Proceedings of A. Razmadze
Mathematical Institute
Vol. **160** (2012), 25–34

SIGNS AND PERMUTATIONS: TWO PROBLEMS OF THE FUNCTION THEORY

S. CHOBANYAN, G. GIORGOBIANI AND V. TARIELADZE

Abstract. We prove a variant of the transference lemma and by applying it we get a refined version of the Garsia inequality for orthogonal systems (the case $p \geq 2$). Moreover we show that the fulfillment of the (σ, θ)-condition on the Fourier series of a continuous periodic Banach space valued function f implies the uniform convergence of a rearrangement of the series to f.

რეზიუმე. დამტკიცებულია გადატანის ლემის ერთი ვარიანტი და მისი გამოყენებით მიღებულია ორთოგონალური სისტემისათვის გარსიას უტოლობის გაუმჯობესებული ვერსია ($p \geq 2$ შემთხვევისათვის). აგრეთვე ნაჩვენებია, რომ ბანახის სივრცეში მნიშვნელობებიანი პერიოდული უწყვეტი f ფუნქციის ფურიეს მწკრივისათვის (σ, θ)-პირობის შესრულება იწვევს მწკრივის f ფუნქციისკენ თანაბრად კრებადი გადანაცვლების არსებობას.

1. Preliminaires

Everywhere in this paper X will stand for a normed space, real or complex, Π_n for all permutations π of $\{1, \dots, n\}$ and Θ_n for all collections of signs $\theta = (\theta_1, \dots, \theta_n)$, $\theta_i = \pm 1$, $i = 1, \dots, n$, $n \in \mathbb{N}$. As usual, for permutations $\pi, \sigma \in \Pi_n$ we write $\pi \circ \sigma$ for their composition.

Given $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, $\pi \in \Pi_n$ and $\theta = (\theta_1, \dots, \theta_n) \in \Theta_n$ we denote

$$\mathbf{x}\theta = (x_1\theta_1, \dots, x_n\theta_n), \ \mathbf{x}_\pi = (x_{\pi(1)}, \dots, x_{\pi(n)}), \ \mathbf{x}_\pi\theta = (x_{\pi(1)}\theta_1, \dots, x_{\pi(n)}\theta_n)$$

and

$$|\mathbf{x}|_n = \max_{1 \leq k \leq n} \|x_1 + \dots + x_k\|.$$

Note that $|\cdot|_n$ is a norm on X^n; this easily verifiable observation will be used below essentially.

2010 *Mathematics Subject Classification.* 42A20, 42A61.
Key words and phrases. Rearrangement, series, transference lemma, orthogonal system, Fourier series, uniform convergence.

We call a permutation $\pi_o \in \Pi_n$ *optimal for* $\mathbf{x} = (x_1, \dots, x_n) \in X^n$, if $|\mathbf{x}_{\pi_o}|_n \le |\mathbf{x}_\pi|_n$ for any $\pi \in \Pi_n$. Note that an optimal permutation always exists.

To every $\theta \in \Theta_n$ we associate a permutation $\sigma_\theta \in \Pi_n$ as follows:

$$\sigma_\theta(1) = u_1, \dots, \sigma_\theta(s) = u_s, \quad \sigma_\theta(s+1) = v_t, \dots, \sigma_\theta(n) = v_1\,,$$

where the integers s, t and the indices $u_1, \dots, u_s$, $v_1, \dots, v_t$ are chosen as follows:

$$\theta_{u_1} = +1, \dots, \theta_{u_s} = +1; \;\; u_1 < u_2 < \dots < u_s,$$
$$\theta_{v_1} = -1, \dots, \theta_{v_t} = -1; \;\; v_1 < v_2 < \dots < v_t; \;\; s + t = n.$$

In proofs below we will use repeatedly the following *transference lemma* (see [1], [2]).

