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SKITOVICH-DARMOIS THEOREM FOR COMPLEX AND QUATERNION CASES

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Abstract. We give an elementary proof of Skitovich-Darmois theorem for the cases of two random variables in each of the two linear forms, when the random variables take values in complex or quaternion fields. The proof is based on the reduction of Skitovich-Darmois theorem to Polya's theorem in complex and quaternion cases respectively.

რეზიუმე. ნაშრომში მოცემულია დარმუა-სკიტოვიჩის თეორემის ელემენტარული დამტკიცება იმ შემთხვევაში, როცა წრფივი ფორმები შედგენილია მხოლოდ ორი, კომპლექსური ან კვატერნიონული შემთხვევითი სიდიდისაგან. თეორემას ვამტკიცებთ პოიას თეორემის გამოყენებით კომპლექსური და კვატერნიონული შემთხვევითი სიდიდეებისათვის.

In the present paper we give an elementary proof of Darmois-Skitovich theorem in the particular case of two linear forms with two random variables in each. The main idea is to reduce the problem to the Polya's characterization theorem. We mean the following well-known theorem of Polya, see e.g. [4], [5].

Theorem 1. *$\xi_1, \xi_2, \dots, \xi_n$, $n \geq 2$ be i.i.d. random variables and $(a_1, a_2, \dots, a_n)$ be nonzero reals that satisfy the condition $\sum_{h=1}^{n} a_h^2 = 1$. If the sum $\sum_{h=1}^{n} a_h \xi_h$ has the same distribution as ξ_1, then ξ_1 is a Gaussian random variable.*

If the random variable takes values in the quaternion algebra then three types of Gaussian random variables are considered: real, complex and quaternion Gaussian random variables. Let us recall the definition of complex and quaternion Gaussian random variables. The usual motivation for these definitions comes from the form of characteristic function of a centered

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Gaussian random variable, see e.g. [2]. For the real case this is given as

$$\exp\Big\{-\frac{1}{2}t^2E\xi^2\Big\}, \quad \forall\, t \in R.$$

For the complex (quaternion) case we would analogously expect the characteristic function to be

$$\exp\big\{-c|q|^2E|\xi|^2|\big\}, \quad \forall\, q \in C, (\forall q \in Q), \quad c > 0. \tag{$*$}$$

The characteristic function of a complex (quaternion) random variable ξ is defined as

$$\chi_\xi(q) = E\exp(i\,\mathrm{Re}(\xi\overline{q}))$$

and if we want the characteristic function of centered complex (quaternion) Gaussian random variable to have the form (*), then the covariance matrix of real two dimensional vector $(\xi^{'}, \xi^{''})$ (four dimensional vector $(\xi^{'}, \xi^{''}, \xi^{'''}, \xi^{IV})$) should be proportional to the identity matrix. Thus the covariance matrices of complex (quaternion) Gaussian random variables have a quite specific form: they are proportional to unit matrices in R^2, (in R^4). Therefore the coordinates of corresponding two dimension (four dimension) random vector $(\xi^{'}, \xi^{''})$ $((\xi^{'}, \xi^{''}, \xi^{'''}, \xi^{IV}))$ are mutually independent and have the same variances.

In [3] there is formulated Polya's theorem for the case of complex random variables.

Theorem 2. *Let ξ be a complex random variable, $\xi_1, \xi_2, \dots, \xi_n$, $n \geq 2$ be independent copies of ξ and $(a_1, a_2, \dots, a_n)$ be nonzero complex numbers such that $\sum\limits_{h=1}^{n} |a_h|^2 = 1$ and at least one of them is not a real number. If $\sum\limits_{h=1}^{n} a_h\xi_h$ has the same distribution as ξ, then ξ is the complex Gaussian random variable.*

As we see in the complex case there is an additional condition on the complex coefficients $(a_1, a_2, \dots, a_n)$, for the Theorem to be true, namely one of these coefficients should be essentially complex number. In [1] there is shown that in the quaternion case such additional condition on the quaternions $(a_1, a_2, \dots, a_n)$ plays condition which we call jointly quaternion system.

