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ON COMPLEX UNIVERSAL SERIES

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Abstract. It is shown, in particular, that for a fixed complex number ζ, $|\zeta| = 1$, $\zeta^2 \neq 1$ the series

$$\sum_{n=1}^{\infty} \frac{1}{n}\zeta^n$$

is universal in the following sense: for each complex number z there exists a rearrangement of the series which converges to z.

რეზიუმე. ნაშრომში, კერძოდ, ნაჩვენებია, რომ ფიქსირებული კომპლექსური რიცხვისთვის ζ, $|\zeta| = 1$, $\zeta^2 \neq 1$, მწკრივი

$$\sum_{n=1}^{\infty} \frac{1}{n}\zeta^n$$

უნივერსალურია შემდეგი აზრით: ყოველი კომპლექსური z რიცხვისთვის არსებობს მწკრივის z-კენ კრებადი გადანაცვლება.

1. Introduction

The famous memoir by Dirichlet (1805 - 1859) entitled "Sur la convergence des séries trigonométriques qui servent à représenter une fonction arbitraire entre des limites données" (1829) contains in particular the following statement:

Theorem 1.1 (Dirichlet's theorem)**.** *Let*

$$\sum_{n=1}^{\infty} z_n \tag{1.1}$$

be an absolutely convergent series of real or complex numbers and $s := \sum_{n=1}^{\infty} z_n$. *Then every rearrangement of* (1.1) *converges to* s.

The Habilitationsschrift of Riemann (1826–1866) "Ueber die Darstellbarkeit durch eine trigonometrische Reihe" (written in 1853 and published in 1867), among other important results contains also the following statement:

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Theorem 1.2 (Riemann's theorem)**.** *Let*

$$\sum_{n=1}^{\infty} x_n \tag{1.2}$$

be a series of real numbers, which converges but the series $\sum\limits_{n=1}^{\infty} |x_n|$ *is not convergent. Then for each real number* x *there exists a rearrangement of* (1.2) *which converges to* x.

Paul Pierre Lévy (1886–1971) in 1905 published a note [3] in which the question of validity of an analogue of Theorem 1.2 for complex series was considered. To formulate his observations it is convenient to introduce the following terminology.

To a sequence (y_n) extracted from a topological vector space Y let us associate a subset $SR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ of Y as follows: an element $y \in Y$ belongs to $SR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ if there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the sequence $\Big(\sum\limits_{k=1}^{n} y_{\pi(k)}\Big)_{n\in\mathbb{N}}$ converges to y. The set $SR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ is called *the sum range* of (y_n) (cf. [4, Definition 2.1.1]).

By using this notation P. Lévy's result can be formulated as follows:

Theorem 1.3. *Let* $\sum\limits_{n=1}^{\infty} z_n$ *be a series of complex numbers, which converges but the series* $\sum\limits_{n=1}^{\infty} |z_n|$ *is not convergent. Then either*

$$SR\Big(\sum_{k=1}^{\infty} z_k\Big) = c_1 + c_2\mathbb{R}$$

for some $c_1 \in \mathbb{C}$ *and* $c_2 \in \mathbb{C} \setminus \{0\}$, *or*

$$SR\Big(\sum_{k=1}^{\infty} z_k\Big) = \mathbb{C}\,.$$

This theorem is not a complete analogue of Theorem 1.2, it leaves open a question: for which complex series $\sum\limits_{n=1}^{\infty} z_n$ one has the equality $SR\Big(\sum\limits_{k=1}^{\infty} z_k\Big) = \mathbb{C}$?

A series $\sum\limits_{k=1}^{\infty} y_k$ in a topological vector space Y is called *universal* if

$$SR\Big(\sum_{k=1}^{\infty} y_k\Big) = Y\,.$$

Using this terminology, the Riemann's theorem can be formulated as follows: every convergent but not absolutely convergent series of real numbers is universal. A similar statement is not true for series of complex numbers. The following theorem gives a characterization of universal complex series.