Lemma 1.1. *Let* $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ *with* $\sum\limits_1^n x_i = 0$ *and* $\theta \in \Theta_n$. *Then*

$$|\mathbf{x}|_n + |\mathbf{x}\theta|_n \ge 2|\mathbf{x}_{\sigma_\theta}|_n\,, \tag{1}$$

and

$$|\mathbf{x}_\pi|_n + |\mathbf{x}_\pi\theta|_n \ge 2|\mathbf{x}_{\pi\circ\sigma_\theta}|_n \quad \forall \pi \in \Pi_n\,. \tag{2}$$

Moreover, for a permutation $\pi_o \in \Pi_n$ *optimal for* $\mathbf{x} = (x_1, \dots, x_n)$

$$|\mathbf{x}_{\pi_o}|_n \le \min_{\theta\in\Theta_n} |\mathbf{x}_{\pi_o}\theta|_n. \tag{3}$$

To make the presentation self-contained we include a simple proof here.

Proof. Using the fact that $|\cdot|_n$ is a norm in X^n we get

$$|(x_1, \dots, x_n)|_n + |(x_1\theta_1, \dots, x_n\theta_n)|_n \ge |(x_1, \dots, x_n) + (x_1\theta_1, \dots, x_n\theta_n)|_n =$$
$$= |(x_1(1+\theta_1), \dots, x_n(1+\theta_n))|_n = 2|(x_{u_1}, \dots, x_{u_s})|_s\,,$$

i.e.

$$|(x_1, \dots, x_n)|_n + |(x_1\theta_1, \dots, x_n\theta_n)|_n \ge 2|(x_{u_1}, \dots, x_{u_s})|_s\,. \tag{4}$$

In a similar way we get also

$$|(x_1, \dots, x_n)|_n + |(x_1\theta_1, \dots, x_n\theta_n)|_n \ge 2|(x_{v_1}, \dots, x_{v_t})|_t\,. \tag{5}$$

From (4) and (5) we conclude:

$$|(x_1, \dots, x_n)|_n + |(x_1\theta_1, \dots, x_n\theta_n)|_n \ge$$
$$\ge 2\max\big(|(x_{u_1}, \dots, x_{u_s})|_s\,, |(x_{v_1}, \dots, x_{v_t})|_t\big)\,. \tag{6}$$

By using the condition $\sum\limits_1^n x_i = 0$ it is easy to make sure that

$$\max\big(|(x_{u_1}, \dots, x_{u_s})|_s\,, |(x_{v_1}, \dots, x_{v_t})|_t\big) = |\mathbf{x}_{\sigma_\theta}|_n\,. \tag{7}$$

The inequalities (6) and (7) give (1).

To prove (2) fix $\pi \in \Pi_n$ and apply (1) to $\mathbf{x}_\pi = (x_{\pi(1)}, \dots, x_{\pi(n)})$.

Let us verify now (3). Fix a permutation $\pi_o \in \Pi_n$ which is optimal for $\mathbf{x} = (x_1, \ldots, x_n)$ and an arbitrary $\theta \in \Theta_n$. An application of inequality (2) for π_o gives:

$$|\mathbf{x}_{\pi_o}|_n + |\mathbf{x}_{\pi_o}\theta|_n \geq 2|\mathbf{x}_{\pi_o \circ \sigma_\theta}|_n \,.$$

Since π_o is optimal, we can write:

$$|\mathbf{x}_{\pi_o \circ \sigma_\theta}|_n \geq |\mathbf{x}_{\pi_o}|_n \,.$$

From the last inequalities we obtain:

$$|\mathbf{x}_{\pi_o}|_n + |\mathbf{x}_{\pi_o}\theta|_n \geq 2|\mathbf{x}_{\pi_o}|_n \,.$$

Hence, $|\mathbf{x}_{\pi_o}\theta|_n \geq |\mathbf{x}_{\pi_o}|_n$ and (3) is proved. □

Corollary 1.2. *Let $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ be a collection (not necessarily summing up to zero) and $\pi_o \in \Pi_n$ be a permutation which is optimal for $\mathbf{x} = (x_1, \ldots, x_n)$. Then*

$$|\mathbf{x}_{\pi_o}|_n \leq |\mathbf{x}_{\pi_o}\theta|_n + 3\Big\|\sum_{i=1}^{n} x_i\Big\|, \quad \forall\, \theta \in \Theta_n \,. \tag{8}$$

