Proceedings of A. Razmadze
Mathematical Institute
Vol. **160** (2012), 143–164

GENERALIZED SPLINE ALGORITHMS AND CONDITIONS OF THEIR LINEARITY AND CENTRALITY

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Abstract. The worst case setting of linear problems, when the error is measured with the help of a metric, is studied. The notions of generalized spline and generalized central algorithms are introduced. Some conditions for a generalized spline algorithm to be linear and generalized central are given. The obtained results are applied to operator equations with positive operators in some Hilbert spaces. Examples of strong degenerated elliptic and their inverse operators satisfying the conditions appearing in the obtained theorems, are given.

რეზიუმე. შესწავლილია წრფივი პრობლემები უარესი დასმის შემთხვევისათვის, როდესაც ცდომილება გაზომილია მეტრიკის საშუალებით. შემოღებულია განზოგადებული სპლაინური და განზოგადებულად ცენტრალური ალგორითმების ცნებები. მიღებულია პირობები იმისა, რომ განზოგადებული სპლაინური ალგორითმი იყოს წრფივი და განზოგადებულად ცენტრალური. მიღებული შედეგები გამოყენებულია ზოგიერთ ჰილბერტის სივრცეში განხილული ოპერატორული განტოლებებისათვის. მოყვანილია მკაცრად გადაგვარებული ელიფსური და მისი შებრუნებული ოპერატორების მაგალითები, რომლებიც აკმაყოფილებენ მიღებული თეორემების პირობებს.

Introduction

In the present paper we use terminology and notations mainly from [1]. Let F be an absolutely convex set in a linear space F_1 over the scalar field of real or complex numbers. Let us consider a linear operator $S : F_1 \to G$ called a solution operator, where G is a local convex metric linear space over the scalar field of real or complex numbers with metric d. Elements f from F are called the problem elements for the solution operator and $S(f)$ are called the solution elements. For f we are required to compute $S(f)$. Let $U(f)$ be a computed approximation. The distance $d(S(f), U(f))$ between $S(f)$ and $U(f)$ is called an absolute error.

How can we compute approximation $U(f)$? First, we gather enough information about the problem element f. Let $y = I(f)$ be nonadaptive

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2010 *Mathematics Subject Classification.* 49N45, 65R32, 97N50.

Key words and phrases. Generalized interpolating spline, problem elements set, spline and central algorithms, Fréchet space, direct and ill-posed problems.

computed information of cardinality m, i.e.,

$$I(f) = \left[L_1(f), \dots, L_m(f)\right] \ , \tag{1}$$

where $L_1, \dots, L_m$ are linear functionals on the space F_1.

Knowing $y = I(f)$, the approximation $U(f)$ is computed by combining the information to produce an element of G, which approximates $S(f)$. That is, $U(f) = \varphi(I(f))$, where φ is a mapping $\varphi : \ I(F_1) \to G$ which is called an algorithm.

The worst case error of U is defined by

$$e(\varphi, I) = \sup\{d(S(f), U(f)), \ f \in F\}. \tag{2}$$

We are interested in algorithm with a minimal error. We say that φ^* is an optimal error algorithm if it realizes inf in (2), i.e., $e(\varphi^*, I) = \inf\{e(\varphi, I) : \varphi \in \Phi\}$, where Φ is the set of all algorithms.

In what follows, an operator S is said to be the solution operator of an operator equation $Au = f$, if $u = Sf$. If there exists an inverse to A, then $S = A^{-1}$. In addition, the central (resp. linear, spline, optimal) algorithm approximating the solution operator $S = A^{-1}$ will be called the central (resp. linear, spline, optimal) algorithm for the equation $Au = f$.

The present work is devoted to the construction of linear central algorithms for various equations in Hilbert and Fréchet spaces. These results are based on the generalization of the best approximation and on the theories of selfadjoint operators for Fréchet spaces.

In [2], the authors consider the case, where in a linear space F_1 there is a decreasing sequence of problem elements sets. In fact, they consider linear problems for a sequence of solution operators. In the present paper we consider the case in which a solution operator acts from a metrizable locally convex space in the same space and this extends essentially the case considered in [2].

In §1, the notions of a generalized interpolating spline and of a generalized spline algorithm are introduced. These notions generalize the corresponding well-known ones [1] for the case where there is not one set of problem element sets on the linear space, but a decreasing sequence of problem element sets on that space. Using this sequence, we construct generalization of the well-known Minkowski's functional which generates a metrizable locally convex topology. The generalized interpolating spline realizes a minimum not only of metric, but also the corresponding Minkowski's functional (Theorem 1). The conditions to realize these minimums with respect to the metric, constructed by D.Zarnadze, are established (Proposition 1).

In §2, the notion of a generalized central algorithm is introduced for a solution operator acting from a Fréchet space into the same space. It is proved (Theorem 2) that if the null-space of the information operators has an orthogonal complement in the Fréchet space, then the corresponding spline algorithm is generalized central.

