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THE STURM TYPE INTEGRAL COMPARISON THEOREMS FOR SINGULAR DIFFERENTIAL EQUATIONS

R. KOPLATADZE

Abstract. For the second order singular differential equations, the comparison theorems are given by applying of which the solvability of some boundary value problems are investigated.

რეზიუმე. მეორე რიგის წრფივი სინგულარული განტოლებებისათვის მოყვანილია შედარების თეორემები და მათი გამოყენებით შესწავლილია ზოგიერთი სასაზღვრო ამოცანის ამოხსნადობის საკითხი.

1. Introduction

Consider the differential equations

$$u'' + p(t)\,u = 0 \tag{1.1}$$

and

$$v'' + q(t)\,v = 0, \tag{1.2}$$

where $p; q \in C((a, b); R)$, $-\infty < a < b < +\infty$. As early as in 1836, for equations (1.1) and (1.2), where $p; q \in C([a, b]; R)$, Sturm [1] proved a comparison theorem, which later was widely used in studying both the boundary value problems and asymptotic behavior of solutions. Some generalizations of Sturm's theorem are given in [2], for the proofs of Sturm's theorems for a singular case see [3,4]).

2. Some Auxiliary Lemmas

Lemma 2.1. *Let $a < b$,*

$$p; q \in C([a, b); R_+) \tag{2.1}$$

and

$$\int_a^t p(s)ds \ge \int_a^t q(s)ds \quad for \quad t \in [a, b). \tag{2.2}$$

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Then

$$\int_a^t p(s)u(s)ds \geq \int_a^t q(s)u(s)ds \quad \textit{for} \quad t \in [a,b), \tag{2.3}$$

where

$$u \in C^{(2)}([a,b);R_+) \quad \textit{and} \quad u'(t) \leq 0 \quad \textit{for} \quad t \in [a,b). \tag{2.4}$$

Proof. Assume the contrary. Let there exist a function u satisfying the condition (2.4) and $t_1 \in (a,b)$ such that

$$\int_a^{t_1} p(s)u(s)ds < \int_a^{t_1} q(s)u(s)ds.$$

Therefore,

$$0 > \int_a^{t_1} \big(p(s)-q(s)\big)u(s)ds = \int_a^{t_1} u(s)d\int_a^s \big(p(\xi)-q(\xi)\big)d\xi =$$

$$= u(t_1)\int_a^{t_1} \big(p(\xi)-q(\xi)\big)d\xi - \int_a^{t_1} u'(s)\int_a^s \big(p(\xi)-q(\xi)\big)d\xi\, ds.$$

Thus, according to (2.3) and (2.4), from the latter inequality we get

$$0 > \int_a^{t_1} \big(p(s)-q(s)\big)ds \geq u(t_1)\int_a^{t_1} \big(p(\xi)-q(\xi)\big)d\xi \geq 0.$$

The obtained contradiction proves the validity of Lemma 2.1. □

Analogously we can proved

Lemma 2.2. *Let $a<b$, $p;q \in C((a,b])$ and*

$$\int_t^b p(s)ds \geq \int_t^b q(s)ds \quad \textit{for} \quad t \in (a,b].$$

Then

$$\int_t^b p(s)u(s)ds \geq \int_t^b q(s)u(s)ds \quad \textit{for} \quad t \in (a,b],$$

where

$$u \in C^{(2)}((a,b];R_+) \quad \textit{and} \quad u'(t) \geq 0 \quad \textit{for} \quad t \in (a,b].$$

3. Integral Comparison Theorems of Sturm Type

Theorem 3.1 (Regular case)**.** *Let* $p; q \in C((a,b); R_+)$*,* $p; q \in L([a,b])$ *and let* $v \in C^{(2)}((a,b); (0,+\infty))$ *be a solution of equation* (1.2)*, which be fulfilled under the conditions*

$$v(t) > 0 \quad \textit{for} \quad t \in (a,b), \quad t_0 \in (a,b), \quad v'(t_0) = 0, \tag{3.1}$$

$$\lim_{t \to a+} v(t) = v(a+) = 0, \quad \lim_{t \to b-} v(t) = v(b-) = 0. \tag{3.2}$$