Proof. For a general collection $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ and a permutation $\pi \in \Pi_n$ let us introduce the collection $(x_{\pi(1)}, \ldots, x_{\pi(n)}, -s) \in X^{n+1}$, where $s = \sum\limits_1^n x_i$. In the same way as in the proof of Lemma 1.1 we get for any $\pi \in \Pi_n$ and $\theta \in \Theta_n$

$$\begin{aligned} &\big|(x_{\pi(1)}, \ldots, x_{\pi(n)}, -s)\big|_{n+1} + \big|(x_{\pi(1)}\theta_1, \ldots, x_{\pi(n)}\theta_n, -s)\big|_{n+1} \geq \\ &\geq 2\big|(x_{\pi(u_1)}, \ldots, x_{\pi(u_s)}, -s, x_{\pi(v_t)}, \ldots, x_{\pi(v_1)})\big|_{n+1} \geq 2\big(|\mathbf{x}_{\pi\circ\sigma_\theta}|_n - \|s\|\big). \end{aligned}$$

If we remark now that $|(x_{\pi(1)}, \ldots, x_{\pi(n)}, -s)|_{n+1} = |\mathbf{x}_\pi|_n$, then we will have for any $\pi \in \Pi_n$ and $\theta \in \Theta_n$

$$|\mathbf{x}_\pi|_n + |\mathbf{x}_\pi\theta|_n + \|s\| \geq 2|\mathbf{x}_{\pi\circ\sigma_\theta}|_n - 2\|s\|. \tag{9}$$

For the optimal permutation π_0 (9) leads to (8). □

It is not clear whether in Corollary 1.2 the constant 3 can be replaced by a smaller one. The following statement shows that in this respect some other permutations may be better than an optimal permutation.

Corollary 1.3. *Let $\mathbf{x} = (x_1, \ldots, x_n) \in X^n$ be a collection (not necessarily summing up to zero). Then there exists a permutation $\pi \in \Pi_n$ such that*

$$|\mathbf{x}_\pi|_n \leq |\mathbf{x}_\pi\theta|_n + 2\Big\|\sum_{i=1}^{n} x_i\Big\| \quad \forall\theta \in \Theta_n \,. \tag{10}$$

Equivalently,

$$\max_{1\le k\le n}\Big\|\sum_1^k x_{\pi(i)}\Big\| \;\le\; \min_{\theta\in\Theta_n}\max_{1\le k\le n}\Big\|\sum_1^k x_{\pi(i)}\theta_i\Big\| + 2\Big\|\sum_1^n x_i\Big\|\,.$$

Proof. Let $s = \sum\limits_{j=1}^n x_j$. If $s = 0$, then the assertion follows from (3) of Lemma 1.1. So, we can suppose that $s \neq 0$. Write $a := \frac{1}{n}s$ and consider the following new collection $\mathbf{y} = (y_1, \dots, y_n)$ elements of X, where $y_i = x_i - a$, $i = 1, \dots, n$. Then $\sum\limits_{j=1}^n y_j = 0$. Let $\pi \in \Pi_n$ be a permutation which is optimal for $\mathbf{y} = (y_1, \dots, y_n)$. Fix arbitrarily $\theta \in \Theta_n$. According to (3) of Lemma 1.1 we have

$$|\mathbf{y}_\pi|_n \le |\mathbf{y}_\pi\theta|_n\,. \tag{11}$$

From this, as $|\cdot|_n$ is a norm, we can write

$$\begin{aligned}|\mathbf{x}_\pi|_n &= |\mathbf{y}_\pi + (a,\dots,a)|_n \le |\mathbf{y}_\pi|_n + |(a,\dots,a)|_n = |\mathbf{y}_\pi|_n + \|s\| \le\\ &\le \big|\mathbf{y}_\pi\theta\big|_n + \|s\| = \big|\mathbf{x}_\pi\theta - (a\theta_1,\dots,a\theta_n)\big|_n + \|s\| \le\\ &\le |\mathbf{x}_\pi\theta|_n + \big|(a\theta_1,\dots,a\theta_n)\big|_n + \|s\| \le \big|\mathbf{x}_\pi\theta\big|_n + 2\|s\|\end{aligned}$$

