the condition (5), along that characteristic we can find values of the derivatives u_x, u_y of the solution $u(x, y)$:

$$u_z(x, x) = \alpha(\nu(x)) - u_y(0, 0) - \log \frac{\alpha(0) - 1}{\alpha(\nu(x)) - 1}, \tag{12}$$

$$u_y(x, x) = u_y(0, 0) + \log \frac{\alpha(0) - 1}{\alpha(\nu(x)) - 1}, \tag{13}$$

where by ν is denoted the inverse function $\mu(x)$. The existence of the unique inverse function is guaraneed by the condition (11).

Integration of the sum $u_x(x, x) + u_y(x, x)$ along the line $y = x$ yields

$$u(x, x) = u(0, 0) + \int\limits_0^x \alpha(\nu(t))dt,$$

due to which from (6) we find the values $u(x,0)$ of the solution of the problem everywhere on the segment J:

$$u(x,0)=\varphi(x)+\beta(x)\cdot\left(u(0,0)+\int\limits_0^x \alpha(t)\cdot\mu'(t)\,dt\right)\equiv f(x). \tag{14}$$

By virtue of (14), we calculate values of the derivative u_x of a solution of the problem at all points of the segment $[0,a]$:

$$u_x(x,0)=\varphi'(x)+\beta'(x)\cdot\left(u(0,0)+\int\limits_0^x \alpha(t)\cdot\mu'(t)dt\right)+\beta(x)\,\alpha(x)\,\mu'(x),$$

and taking into account (5), we find also the values

$$u_y(x,0)=\alpha(x)-\varphi'(x)-\beta'(x)\cdot\left(u(0,0)+\int\limits_0^x \alpha(t)\cdot\mu'(t)\,dt\right)-$$
$$-\beta(x)\,\alpha(x)\,\mu'(x)\equiv g(x). \tag{15}$$

As is seen from the above expressions, the derivatives of an unknown solution u are defined completely at the origin. Moreover, all these values can be defined along the characteristic $y=\alpha(a)(x-a)$. Indeed, (14) and (15) allow one to calculate values of the invariants ξ,ξ_1 at every point $(x_0,0)\in J$:

$$\xi\big|_{x=x_0,y=0}=-x_0,$$
$$\xi_1=e^{g(x_0)}(\alpha(x_0)-1),$$

which are retained along the entire characteristic $y=x-x_0$. On the other hand, the values of another invariants of the characteristic η,η_1 on

$$\eta=y-\alpha(a)x \quad \text{and} \quad \eta_1=\alpha(a)$$

are known.

In view of the fact that all four equalities are fulfilled on the above characteristic at the point $(\mu(a),\mu(a))$, we can define the values u,u_x,u_y at every its point. Thus the problem (1) is, in fact, solved. The reason for such a conclusion is the following fact: all characteristics

$$y=\alpha(\nu(x_1))(x-\nu(x_1)),$$

of the family λ_2 emanated from the points (x_1,x_1), $x_1\in[0,\mu(a)]$ as well as the characteristics

$$y=x+x_2(\alpha(a)-1)-a\,\alpha(a)$$

of the family λ_1 emanated from the points $(x_2,\alpha(a)(x_2-a))$, are known. A set of points of intersection of these characteristics allows one to define the domain of propagation of a solution of the problem under consideration. In the given case we are interested in the characteristic triangle Δ which is bounded by the data supports and characteristic $y = \alpha(a)(x-a)$. The values of all invariants along the corresponding characteristics allowing us to get a solution of the problem everywhere in the triangle Δ, are known. $\square$

But if, nevertheless, we are required to have not only a numerical, but also an analytic representation of a solution of the problem, for this purpose we have several variants.

The first variant dealing with the consideration of the Cauchy problem with data

$$u\big|_{y=0} = f(x), \quad u_y\big|_{y=0} = g(x), \quad x \in [0,a],$$

where the functions f and g are given by formulas (14) and (15), has been considered by us by using a general solution of equation (1). Here we consider another variant of investigation of the problem (1), (5,6) which turns out to be considerably simpler even without a general solution of the equation.