Theorem 1.4. *For a convergent but not absolutely convergent series of complex numbers* $\sum\limits_{n=1}^{\infty} z_n$ *the following statements are equivalent:*

(i) *The series* $\sum\limits_{n=1}^{\infty} z_n$ *is universal.*

(ii) $\sum\limits_{n=1}^{\infty} |Re(wz_n)| = \infty$ *for every* $w \in \mathbb{C} \setminus \{0\}$.

From this theorem we derive the following statement, which covers the assertion from the abstract and which gives many examples of complex universal series.

Theorem 1.5. *Let* $\zeta \notin \{-1, 1\}$ *be a complex number with* $|\zeta| = 1$ *and* $(a_n)_{n\in\mathbb{N}}$ *a sequence of non-negative real numbers such that*

$$a_n \to 0, \quad \sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} a_n = \infty\,.$$

Then the power series

$$\sum_{n=1}^{\infty} a_n \zeta^n \tag{1.3}$$

is universal.

Theorem 1.4 together with some other statements, which may have an independent interest (cf. Proposition 2.1) is proved in Section 2. Section 3 contains the proof of Theorem 1.5.

2. Steinitz's Range and Proof of Theorem 1.4.

We fix a topological vector space Y over $\mathbb{R}$ with the topological dual space Y^*.

For a non-empty $\Gamma \subset Y^*$ we write:

$$^{\perp}\Gamma = \left\{ y \in Y : y^*(y) = 0, \quad \forall\, y^* \in \Gamma \right\}.$$

Moreover, for a sequence $\mathbf{y} = (y_n)$ in Y we write:

$$\Gamma_{\mathbf{y}} = \Big\{ y^* \in Y^* : \sum_{k=1}^{\infty} |y^*(y_k)| < \infty \Big\}.$$

To a sequence (y_n) in Y let us associate two subsets

$$WSR\Big(\sum_{k=1}^{\infty} y_k\Big) \quad \text{and} \quad StR\Big(\sum_{k=1}^{\infty} y_k\Big)$$

of Y as follows:

- an element $y \in Y$ belongs to $WSR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ if there exists a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that the sequence $\Big(\sum\limits_{k=1}^{n} y_{\pi(k)}\Big)_{n\in\mathbb{N}}$ converges *weakly* to y. The set $WSR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ is called *the weak sum range* of (y_n).
- an element $y \in Y$ belongs to $StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ if

$$y^*(y) \in SR\Big(\sum_{k=1}^{\infty} y^*(y_k)\Big), \quad \forall\, y^* \in Y^*.$$

Let us call the set $StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ *the Steinitz range* of (y_n).

Proposition 2.1. *For a sequence* $\mathbf{y} = (y_n)$ *in a topological vector space* Y *over* $\mathbb{R}$ *we have:*

(a) $SR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) \subset WSR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) \subset StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$.

(b) *If* $StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) \neq \emptyset$ *and* $s \in StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$, *then*

$$StR\Big(\sum_{k=1}^{\infty} y_k\Big) = s + {}^{\perp}\Gamma_{\mathbf{y}}\,.$$

(c) *If* $StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) \neq \emptyset$, *then* $StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ *is a weakly closed affine subset of* Y*.*

Proof. (a) is evident.

(b) Let us see first that

$$StR\Big(\sum_{k=1}^{\infty} y_k\Big) \subset {}^{\perp}\Gamma_{\mathbf{y}} + s\,.$$

Fix $y \in StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ and $y^* \in \Gamma_{\mathbf{y}}$. Then for some permutation $\pi : \mathbb{N} \to \mathbb{N}$ we have:

$$y^*(y) = \sum_{k=1}^{\infty} y^*(y_{\pi(k)})\,.$$

As $s \in StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ as well, for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$ we have also:

$$y^*(s) = \sum_{k=1}^{\infty} y^*(y_{\sigma(k)})\,.$$

Since $\sum\limits_{k=1}^{\infty} |y^*(y_k)| < \infty$, by Theorem 1.1 we have:

$$\sum_{k=1}^{\infty} y^*(y_{\pi(k)}) = \sum_{k=1}^{\infty} y^*(y_{\sigma(k)})\,.$$

Hence, $y^*(y) = y^*(s)$. Therefore $y^*(y - s) = 0$. As $y^* \in \Gamma_{\mathbf{y}}$ is arbitrary, we conclude that $y - s \in {}^{\perp}\Gamma_{\mathbf{y}}$ and so, $y \in {}^{\perp}F + s$.