Moreover, if

$$\begin{aligned} &\int_t^{t_0} \big(p(s) - q(s)\big) ds \geq 0 \quad \textit{for} \quad t \in (a, t_0], \\ &\int_{t_0}^{t} \big(p(s) - q(s)\big) ds \geq 0 \quad \textit{for} \quad t \in [t_0, b) \end{aligned} \tag{3.3}$$

and $u \in C^{(2)}((a,b); R)$ *is a solution of equation* (1.1)*, then at least one of the conditions*

$$1) \qquad \textit{there exists} \quad t_* \in (a,b) \quad \textit{such that} \quad u(t_*) = 0 \tag{3.4}$$

or

$$2) \qquad u(a+) = u(b-) = u'(t_0) = 0. \tag{3.5}$$

is fulfilled. Besides, if

$$\int_a^b \big(p(s) - q(s)\big) ds > 0, \tag{3.6}$$

then (3.4) *holds.*

Proof. Let $u \in C^{(2)}((a,b); R)$ be a solution of equation (1.1) and $u(t) \neq 0$ for $t \in (a,b)$. Show that the condition 2) is fulfilled. Without loss of generality we assume that $u(t) > 0$ for $t \in (a,b)$. Show that $u'(t_0) = 0$. Otherwise, $u'(t_0) > 0$ or $u'(t_0) < 0$. Since $p \in L([a,b])$, it is obvious that

$$\sup \big\{ |u'(t)| : t \in (a,b) \big\} < +\infty. \tag{3.7}$$

Let $u'(t_0) > 0$. By (2.1), it is obvious that $u'(t) > 0$ and $v'(t) \geq 0$ for $t \in (a, t_0]$. Therefore, according to Lemma 2.2, we have

$$\int_t^{t_0} \big(p(s) - q(s)\big) u(s) v(s) ds \geq 0 \quad \text{for} \quad t \in (a, t_0]. \tag{3.8}$$

From the equality

$$\int_t^{t_0} \big(u''(s)v(s) - v''(s)u(s)\big)ds + \int_t^{t_0} \big(p(s) - q(s)\big)u(s)v(s)ds = 0$$

by (3.1) we obtain

$$u'(t_0)v(t_0) + v'(t)u(t) - u'(t)v(t) + \int_t^{t_0} \big(p(s) - q(s)\big)u(s)v(s)ds = 0. \quad (3.9)$$

According to (3.5), since $v(a+) = 0$, we get

$$u'(t_0)v(t_0) + v'(a+)u(a+) + \int_a^{t_0} \big(p(s) - q(s)\big)u(s)v(s)ds = 0. \quad (3.10)$$

Since $u'(t_0)v(t_0) > 0$, by (3.8) we have a contradiction. The obtained contradiction proves that $u'(t_0) \leq 0$.

Let $u'(t_0) < 0$. Then, by (2.1), $u'(t) < 0$ and $v'(t) \leq 0$ for $t \in [t_0, b)$. Therefore, by Lemma 2.1,

$$\int_{t_0}^{t} \big(p(s) - q(s)\big)u(s)v(s)ds \geq 0 \quad \text{for} \quad t \in [t_0, b). \quad (3.11)$$

From the equality

$$\int_{t_0}^{t} \big(u''(s)v(s) - v''(s)u(s)\big)ds + \int_{t_0}^{t} \big(p(s) - q(s)\big)u(s)v(s)ds = 0,$$

by (3.1) we have

$$u'(t)v(t) - v'(t)u(t) - u'(t)v(t_0) + \int_{t_0}^{t} \big(p(s) - q(s)\big)u(s)v(s)ds = 0.$$

Thus according to (3.6), since $v(b-) = 0$, we get

$$-v'(b-)u(b-) - u'(t_0)v(t_0) + \int_{t_0}^{b} \big(p(s) - q(s)\big)u(s)v(s)ds = 0. \quad (3.12)$$

Since $u'(t_0)v(t_0) < 0$, by (3.11) we have contradiction. The obtained contradiction proves that $u'(t_0) \geq 0$. Consequently $u'(t_0) = 0$. Therefore, according to (3.10) and (3.12), $u(a+) = u(b-) = 0$.