Definition. We say that a collection of n quaternions $(a_1, a_2, \dots, a_n)$, $n \geq 2$, constitutes a jointly quaternion system (JQS) if there does not exist imaginary number $\tilde{i} = \alpha i + \beta j + \gamma k$, with real α, β, γ, such that the following expressions hold: $a_1 = a_1^{'} + a_1^{''}\tilde{i}$, $a_2 = a_2^{'} + a_2^{''}\tilde{i}, \dots$, $a_n = a_n^{'} + a_n^{''}\tilde{i}$, $a_i^{'}, a_i^{''} \in R$, $1 \leq i \leq n$.

Theorem 3. *Let ξ be a quaternion random variable, $\xi_1, \xi_2, \dots, \xi_n$, $n \geq 2$ be independent copies of ξ, and $(a_1, a_2, \dots, a_n)$ be nonzero quaternions that* *form jointly quaternion system and satisfy the condition* $\sum\limits_{h=1}^{n}|a_h|^2=1$. *Then, if the sum* $\eta=\sum\limits_{h=1}^{n}a_h\xi_h$ *has the same distribution as* ξ, ξ *is the quaternion Gaussian random variable.*

Now using these results we will prove Skitovich-Darmois theorem for complex and quaternion cases then the two linear forms consisting with only two random variables.

Theorem (Skitovich-Darmois, complex case). *Let* ξ_1 *and* ξ_2 *be independent complex random variables* (*not necessarily identically distributed*) *and* a_1, a_2, b_1, b_2 *be non-zero complex numbers, such that* $\overline{a_1}a_2b_1\overline{b_2}$ *or* $\sqrt{-\frac{a_1b_1}{a_2b_2}}\left(\frac{b_2}{b_1}-\frac{a_2}{a_1}\right)$ *is not a real number. If the random variables* $\eta_1=a_1\xi_1+a_2\xi_2$ *and* $\eta_2=b_1\xi_1+b_2\xi_2$ *are also independent then* ξ_1 *and* ξ_2, *and hence* η_1 *and* η_2, *are complex Gaussian random variables.*

Proof. Without loss of generality we can assume that $a_1b_1=-a_2b_2$. Indeed, if $a_1b_1\neq-a_2b_2$ we could consider the linear forms $\eta_1=a_1\xi_1+a_2c\xi_2'$ and $\eta_2=b_1\xi_1+b_2c\xi_2'$, where c is one of the complex root from $-\frac{a_1b_1}{a_2b_2}$, i.e. c is such complex number that $c^2=-\frac{a_1b_1}{a_2b_2}$, and $\xi_2'=\frac{\xi_2}{c}$. Nothing will be changed this way: η_1 and η_2 will remain the same and ξ_2' will be independent of ξ_1. Therefore, in what follows in the linear forms η_1 and η_2 we assume that $a_1b_1=-a_2b_2$.

Denote the characteristic functions of ξ_1,ξ_2 and (η_1,η_2) by χ_1,χ_2 and $\chi_{(\eta_1,\eta_2)}$ respectively. Using independence of ξ_1 and ξ_2 we get the equality

$$\chi_{(\eta_1,\eta_2)}(t_1,t_2)=Ee^{i\mathrm{Re}[\eta_1\overline{t_1}+\eta_2\overline{t_2}]}=Ee^{i\mathrm{Re}[(a_1\xi_1+a_2\xi_2)\overline{t_1}+(b_1\xi_1+b_2\xi_2)\overline{t_2}]}=$$
$$=\chi_1(\overline{a_1}t_1+\overline{b_1}t_2)\chi_2(\overline{a_2}t_1+\overline{b_2}t_2).$$

If we use independence of η_1 and η_2 at first and then that of ξ_1 and ξ_2, we get

$$\chi_{(\eta_1,\eta_2)}(t_1,t_2)=\chi_1(\overline{a_1}t_1)\chi_2(\overline{a_2}t_1)\chi_1(\overline{b_1}t_2)\chi_2(\overline{b_2}t_2)$$

and therefore the following equality holds:

$$\chi_1(\overline{a_1}t_1+\overline{b_1}t_2)\chi_2(\overline{a_2}t_1+\overline{b_2}t_2)=\chi_1(\overline{a_1}t_1)\chi_2(\overline{a_2}t_1)\chi_1(\overline{b_1}t_2)\chi_2(\overline{b_2}t_2).$$