and corollary is proved. □

Below $(r_n)_{n\in\mathbb{N}}$ is a sequence of independent random variables on a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ such that $\mathbb{P}[r_n = -1] = \mathbb{P}[r_n = 1] = \frac{1}{2}$, $n = 1, 2, \dots$ $\mathbb{E}$ is the related expectation.

Proposition 1.4. *Let $\mathbf{x} = (x_1, \dots, x_n)$ be a collection of elements of X (not necessarily summing up to zero) and $\Phi : [0,\infty) \to [0,\infty)$ be an increasing convex function. Then the following inequality holds:*

$$\begin{aligned}\frac{1}{n!}\sum_\pi \max_{1\le k\le n}\Phi\Big(\Big\|x_{\pi(1)} + \ldots + x_{\pi(k)} - \frac{k}{n}\sum_{j=1}^n x_j\Big\|\Big) \le\\ \le 2\mathbb{E}\Phi\Big(\Big\|\sum_{i=1}^n\Big(x_i - \frac{1}{n}\sum_{j=1}^n x_j\Big)r_i\Big\|\Big).\end{aligned} \tag{12}$$

Proof. Let us introduce a new collection $\mathbf{y} = (y_1, \dots, y_n)$ elements of X as follows: $y_i = x_i - \frac{1}{n}\sum\limits_{j=1}^n x_j$. Then $\sum\limits_{j=1}^n y_j = 0$ and according to Lemma 1.1 we have for each $\pi \in \Pi_n$ and $\theta \in \Theta_n$

$$|\mathbf{y}_{\pi\circ\sigma_\theta}|_n \le \frac{1}{2}|\mathbf{y}_\pi|_n + \frac{1}{2}|\mathbf{y}_\pi\theta|_n.$$

From this, since Φ is increasing and convex, we get

$$\Phi\big(|\mathbf{y}_{\pi\circ\sigma_\theta}|_n\big) \le \frac{1}{2}\Phi\big(|\mathbf{y}_\pi|_n\big) + \frac{1}{2}\Phi\big(|\mathbf{y}_\pi\theta|_n\big).$$

Therefore,

$$\sum_{\pi\in\Pi_n}\Phi(|\mathbf{y}_{\pi\circ\sigma_\theta}|_n)\le\frac{1}{2}\sum_{\pi\in\Pi_n}\Phi(|\mathbf{y}_\pi|_n)+\frac{1}{2}\sum_{\pi\in\Pi_n}\Phi(|\mathbf{y}_\pi\theta|_n).$$

From this, as $\{\pi\circ\sigma_\theta:\pi\in\Pi_n\}=\Pi_n$, we get:

$$\sum_{\pi\in\Pi_n}\Phi(|\mathbf{y}_\pi|_n)\le\frac{1}{2}\sum_{\pi\in\Pi_n}\Phi(|\mathbf{y}_\pi|_n)+\frac{1}{2}\sum_{\pi\in\Pi_n}\Phi(|\mathbf{y}_\pi\theta|_n).$$

Hence,

$$\sum_{\pi\in\Pi_n}\Phi(|\mathbf{y}_\pi|_n)\le\sum_{\pi\in\Pi_n}\Phi(|\mathbf{y}_\pi\theta|_n).$$

Since $\theta\in\Theta_n$ is arbitrary, this inequality can be rewritten as follows:

$$\sum_{\pi\in\Pi_n}\Phi\big(\big|\mathbf{y}_\pi\big|_n\big)\le\sum_{\pi\in\Pi_n}\Phi\big(\big|(y_{\pi(1)}r_1,\dots,y_{\pi(n)}r_n)\big|_n\big)\ a.s.\ .$$