Let us see now that

$$StR\Big(\sum_{k=1}^{\infty} y_k\Big) \supset {}^{\perp}\Gamma_{\mathbf{y}} + s\,.$$

Fix $y \in {}^{\perp}\Gamma_{\mathbf{y}} + s$ and $y^* \in Y^*$. We need to verify that

$$y^*(y) \in SR\Big(\sum_{k=1}^{\infty} y^*(y_k)\Big)\,.$$

Let first $y^* \in \Gamma_{\mathbf{y}}$; then (as $y \in {}^{\perp}\Gamma_{\mathbf{y}} + s$) we have $y^*(y) = y^*(s)$. As $s \in StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$, for some permutation $\sigma : \mathbb{N} \to \mathbb{N}$ we have:

$$y^*(s) = \sum_{k=1}^{\infty} y^*(y_{\sigma(k)})\,.$$

From this (as $y^*(y) = y^*(s)$), we get

$$y^*(y) = \sum_{k=1}^{\infty} y^*(y_{\sigma(k)})$$

and so

$$y^*(y) \in SR\Big(\sum_{k=1}^{\infty} y^*(y_k)\Big)\,.$$

Let now $y^* \in Y^* \backslash \Gamma_{\mathbf{y}}$; then as $StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) \neq \emptyset$, we have $SR\Big(\sum\limits_{k=1}^{\infty} y^*(y_k)\Big) \neq \emptyset$ and $\sum\limits_{k=1}^{\infty} |y^*(y_k)| = \infty$. So, *by Riemann's theorem* (only in this place we use the assumption that Y is a vector space over $\mathbb{R}$) there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that

$$y^*(y) = \sum_{k=1}^{\infty} y^*(y_{\pi(k)})\,.$$

Therefore

$$y^*(y) \in SR\Big(\sum_{k=1}^{\infty} y^*(y_k)\Big)$$

in this case too.

(c) follows from (b). □

Remark 2.2. For a convergent series $\sum\limits_{k=1}^{\infty} y_k$ in $Y = l_2$ the inclusions in Proposition 2.1(a) may be strict. Namely

(1) The set $SR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ may not be convex (Marcinkiewicz-Nikishin-Kornilov), but $WSR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) = Y$ [4, Example 3.1.2 and Exercise 3.1.6],

(2) The set $WSR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ may not be convex [4, Theorem 8.1.1], while $StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big)$ always is convex by Proposition 2.1(c).

Corollary 2.3. *For a sequence* (y_n) *in a topological vector space* Y *over* $\mathbb{R}$ *the implication*

$$SR\Big(\sum_{k=1}^{\infty} y_k\Big) = Y \implies \sum_{k=1}^{\infty} |y^*(y_k)| = \infty \quad \forall y^* \in Y^* \setminus \{0\} \tag{2.1}$$

is true.

Proof. Let $SR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) = Y$. Then by Proposition 2.1 (a) we have $StR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) = Y$. The last equality by Proposition 2.1 (b) implies ${}^{\perp}\Gamma_{\mathbf{y}} = Y$. Take an arbitrary $y^* \in \Gamma_{\mathbf{y}}$; we have $y^*(y) = 0$ for every $y \in Y$, i.e., $y^* = 0$. Consequently, $\Gamma_{\mathbf{y}} = \{0\}$. □

Corollary 2.4. *For a sequence* (y_n) *in a topological vector space* Y *over* $\mathbb{C}$ *the implication*

$$SR\Big(\sum_{k=1}^{\infty} y_k\Big) = Y \implies \sum_{k=1}^{\infty} \big|\mathrm{Re}(y^*(y_k))\big| = \infty \quad \forall y^* \in Y^* \setminus \{0\} \tag{2.2}$$

is true.