Denoting $\overline{a_1}t_1+\overline{b_1}t_2=x$ and $\overline{a_2}t_1+\overline{b_2}t_2=y$ and solving this simple algebraic system we come to the following equality in which Δ denotes the determinant of the system, i.e. $\Delta=\overline{a_1}\overline{b_2}-\overline{a_2}\overline{b_1}$,

$$\chi_1(x)\chi_2(y)=\chi_1\left(\frac{\overline{a_1}\overline{b_2}x-\overline{a_1}\overline{b_1}y}{\Delta}\right)\chi_2\left(\frac{\overline{a_2}\overline{b_2}x-\overline{a_2}\overline{b_1}y}{\Delta}\right)\times$$
$$\times\chi_1\left(\frac{-\overline{a_2}\overline{b_1}x+\overline{a_1}\overline{b_1}y}{\Delta}\right)\chi_2\left(\frac{-\overline{a_2}\overline{b_2}x+\overline{a_1}\overline{b_2}y}{\Delta}\right).$$

Note that $\Delta \neq 0$. Indeed, if $\overline{a_1}\overline{b_2} = \overline{a_2}\overline{b_1}$ then $a_1 b_2 = a_2 b_1$ and it follows that $a_1 b_2 \xi_1 + a_2 b_2 \xi_2 = a_2 b_1 \xi_1 + a_2 b_2 \xi_2$ i.e. $b_2 \eta_1 = a_2 \eta_2$ in contradiction with independence of η_1 and η_2. Putting in the equality above first $y = 0$ and then $x = 0$ we get, respectively, the following two equalities:

$$\chi_1(x) = \chi_1\left(\frac{\overline{a_1}\overline{b_2}}{\Delta}x\right)\chi_2\left(\frac{\overline{a_2}\overline{b_2}}{\Delta}x\right)\chi_1\left(\frac{-\overline{a_2}\overline{b_1}}{\Delta}x\right)\chi_2\left(\frac{-\overline{a_2}\overline{b_2}}{\Delta}x\right), \tag{1}$$

$$\chi_2(x) = \chi_1\left(\frac{-\overline{a_1}\overline{b_1}}{\Delta}x\right)\chi_2\left(\frac{-\overline{a_2}\overline{b_1}}{\Delta}x\right)\chi_1\left(\frac{\overline{a_1}\overline{b_1}}{\Delta}x\right)\chi_2\left(\frac{\overline{a_1}\overline{b_2}}{\Delta}x\right). \tag{2}$$

As we notices above, without loss of generality we can suppose that $a_1 b_1 = -a_2 b_2$ and hence we have $\overline{a_1}\overline{b_1} = -\overline{a_2}\overline{b_2}$. Taking into account independence of ξ_1 and ξ_2, denoting $\chi = \chi_1\chi_2$ and multiplying equalities (1) and (2), we get the equality

$$\chi(x) = \chi\left(\frac{\overline{a_1}\overline{b_2}}{\Delta}x\right)\chi\left(\frac{-\overline{a_2}\overline{b_1}}{\Delta}x\right)\chi\left(\frac{\overline{a_1}\overline{b_1}}{\Delta}x\right)\chi\left(\frac{-\overline{a_1}\overline{b_1}}{\Delta}x\right). \tag{3}$$

Observe that

$$\left|\frac{\overline{a_1}\,\overline{b_2}}{\Delta}\right|^2 + \left|\frac{-\overline{a_2}\,\overline{b_1}}{\Delta}\right|^2 + \left|\frac{\overline{a_1}\,\overline{b_1}}{\Delta}\right|^2 + \left|\frac{-\overline{a_1}\,\overline{b_1}}{\Delta}\right|^2 = 1$$