This inequality implies

$$\sum_{\pi\in\Pi_n}\Phi\big(|\mathbf{y}_\pi|_n\big)\le\sum_{\pi\in\Pi_n}\mathbb{E}\Phi\big(|(y_{\pi(1)}r_1,\dots,y_{\pi(n)}r_n)|_n\big).$$

Fix again $\pi\in\Pi_n$; it is standard to derive from Levy's inequality that

$$\mathbb{E}\Phi\big(\big|(y_{\pi(1)}r_1,\dots,y_{\pi(n)}r_n)\big|_n\big)\le 2\mathbb{E}\Phi\bigg(\Big\|\sum_{i=1}^n y_{\pi(i)}r_i\Big\|\bigg)=$$
$$=2\mathbb{E}\Phi\bigg(\Big\|\sum_{i=1}^n y_{\pi(i)}r_{\pi(i)}\Big\|\bigg)=2\mathbb{E}\Phi\bigg(\Big\|\sum_{i=1}^n y_ir_i\Big\|\bigg).$$

Finally, we obtain

$$\sum_{\pi\in\Pi_n}\Phi\big(|\mathbf{y}_\pi|_n\big)\le 2n!\mathbb{E}\Phi\bigg(\Big\|\sum_{i=1}^n y_ir_i\Big\|\bigg)$$

and (12) is proved. □

Corollary 1.5 (Garsia [3,4]). *Let $1\le p<\infty$ and $\mathbf{x}=(x_1,\dots,x_n)$ be a collection of real or complex numbers (not necessarily summing up to zero). Then the following inequalities hold:*

$$\frac{1}{n!}\sum_\pi\max_{1\le k\le n}\bigg(\Big|x_{\pi(1)}+\dots+x_{\pi(k)}-\frac{k}{n}\sum_{j=1}^n x_j\Big|^p\bigg)\le$$
$$\le 2C_p^p\bigg(\sum_1^n\Big|x_i-\frac{1}{n}\sum_{j=1}^n x_j\Big|^2\bigg)^{p/2}\tag{13}$$

and

$$\frac{1}{n!}\sum_{\pi}\max_{1\le k\le n}\left(\left|x_{\pi(1)}+\cdots+x_{\pi(k)}-\frac{k}{n}\sum_{j=1}^{n}x_j\right|^p\right)\le 2C_p^p\left(\sum_{1}^{n}|x_i|^2\right)^{p/2}, \quad (14)$$

where C_p is the Khinchine constant.

Proof. An application of (12) to the function $t\mapsto\Phi(t):=t^p$ gives:

$$\frac{1}{n!}\sum_{\pi}\max_{1\le k\le n}\left(\left|x_{\pi(1)}+\cdots+x_{\pi(k)}-\frac{k}{n}\sum_{j=1}^{n}x_j\right|^p\right)\le 2\mathbb{E}\left|\sum_{i=1}^{n}\Big(x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\Big)r_i\right|^p.$$

By the Khinchine inequality

$$\mathbb{E}\left|\sum_{i=1}^{n}\Big(x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\Big)r_i\right|^p\le C_p^p\left(\sum_{1}^{n}\left|x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\right|^2\right)^{p/2}.$$

These inequalities imply (13).

It is easy to verify that

$$\sum_{i=1}^{n}\left|x_i-\frac{1}{n}\sum_{j=1}^{n}x_j\right|^2\le\sum_{1}^{n}|x_i|^2.$$

Hence, (14) follows from (13). □

2. The Garsia Theorem on Orthogonal Systems, the Case $p\ge 2$

Here we apply Corollary 1.5 to get the following famous result belonging to Garsia [3,4].