Proof. Denote by $Y_{\mathbb{R}}$ the space Y considered as a topological vector space over $\mathbb{R}$. Clearly if $y^* \in Y^*$, then $\mathrm{Re}y^* \in (Y_{\mathbb{R}})^*$. Note that if $y^* \in Y^* \setminus \{0\}$, then $\mathrm{Re}y^* \in (Y_{\mathbb{R}})^* \setminus \{0\}$. (In fact, suppose that for some $y^* \in Y^*$ we have $\mathrm{Re}y^* = 0$, i.e., $\mathrm{Re}y^*(y) = 0\,\forall y \in Y$; then $\mathrm{Im}y^*(y) = -\mathrm{Re}y^*(iy) = 0\,\forall y \in Y$. Hence, $\mathrm{Im}y^* = 0$. Now from $\mathrm{Re}y^* = 0$ and $\mathrm{Im}y^* = 0$ we get $y^* = 0$.)

Let $SR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) = Y$. Fix $y^* \in Y^* \setminus \{0\}$. Then $\mathrm{Re} y^* \in (Y_{\mathbb{R}})^* \setminus \{0\}$. Hence, by Corollary 2.3, $\sum\limits_{k=1}^{\infty} |\mathrm{Re}(y^*(y_k))| = \infty$. $\square$

We will use the following remarkable result.

Theorem 2.5 (Steinitz's theorem, see [4, Theorem 2.1.1])**.** *Let* $\mathbf{y} = (y_n)$ *be a sequence in a finite-dimensional Hausdorff topological vector space* Y *over* $\mathbb{R}$ *for which the series* $\sum\limits_{n=1}^{\infty} y_n$ *converges and* $s := \sum\limits_{n=1}^{\infty} y_n$*. Then*

$$SR\Big(\sum_{k=1}^{\infty} y_k\Big) = s + {}^{\perp}\Gamma_{\mathbf{y}}\,.$$

Remark 2.6. Theorem 2.5 appeared in E. Steinitz' cycle of articles [6], where the bases of the finite-dimensional linear algebra is developed as well. Later in [1] W. Banaszczyk succeeded to show that the theorem remains true for all nuclear Frechet spaces over $\mathbb{R}$ (the result of [1] is exposed also in [4, pp. 110–117]).

Using Theorem 2.5 we obtain the following criterion of universality of series in finite-dimensional case.

Proposition 2.7. *For a sequence* $\mathbf{y} = (y_n)$ *in a finite-dimensional Hausdorff topological vector space* Y *over* $\mathbb{R}$ *for which the series* $\sum\limits_{n=1}^{\infty} y_n$ *converges the following are equivalent:*

(i) $SR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) = Y$.

(ii) $\Gamma_{\mathbf{y}} = \{0\}$.

(iii) $\sum\limits_{k=1}^{\infty} |y^*(y_k)| = \infty \quad \forall\, y^* \in Y^* \setminus \{0\}\,.$

Proof. The equivalence (ii)$\Longleftrightarrow$(iii) is evident.

The implication (i)$\Longrightarrow$(iii) is proved in Corollary 2.3.

(ii)$\Longrightarrow$(i). Let $s := \sum\limits_{n=1}^{\infty} y_n$. As ${}^{\perp}\{0\} = Y$, By Theorem 2.5

$$SR\Big(\sum_{k=1}^{\infty} y_k\Big) = s + {}^{\perp}\Gamma_{\mathbf{y}} = s + {}^{\perp}\{0\} = s + Y = Y$$

and (i) is proved. $\square$

Remark 2.8. In [5] it was posed a question of validity of implication (iii)$\Longrightarrow$(i) of Proposition 2.7 when Y is an infinite-dimensional real separable Banach space. In [2] it was provided a negative answer to this question when $Y = l_2$.