The operator equation $Au = f$ with a selfadjoint and positive definite operator in the Hilbert space H is considered in §3. We transfer this equation into the well-known Fréchet space $D(A^\infty)$. The operator $A^\infty : D(A^\infty) \to D(A^\infty)$ [3] is introduced which coincides with the restriction of A^N from the Fréchet-Hilbert space H^N to $D(A^\infty)$. To solve the operator equation $A^\infty u = f$ in this space, we construct a linear and generalized central algorithm (Theorem 3). We present examples of some differential operators for which Theorem 3 can be applied.

In §4, we consider the equation $Ku = f$ in the Hilbert space H with a selfadjoint, positive, one-to-one compact operator, possessing dense image. We introduce the Fréchet space $D(K^{-\infty})$ and transfer this equation in the same space. We obtain the equation $K_\infty u = f$, where K_∞ is the restriction of the operator K^N from the space H^N in $D(K^{-\infty})$. For this equation we construct a linear, generalized spline and central algorithm (Theorem 4). Examples of inverses of strong elliptic operators and of some integral operators of the first kind for which Theorem 4 can be used, are given.

1. The Generalized Spline Algorithm

Let F_1 be a linear space, G be a metrizable locally convex space (lcs), $S : F_1 \to G$ be a linear solution operator and $\{V_n\}$ be a decreasing sequence of absolutely convex subsets of the space F_1, i.e., $V_1 \supset V_2 \supset \cdots \supset V_n \supset \cdots$. Consider the sets $K_r = rV_n$ of F_1, where

$$r \in I_n = \begin{cases} [1, \infty[& \text{for } n = 1, \\ [2^{-n+1}, 2^{-n+2}[& \text{for } n \geq 2. \end{cases} \tag{3}$$

Consider the functional $\mu_{\{V_n\}}$ defined on F_1 as

$$\mu_{\{V_n\}}(f) = \inf\{r > 0;\ f \in K_r\}. \tag{4}$$

If $V_1 = V_2 = \cdots = V_n = \cdots = F$, then $K_r = rF$ and $\mu_{\{V_n\}}$ coincides with Minkowski's functional μ_F of F. We reduce the following properties of the functional (4):

1. It is clear that $\mu_{\{V_n\}}(f) \geq 0,\ f \in F_1$.

2. If $f_1, f_2 \in F_1$, then $\mu_{\{V_n\}}(f_1 + f_2) \leq \mu_{\{V_n\}}(f_1) + \mu_{\{V_n\}}(f_2)$. To prove this fact, we note first that $K_r + K_s \subset K_{r+s}$. Consider the following three special cases: a) $r \in [2^{-n}, 2^{-n+1}[$ and $s \in [2^{-m}, 2^{-m+1}[$, where $1 \leq n \leq m$. Then $r + s \in [2^{-n} + 2^{-m}, 2^{-n+1} + 2^{-m+1}[\subset [2^{-n}, 2^{-n+1}[\cup [2^{-n+1}, 2^{-n+2}[$. If $r + s \in [2^{-n}, 2^{-n+1}[$, then $K_r + K_s = rV_{n+1} + sV_{m+1} \subset rV_{n+1} + sV_{n+1} = (r+s)V_{n+1} = K_{r+s}$. If $r + s \in [2^{-n+1}, 2^{-n+2}[$, then $K_r + K_s = rV_{n+1} + sV_{m+1} \subset (r+s)V_n = K_{r+s}$; b) Let $r \in [1, \infty[$ and $s \in [2^{-m}, 2^{-m+1}[,\ m \in N$, then $r + s \in [1, \infty[$ and we have $K_r + K_s =$

$rV_1 + sV_{m+1} \subset (r+s)V_1 = K_{r+s}$; c) If $r, s \in [1, \infty[$, then it is clear that $K_r + K_s \subset K_{r+s}$. Now, let $\mu_{\{V_n\}}(f_1) = r$ and $\mu_{\{V_n\}}(f_2) = s$. Then for a sufficiently small $\varepsilon > 0$ we will have $f_1 \in K_{r+\varepsilon/2}$ and $f_2 \in K_{s+\varepsilon/2}$, i.e. $\mu_{\{V_n\}}(f_1 + f_2) \le r + s + \varepsilon$ for sufficiently small $\varepsilon > 0$ and consequently $\mu_{\{V_n\}}(f_1 + f_2) \le r + s = \mu_{\{V_n\}}(f_1) + \mu_{\{V_n\}}(f_2)$.