Theorem 2.1. *Let $(\varphi_1,\dots,\varphi_n)$ be an orthonormal system of $L_2(\Omega,\mathcal{A},\mu)$, $(\alpha_1,\dots,\alpha_n)$ be a collection of reals and $2\le p<\infty$. Assume that $\varphi_i\in L_p(\Omega,\mathcal{A},\mu)$, $i=1,2,\dots,n$ and $M:=\max_{1\le i\le n}\|\varphi_i\|_{L_p}$. Then the following inequality holds*

$$\frac{1}{n!}\sum_{\pi}\int_{\Omega}\max_{1\le k\le n}\left|\alpha_{\pi(1)}\varphi_{\pi(1)}+\cdots+\alpha_{\pi(k)}\varphi_{\pi(k)}-\frac{k}{n}f\right|^p d\mu\le$$

$$\le 2C_p^pM^p\left(\int_{\Omega}f^2d\mu\right)^{p/2}, \quad (15)$$

where $f=\sum_{1}^{n}\alpha_i\varphi_i$ and C_p is the Khinchine constant.

Proof. Fix $\omega \in \Omega$. The application of the inequality (14) of Corollary 1.5 for the collection $(\alpha_1\varphi_1(\omega), \dots, \alpha_n\varphi_n(\omega))$ gives:

$$\frac{1}{n!}\sum_{\pi}\max_{1\le k\le n}\left(\Big|\sum_{i=1}^{k}\alpha_{\pi(i)}\varphi_{\pi(i)}(\omega)-\frac{k}{n}f(\omega)\Big|^p\right)\le$$
$$\le 2C_p^p\Big[\sum_{1}^{n}\big|\alpha_i\varphi_i(\omega)\big|^2\Big]^{p/2}. \tag{16}$$

Integrating both sides of (16) we get

$$\frac{1}{n!}\sum_{\pi}\int_{\Omega}\max_{1\le k\le n}\left(\Big|\sum_{i=1}^{k}\alpha_{\pi(i)}\varphi_{\pi(i)}-\frac{k}{n}f\Big|^p\right)d\mu\le$$
$$\le 2C_p^p\int_{\Omega}\Big(\sum_{1}^{n}|\alpha_i\varphi_i|^2\Big)^{p/2}d\mu\,. \tag{17}$$

As $p/2\ge 1$, the application of Minkowski's inequality gives:

$$\Big(\int_{\Omega}\Big(\sum_{1}^{n}|\alpha_i\varphi_i|^2\Big)^{2/p}d\mu\Big)^{2/p}\le$$
$$\le\sum_{1}^{n}|\alpha_i|^2\int_{\Omega}\Big(\int_{\Omega}|\varphi_i|^p d\mu\Big)^{2/p}\le M^2\int_{\Omega}f^2d\mu\,. \tag{18}$$

From (18) and (17) we get (15). □

Remark 2.2. Garsia [4] stated Theorem 2.1 under the assumption that $\varphi_i\in L_\infty(\Omega,\mathcal{A},\mu)$ $i=1,2,\dots,n$ and with $M:=\max_{1\le i\le n}\|\varphi\|_{L_\infty}$.

Apparently, our proof is simpler and the constants are smaller.

3. On the Ulyanov Conjecture

In this section Y will stand for a real or complex Banach space and $C([-\pi,\pi],Y)$ for the set of all continuous functions $f:[-\pi,\pi]\to Y$ such that $f(-\pi)=f(\pi)$. This set with respect to the point-wise operations and norm $f\mapsto\|f\|:=\sup_{t\in[-\pi,\pi]}\|f(t)\|_Y$ is a Banach space.

For $f\in C([-\pi,\pi],Y)$ we write:

$$a_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\cos nt\,dt,\quad b_n(f)=\frac{1}{\pi}\int_{-\pi}^{\pi}f(t)\sin nt\,dt,\quad n=0,1,\dots\,,$$

and define the functions $A_n(f) : [-\pi, \pi] \to Y$, $n = 0, 1, \dots$ by the equalities:

$$A_0(f)(t) = \frac{1}{2} a_0(f), \quad A_n(f)(t) = a_n(f) \cos nt + b_n(f) \sin nt,$$
$$\forall\, n \in \mathbb{N}, \ \ \forall\, t \in [-\pi, \pi]$$

and call

$$A_0(f) + \sum_{n=1}^{\infty} A_n(f)$$

the (trigonometric) Fourier series of f.