It seems to be unknown whether for a convergent series $\sum\limits_{k=1}^{\infty} y_k$ in $Y = l_2$ the condition (iii) of Proposition 2.7 implies the equality $WSR\Big(\sum\limits_{k=1}^{\infty} y_k\Big) = Y$.

Proof of Theorem 1.4. For a fixed $w \in \mathbb{C}$ define $l_w : \mathbb{C} \to \mathbb{C}$ by the equality:

$$l_w(z) = zw \quad \forall\, z \in \mathbb{C}\,.$$

Then

$$\mathbb{C}^* = \{l_w : w \in \mathbb{C}\}\,. \tag{2.3}$$

The implication $(i) \implies (ii)$ of Theorem 1.4 follows from Corollary 2.4 applied for $Y = \mathbb{C}$ and the equality (2.3).

Denote by $\mathbb{C}_{\mathbb{R}}$ the space $\mathbb{C}$ considered as a topological vector space over $\mathbb{R}$. It is easy to see that

$$(\mathbb{C}_{\mathbb{R}})^* = \{\mathrm{Re} l_w : w \in \mathbb{C}\}\,. \tag{2.4}$$

The implication (ii)$\Longrightarrow$(i) of Theorem 1.4 follows from the implication (iii)$\Longrightarrow$(i) of Proposition 2.7 applied for $Y = \mathbb{C}_{\mathbb{R}}$ and the equality (2.4). $\square$

3. Dirichlet's Test for Series and Proof of Theorem 1.5

We need the following known statement. For the sake of completeness we include its proof.

Proposition 3.1 (Dirichlet)**.** *Let (a_n) and (b_n) be the sequences of complex numbers such that $\lim_n a_n = 0$, $\sum\limits_{n=1}^{\infty} |a_n - a_{n+1}| < \infty$ and $\beta = \sup_{n\in\mathbb{N}} \Big|\sum\limits_{k=1}^{n} b_k\Big| < \infty$. Then the series $\sum\limits_{n=1}^{\infty} a_n b_n$ converges.*

Proof. Write $B_0 = 0$, $B_n = \sum\limits_{k=1}^{n} b_k$, $n = 1, 2, \dots$. Fix natural numbers $n > 2$ and $m \le n-2$. We have:

$$\sum_{k=m+1}^{n} a_k b_k = \sum_{k=m+1}^{n} a_k (B_k - B_{k-1}) = \sum_{k=m+1}^{n} a_k B_k - \sum_{k=m+1}^{n} a_k B_{k-1} =$$
$$= \sum_{k=m+1}^{n} a_k B_k - \sum_{k=m}^{n-1} a_{k+1} B_k = \sum_{k=m+1}^{n-1} (a_k - a_{k+1}) B_k + a_n B_n - a_{m+1} B_m\,.$$

Consequently we have the *Abel's formula*:

$$\sum_{k=m+1}^{n} a_k b_k = \sum_{k=m+1}^{n-1} (a_k - a_{k+1}) B_k + a_n B_n - a_{m+1} B_m\,. \tag{3.1}$$

From (3.1) we get:

$$\Big|\sum_{k=m+1}^{n} a_k b_k\Big| \le \beta\Big(\sum_{k=m+1}^{n-1} |a_k - a_{k+1}| + |a_n| + |a_{m+1}|\Big)\,. \tag{3.2}$$

From (3.2) as $\sum\limits_{n=1}^{\infty} |a_n - a_{n+1}| < \infty$ and $\lim\limits_{n} a_n = 0$, we obtain that $\lim\limits_{m,n} \Big|\sum\limits_{k=m+1}^{n} a_k b_k\Big| = 0$. Hence the series $\sum\limits_{n=1}^{\infty} a_n b_n$ converges. □

We need also the following lemma.