and by virtue of the complex Polya's theorem we get that $\xi_1 + \xi_2$ is a complex Gaussian random variable, if at least one of the coefficient $\frac{\overline{a_1}\,\overline{b_2}}{\Delta}$, $\frac{-\overline{a_2}\,\overline{b_1}}{\Delta}$, $\frac{\overline{a_1}\,\overline{b_1}}{\Delta}$, $\frac{-\overline{a_1}\,\overline{b_1}}{\Delta}$ is not a real number. Now using the restriction on the coefficients a_1, a_2, b_1, b_2 we will show that one of the numbers $\frac{\overline{a_1}\,\overline{b_2}}{\Delta}$ or $\frac{\overline{a_1}\,\overline{b_1}}{\Delta}$ is not a real number. Let us note that, in these expressions, due to our assumption, we should put instead of a_2 and b_2, $a_2 c$ and $b_2 c$ respectively, where $c^2 = -\frac{a_1 b_1}{a_2 b_2}$. We have,

$$\frac{\overline{a_1}\,\overline{b_2 c}}{\overline{a_1}\,\overline{b_2 c} - \overline{a_2 c}\,\overline{b_1}} = \frac{\overline{a_1}\,\overline{b_2}(a_1 b_2 - a_2 b_1)}{|a_1 b_2 - a_2 b_1|^2} = \frac{|a_1|^2|b_2|^2 - \overline{a_1}\,\overline{b_2} a_2 b_1}{|a_1 b_2 - a_2 b_1|^2}$$

and since $\overline{a_1} a_2 b_1 \overline{b_2}$ is not a real number it is clear that this ratio is a complex number with nonzero imaginary part. It is also easy to check that $\frac{\overline{a_1}\overline{b_1}}{\Delta}$ and $\sqrt{-\frac{a_1 b_1}{a_2 b_2}}\left(\frac{b_2}{b_1} - \frac{a_2}{a_1}\right)$ are proportional numbers and hence $\frac{\overline{a_1}\overline{b_1}}{\Delta}$ will be complex number with nonzero imaginary part if $\sqrt{-\frac{a_1 b_1}{a_2 b_2}}\left(\frac{b_2}{b_1} - \frac{a_2}{a_1}\right)$ is not a real number. Therefore, $\xi_1 + \xi_2$ is a complex Gaussian random variable. Now using Cramer's theorem we see that the proof is finished. □

Now passing to the quaternion case we obtain the following:

Theorem (Skitovich-Darmois, quaternion case). *Let ξ_1 and ξ_2 be independent quaternion random variables* (*not necessarily identically distributed*) *and a_1, a_2, b_1, b_2 be non-zero quaternion numbers, such that $\overline{a_1} a_2 b_1 \overline{b_2}$ and*

$\sqrt{-\frac{a_1 b_1}{a_2 b_2}}\big(\frac{b_2}{b_1} - \frac{a_2}{a_1}\big)$ *constitute jointly quaternion system. If the random variables* $\eta_1 = a_1\xi_1 + a_2\xi_2$ *and* $\eta_2 = b_1\xi_1 + b_2\xi_2$ *are also independent then* ξ_1 *and* ξ_2, *and hence* η_1 *and* η_2, *are quaternion Gaussian random variables.*

Proof. As in the complex case we analogously obtain equation (3) from which it will be followed that if the collection of numbers $\frac{\overline{a_1}\overline{b_2}}{\Delta}$, $\frac{-\overline{a_2}\,\overline{b_1}}{\Delta}$, $\frac{\overline{a_1}\,\overline{b_1}}{\Delta}$, $\frac{-\overline{a_1}\,\overline{b_1}}{\Delta}$ constitute jointly quaternion system, then using Polya's theorem for quaternion case, we will obtain that $\xi_1 + \xi_2$ and hence ξ_1, ξ_2 are quaternion Gaussian random variables. It is easy to check that if $\overline{a_1} a_2 b_1 \overline{b_2}$ and $\sqrt{-\frac{a_1 b_1}{a_2 b_2}}\big(\frac{b_2}{b_1} - \frac{a_2}{a_1}\big)$ constitute jointly quaternion system then the collection $\frac{\overline{a_1}\,\overline{b_2}}{\Delta}$, $\frac{-\overline{a_2}\,\overline{b_1}}{\Delta}$, $\frac{\overline{a_1}\,\overline{b_1}}{\Delta}$, $\frac{-\overline{a_1}\,\overline{b_1}}{\Delta}$ is also a jointly quaternion system. The proof is finished. $\square$

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