Let us denote

- by $\mathbf{U}([-\pi, \pi], Y)$ the set of all $f \in C([-\pi, \pi], Y)$ for which the corresponding Fourier series converges uniformly (to f);
- by $\mathbb{U}_{\mathrm{ul}}([-\pi, \pi], Y)$ the set of all $f \in C([-\pi, \pi], Y)$ for which there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that the series $A_0(f) + \sum\limits_{n=1}^{\infty} A_{\sigma(n)}(f)$ converges uniformly (to f).

It is well-known that $\mathbf{U}([-\pi, \pi], \mathbb{R}) \neq C([-\pi, \pi], \mathbb{R})$. Ulyanov [5] conjectured that $\mathbb{U}_{\mathrm{ul}}([-\pi, \pi], \mathbb{R}) = C([-\pi, \pi], \mathbb{R})$.

This conjecture remains open so far. There are several results dealing with the Ulyanov conjecture. Konyagin [6] has proved that if the modulus of continuity of a function $f \in C([-\pi, \pi], \mathbb{R})$ satisfies a weakened Dini-Lipschitz type condition, then $f \in \mathbb{U}_{\mathrm{ul}}([-\pi, \pi], \mathbb{R})$. The following general result by Revesz [7] seems to be a very important related result.

Theorem 3.1 (7, Theorem 1)**.** *For any $f \in C([-\pi, \pi], \mathbb{R})$ there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ and a sequence of integers $N_k \uparrow \infty$ such that the sequence*

$$\left(A_0(f) + \sum_{1}^{N_k} A_{\sigma(i)}(f) \right)_{k \in \mathbb{N}}$$

converges to f uniformly.

Definition 3.2. We say that a sequence $(x_k)_{k \in \mathbb{N}}$ of elements of a normed space X *satisfies the Rademacher condition*, if the series $\sum\limits_{k=1}^{\infty} x_k r_k(\omega)$ converges in X for $\mathbb{P}$-almost every $\omega \in \Omega$.

Theorem 3.3 ([8, Theorem 1] and [9, Theorem 2])**.** *Let $f \in C([-\pi, \pi], \mathbb{R})$ be such that the sequence $(A_k(f))_{k \in \mathbb{N}}$ satisfies the Rademacher condition in $C([-\pi, \pi], \mathbb{R})$. Then $f \in \mathbb{U}_{\mathrm{ul}}([-\pi, \pi], \mathbb{R})$.*

To state the main result of this section we need one more definition.

Definition 3.4. We say that a sequence $(x_k)_{k \in \mathbb{N}}$ of elements of a normed space X *satisfies the (σ,θ)-condition*, if for any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ there

exists a collection of signs $\theta = (\theta_1, \theta_2, \ldots)$ such that the series $\sum\limits_{i=1}^{\infty} x_{\sigma(i)}\theta_i$ converges in X.

Since the sequences satisfying the Rademacher condition satisfy the (σ,θ)-condition (cf. [9, Proposition 1]), the following result formally is a refinement of Theorem 3.3 even for $Y = \mathbb{R}$.

Theorem 3.5. *Let Y be a Banach space and $f \in C([-\pi,\pi], Y)$ be such that the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the (σ,θ)- condition in $C([-\pi,\pi], Y)$. Then $f \in \mathbb{U}_{\mathrm{ul}}([-\pi,\pi], Y)$.*

Proof. Fix a function $f \in C([-\pi,\pi], Y)$ such that

(1) *the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the (σ,θ)- condition in $C([-\pi,\pi], Y)$.*

Write $S_n(f) = \sum\limits_{i=0}^{n} A_n(f)\, n = 0, 1, \ldots$. By the Fejer theorem,

(2) *the sequence*

$$\frac{1}{n+1}\sum_{k=0}^{n} S_k(f),\ n = 1, 2, \ldots$$

converges in $C([-\pi,\pi], Y)$ to f.