Theorem 2. *Let the conditions* (7) *be fulfilled and the system*

$$x = \frac{z - t\alpha(t)}{1-\alpha(t)}, \quad y = \frac{(z-t)\alpha(t)}{1-\alpha(t)} \tag{16}$$

define a unique inverse transformation from the arguments z, t to the arguments x, y:

$$z = y - x, \quad t = R(x,y), \tag{17}$$

then in the characteristic triangle Δ there exists a unique regular solution of the nonlocal problem (1), (5,6) *which is given by the formula*

$$u(x,y) = \int_0^x (\alpha(\tau) - g(\tau))d\tau + \int_0^y \left(g(x-\tau) + \log \frac{\alpha(x-\tau)-1}{\alpha(R(x,\tau))-1} \right) d\tau. \tag{18}$$

Proof. The conditions (5,6) of the problem allow us to find the values of an unknown solution and also of its first order derivatives on the whole data support. They are represented by formulas (12)–(15). Using these values, we can determine all characteristics emanated from points of the support $J \cup \{(x,y) : x = y,\, 0 \le x \le a\}$. All of them are the segments of the lines lying in the triangle Δ. If in that triangle the conditions (7) are fulfilled, then standard requirements regarding characteristic families are not violated: family characteristics of the root λ_2 are mutially disjoint, and curves of different families cannot intersect more than at one point.

For a more complete description of a family of characteristics of the root λ_2 we choose arbitrarily on the segment J a pair of points $(z,0)$ and $(t,0)$, where $0 < z < t < a$. Through these points we draw the characteristics $\ell_1 : y = x - z$, of the family of the root λ_1 and $\lambda_2 : y\alpha(t)(x-t)$, of the family of the root λ_2. Since for $\alpha(x) > 1$ the conditions (7) are fulfilled, all the above arguments are true, and the intersection of the segments ℓ_1 and ℓ_2 is guaranteed. The point of that intersection is defined completely,

$$x^* = \frac{z - t\alpha(t)}{1 - \alpha(t)}\,, \quad y^* = \frac{(z-t)\alpha(t)}{1-\alpha(t)}\,. \tag{19}$$

Along the characteristics ℓ_1 and ℓ_2, we have the relations

$$\begin{aligned} \xi_1\big|_{\ell_1} &= e^{u_y} \cdot [u_x + u_y - 1]\big|_{y=x-z}\,, \\ \eta_1\big|_{\ell_2} &= [u_x + u_y]\big|_{y=\alpha(t)(x-t)}\,, \end{aligned}$$

in which there appear traces of the derivatives u_x and u_y taken respectively on ℓ_1 and ℓ_2. These traces must coincide at the point (x^*, y^*). Consequently, we can consider them as a system for finding values of derivatives at the point of intersection. They are really defined in terms of new arguments z, t:

$$\begin{aligned} u_z(x^*, y^*) &= \alpha(t) - g(z) - \log\left(\frac{\alpha(z)-1}{\alpha(t)-1}\right), \\ u_y(x^*, y^*) &= g(z) + \log\left(\frac{\alpha(z)-1}{\alpha(t)-1}\right). \end{aligned}$$

We represent a solution of the problem $u(x(z,t), y(z(t))$ in new variables in terms of the function v of the arguments z, t which in their turn are the functions of (x, y), $v = v(z(x,y), t(x,y))$. Obviously, the first order derivatives of a solution of the problem with respect to parameters z, t will have the form

$$\begin{aligned} v_t &= \alpha(t)\left(\frac{\alpha(t)}{\alpha(t)-1} + \frac{(z-t)\alpha'(t)}{(\alpha(t)-1)^2}\right), \\ v_z &= \frac{\alpha(t)}{1-\alpha(t)} - g(z) - \log\left(\frac{\alpha(z)-1}{\alpha(t)-1}\right). \end{aligned}$$

It can be directly verified that the right-hand sides of the above equalities satisfy the conditions of Schwarz theorem. Therefore, constructing the expression of a full differential in terms of z, t, we can, by means of integration, uniquely define a certain function $H(z,t)$. Moreover, this function can be expressed in terms of the values x^*, y^*. These values are defined by intersection of arbitrarily taken characteristics ℓ_1, ℓ_2 and, hence, they may also be assumed as arbitrary. In other words, they may be designed as arguments. On the strength of the above-said, the stars in the system (19) can be omitted. We denote this inversion by (17) which finally results in (18).

There arises the question whether the function (18) is really a solution of the problem under consideration. A positive answer to the question becomes clear after the remark that $R(x, 0) = x$. Such a property can easily be perceived from the first relation in (16) by substituting (17) into these relations. □

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Author's addresses:

Niko Muskhelishvili Institute of Computational
Mathematics of the Georgian Technical University
8a, Akuri Str., Tbilisi 0160
Georgia

Sokhumi State University
9, Anna Politkovskaia str., Tbilisi 0186, Georgia
E-mail: Menteshashvili_m@mail.ru