Now we show that if (3.6) is fulfilled, then (3.4) holds. Indeed, let (3.6) hold, then

$$\int_a^{t_0} \big(p(s) - q(s)\big) ds > 0 \quad \text{or} \quad \int_{t_0}^{t} \big(p(s) - q(s)\big) ds > 0. \tag{3.13}$$

Without loss of generality assume that the first condition of (3.13) is fulfilled. Show that

$$\int_a^{t_0} \big(p(s) - q(s)\big) u(s) v(s) ds > 0. \tag{3.14}$$

According to (3.13), there exist $t_* \in (a, t_0)$ and $c > 0$ such that

$$\int_t^{t_0} \big(p(s) - q(s)\big) ds \geq c \quad \text{for} \quad t \in [a, t_0].$$

Therefore, since $u(a+) = 0$, we have

$$\begin{aligned}
\int_a^{t_0} \big(p(s) - q(s)\big) u(s) v(s) ds &= - \int_a^{t_0} u(s) v(s) d \int_s^{t_0} \big(p(\xi) - q(\xi)\big) d\xi = \\
&= \int_a^{t_0} \big(u(s) v(s)\big)' \int_s^{t_0} \big(p(\xi) - q(\xi)\big) d\xi \, ds \geq \\
&\geq \int_a^{t_*} \big(u(s) v(s)\big)' \int_s^{t_0} \big(p(\xi) - q(\xi)\big) d\xi \, ds \geq c u(t_*) v(t_*) > 0.
\end{aligned}$$

Consequently (3.14) is fulfilled, which contradicts to the equality (3.10). The obtained contradiction proves the validity of the theorem. □

Our next theorem is proved similarly.

Theorem 3.2 (Singular case). *Let $p; q \in C((a, b); R_+)$ and let $v \in C^{(2)}((a, b); (0, +\infty))$ be a solution of equation* (1.2)*, satisfying the conditions* (3.1) *and* (3.2)*. Moreover, if* (3.3) *and*

$$\int_a^{t_0} \frac{ds}{v^2(s)} = \int_{t_0}^{b} \frac{ds}{v^2(s)} = +\infty,$$

are fulfilled, then for any $u \in C^{(2)}((0, b); R)$, which is a solution of equation (1.1)*, at least one of the conditions* (3.4) *or $u'(t_0) = 0$ holds. Besides, if* (3.6) *is fulfilled then* (3.4) *holds.*

Corollary 3.1. *Let* $p \in C((a,b);R)$ *and let* $u_1, u_2 \in C^{(2)}((a,b);R)$ *be linearly independent solutions of the equation* (1.1), *and*

$$u_1(a+) = u_1(b-) = 0.$$

Then there exists $t_* \in (a,b)$ *such that* $u_2(t_*) = 0$.

Corollary 3.2. *Let* $p \in C((0,+\infty);R)$, $tp(t) \in L([0,1])$ *and*

$$p(t) \le \frac{1}{4t^2} \quad \text{for} \quad t \in (0,+\infty). \tag{3.15}$$

Then the problem

$$u'' + p(t)\,u = f(t),$$
$$u(0) = \alpha, \quad u(a) = \beta,$$

for any $f \in C(R_+;R)$, $\alpha;\beta \in R$ *and* $a \in (0,+\infty)$, *has only one solution.*

Remark. In Corollary 3.2, the condition (3.15) for any $\varepsilon > 0$ cannot be replaced by

$$p(t) \le \frac{1+\varepsilon}{4t^2} \quad \text{for} \quad t \in (0,+\infty).$$

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Author's address:

I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University
2, University Str. Tbilisi 0186
Georgia
E-mail: r_koplatadze@yahoo.com