From (1) and (2) according to [9, Corollary 4] it follows that there is a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that the series $A_0(f) + \sum\limits_{n=1}^{\infty} A_{\sigma(n)}(f)$ converges in $C([-\pi,\pi], Y)$ to f. Consequently, $f \in \mathbb{U}_{\mathrm{ul}}([-\pi,\pi], Y)$. □

Remark 3.6. Let

- $C_{\mathrm{rad}}([-\pi,\pi], Y)$ be the set of all $f \in C([-\pi,\pi], Y)$ such that the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the Rademacher condition in $C([-\pi,\pi], Y)$,
- $C_{\sigma,\theta}([-\pi,\pi], Y)$ be the set of all $f \in C([-\pi,\pi], Y)$ such that the sequence $(A_k(f))_{k\in\mathbb{N}}$ satisfies the (σ,θ)- condition in $C([-\pi,\pi], Y)$.

Then

(a) Theorem 3.3 asserts that $C_{\mathrm{rad}}([-\pi,\pi], \mathbb{R}) \subset \mathbb{U}_{\mathrm{ul}}([-\pi,\pi], \mathbb{R})$, while [8, Theorem 2] tells us that this inclusion is strict.

(b) $C_{\mathrm{rad}}([-\pi,\pi], Y) \subset C_{\sigma,\theta}([-\pi,\pi], Y)$ and *we conjecture that* $C_{\sigma,\theta}([-\pi,\pi], \mathbb{R}) = C([-\pi,\pi], \mathbb{R})$.

(c) If the conjecture from (b) is true, then Theorem 3.5 would imply the positive answer to Ulyanov's conjecture.

Acknowledgements

We are grateful to professors S. Kwapien and S. V. Konyagin for their useful comments during preparation of this paper.

Supported in part by Georgian National Science Foundation grant /ST08/3-384; the third named author was partially supported also by grant GNSF/ST09_99_3 − 104.

References

1. S. Chobanyan and G. Giorgobiani, A problem on rearrangements of summands in normed spaces and Rademacher sums. *Lecture Notes in Math.* **1381** (1987), 33–46.
2. S. A. Chobanyan, Convergence a.s. of rearranged series in Banach spaces and associated inequalities. *Prog. Probab.* **35** (1994), 3–29.
3. A. Garsia, Existence of almost everywhere convergent rearrangements for Fourier series of L_2 functions. *Ann. of Math.* **22** (1964), 286–309.
4. A. Garsia, Topics in almost everywhere convergence. *Topics in almost everywhere convergence. Lectures in Advanced Mathematics,* 4 *Markham Publishing Co., Chicago*, Ill. 1970.
5. P. L. Ulyanov, Solved and unsolved problems of the theory of trigonometric and orthogonal series. *Uspehi Mat. Nauk* **19** (1964), 3–69.
6. S. V. Konyagin, On uniformly convergent rearrangements of trigonometric Fourier series. *J. Math. Sci., New York* **155** (2008), No. 1, 81–88.
7. Sz. Gy. Révész, Rearrangement of Fourier Series. *J. Approx. Theory* **60** (1990), No. 1, 101–121.
8. Sz. Gy. Révész, Rearrangement of Fourier series and Fourier series whose terms have random signs. *Acta Math. Hungar.* **63** (1994), No. 4, 395–402.
9. D. V. Pecherskij, Rearrangements of series in Banach spaces and arrangements of signs. *Matem. Sb.* **135** (177) (1988), No. 1, 24–35; English transl.: *Math. USSR Sb.* **63** (1989), No. 1, 23–33.

(Received 25.04.2012)

Authors' address:

Niko Muskhelishvili Institute of Computational
Mathematics of the Georgian Technical University
8a, Akuri Str., Tbilisi 0160
Georgia
E-mail: chobanya@stt.msu.edu
bachanabc@yahoo.com
vajatarieladze@yahoo.com