Lemma 3.2. *Let ζ be a complex number with $|\zeta| = 1$. Then*
(a) $2|\mathrm{Re}(\zeta)| \ge 1 + \mathrm{Re}(\zeta^2)$.
(b) *For a complex number w with $|w| = 1$ we have:*

$$2\big|\mathrm{Re}(\zeta^n w)\big| \ge 1 + \mathrm{Re}(\zeta^{2n} w^2), \quad n = 1, 2, \ldots.$$

Proof. (a) We have: $2\mathrm{Re}(\zeta) = \zeta + \bar{\zeta}$. This implies:

$$4(\mathrm{Re}(\zeta))^2 = (\zeta + \bar{\zeta})^2 = \zeta^2 + 2\zeta\bar{\zeta} + (\bar{\zeta})^2 = 2 + \zeta^2 + (\bar{\zeta})^2 = 2 + 2\mathrm{Re}(\zeta^2).$$

Hence,

$$2(\mathrm{Re}(\zeta))^2 = 1 + \mathrm{Re}(\zeta^2).$$

From this equality, as $|\mathrm{Re}(\zeta)| \le 1$, we get

$$2|\mathrm{Re}(\zeta)| \ge 2(\mathrm{Re}(\zeta))^2 = 1 + \mathrm{Re}(\zeta^2).$$

(b) An application of (a) for ζw instead of ζ gives:

$$2|\mathrm{Re}(\zeta w)| \ge 1 + \mathrm{Re}(\zeta^2 w^2)\,. \tag{3.3}$$

Fix a natural number n; an application of (3.3) for ζ^n instead of ζ proves (b). □

Lemma 3.3. *Let $\zeta \ne 1$ be a complex number with $|\zeta| = 1$ and (a_n) be a sequence of real numbers such that*

$$a_n \to 0, \quad \sum_{n=1}^{\infty} |a_n - a_{n+1}| < \infty\,.$$

Then
(a) *The series $\sum\limits_{n=1}^{\infty} a_n \zeta^n$ is convergent.*
(b) *If $\zeta^2 \ne 1$ as well, then the series $\sum\limits_{n=1}^{\infty} a_n \zeta^{2n}$ is convergent too.*

Proof. (a) For a fixed natural number n we have:

$$\sum_{k=1}^{n} \zeta^k = \frac{\zeta - \zeta^{n+1}}{1 - \zeta}\,.$$

Hence,

$$\beta := \sup_{n\in\mathbb{N}} \Big|\sum_{k=1}^{n} \zeta^k\Big| \le \frac{2}{|1-\zeta|} < \infty\,.$$

So, an application of Proposition 3.1 for $b_n := \zeta^n$, $n = 1, 2, \dots$ proves (a).

(b) follows from (a) applied to $\zeta^2 \neq 1$ instead of ζ. □

Proof of Theorem 1.5. The series (1.3) converges by Lemma 3.3(a). To show that this series is universal, by Theorem 1.4 it is sufficient to verify that for each $w \in \mathbb{C}\setminus\{0\}$

$$\sum_{n=1}^{\infty} \big|\mathrm{Re}(wa_n\zeta^n)\big| = \infty\,.$$

So, fix $w \in \mathbb{C}\setminus\{0\}$. We can suppose without loss of generality that $|w| = 1$. Assume on the contrary, that

$$\sum_{n=1}^{\infty} \big|\mathrm{Re}(wa_n\zeta^n)\big| < \infty\,. \tag{3.4}$$

Since a_n, $n = 1, 2, \dots$ *are non-negative real numbers*, from (3.4) we get:

$$\sum_{n=1}^{\infty} a_n\big|\mathrm{Re}(w\zeta^n)\big| < \infty\,. \tag{3.5}$$

From (3.5) by Lemma 3.2(b) we obtain

$$\sum_{n=1}^{\infty} a_n\big(1 + \mathrm{Re}(\zeta^{2n}w^2)\big) < \infty\,. \tag{3.6}$$

As $\zeta^2 \neq 1$, from Lemma 3.3(b) we have that the series

$$\sum_{n=1}^{\infty} a_n\mathrm{Re}(\zeta^{2n}w^2)) \tag{3.7}$$

converges. Clearly, (3.6) and the convergence of (3.7) imply $\sum\limits_{n=1}^{\infty} a_n < \infty$. This leads to the contradiction.

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