3. If $\cap_{r\in R^+} K_r \ne \{0\}$, then $\mathrm{Ker}\mu_{\{V_n\}} = \{f \in F_1,\ \mu_{\{V_n\}}(f) = 0\} \ne \{0\}$ and $\mu_{\{V_n\}}(f-g) = d(f,g)$ is a translation invariant submetric on F_1. If $\cap_{r\in R^+} K_r = \{0\}$, then $\mathrm{Ker}\mu_{\{V_n\}}(\cdot) = \{0\}$ and $\mu_{\{V_n\}}(f-g) = d(f,g)$ is a metric on F_1. Really, it is easy to see that if $x_0 \in \cap_{r\in R^+} K_r$ and $x_0 \ne 0$, then $\mu_{\{V_n\}}(x_0) = 0$.

Denote Minkowski's functional of K_r by $q_r(\cdot)$. It is clear that if $r \in I_n$, then $q_r(\cdot) = r^{-1}\|\cdot\|_n$, where $\|\cdot\|_n$ is the Minkowski functional for V_n.

Let $I : F_1 \to R^m$ and $T : F_1 \to E$ be two linear operators, where F_1 is a linear space, $\{V_n\}$ be a decreasing sequence of absolutely convex subsets of the space F_1 and E be a metrizable lcs defined by a sequence of $\{V_n\}$ as follows.

Let $F_1 = \mathrm{Ker}\mu_{\{V_n\}} + \mathrm{Ker}\mu_{\{V_n\}}{}^{\perp}$, where the second summand is the algebraic complement linear subspace of $\mathrm{Ker}\mu_{\{V_n\}}$ in F_1. Then $f = f_1 + f_2$, for any $f \in F_1$, where $f_1 \in \mathrm{Ker}\mu_{\{V_n\}}$ and $f_2 \in \mathrm{Ker}\mu_{\{V_n\}}{}^{\perp}$. Define $E = \mathrm{Ker}\mu_{\{V_n\}}{}^{\perp}$ and $d(f_2, g_2) = \mu_{\{V_n\}}(f_2 - g_2)$ for $f_2, g_2 \in E$. Since $\mu_{\{V_n\}}(f_2) = 0$ implies that $f_2 \in \mathrm{Ker}\mu_{\{V_n\}}$, $f_2 = 0$, the functional $d(f_2, g_2) = \mu_{\{V_n\}}(f_2 - g_2)$ is a metric on E. Thus, E is a linear metrizable lcs. Define a linear operator T as $Tf = f_2$. In fact, T is an algebraic projection of the space F_1 onto the subspace $\mathrm{Ker}\mu_{\{V_n\}}{}^{\perp}$. We find that

$$d(Tf, Tg) = d(f_2, g_2) = \mu_{\{V_n\}}(f_2 - g_2) = \mu_{\{V_n\}}(f - g). \tag{5}$$

Let $y \in I(F_1)$, $T : F_1 \to E$ be the above-mentioned linear operator and I be a nonadaptive information of cardinality m. An element $\sigma = \sigma(y)$ is called a generalized spline interpolating y (briefly, a generalized spline), iff

(i) $I(\sigma) = y$,

(ii) $d(T\sigma, 0) = \inf\{d(Tz, 0);\ z \in F_1 \ \text{ and } \ I(z) = y\} = r$,

(iii) $\overline{q_r}(T\sigma) = \inf\{\overline{q_r}(Tz);\ z \in F_1 \text{ and } I(z) = y\}$ if $r > 0$, where $\overline{q_r}$ is the Minkowski's functional of the set $\{x \in E;\ d(x, 0) \le r\}$.

The above definition deals with the problem of existence of functionals minimums on the set $\{z \in F_1;\ I(z) = y\}$, whose closure is unknown because the space F_1 is, in general, only linear one. This is also the case in the classical definition [1]. Let us now prove that

$$\overline{q_r}(Tx) = q_r(x),\ x \in F_1,\ r > 0. \tag{6}$$

By the definition of the functional $\mu_{\{V_n\}}$ we find that if $\mu_{\{V_n\}}(x) \le r$, then $x \in K_{r+\varepsilon}$ for all $\varepsilon > 0$ i.e. $x \in (r+\varepsilon)V_n$, when $r \in I_n$. This implies that $\|x\|_n \le r + \varepsilon$ for arbitrary $\varepsilon > 0$, i.e., $\|x\|_n \le r$. Analogously we conclude that if $\|x\|_n \leq r$, then $\mu_{\{V_n\}}(x) \leq r$. That is $\mu_{\{V_n\}}(x) \leq r \Leftrightarrow \|x\|_n \leq r$. We have further that $\overline{q_r}(Tx) = \inf\{\alpha > 0,\ d(Tx/\alpha, 0) \leq r\} = \inf\{\alpha > 0,\ \mu_{\{V_n\}}(Tx/\alpha) \leq r\}$. Let α_0 be an arbitrary number with the property $\alpha_0 > \overline{q_r}(Tx)$. Then $d(Tx/\alpha_0, 0) \leq r$ i.e. $\mu_{\{V_n\}}(Tx/\alpha_0) \leq r\}$ and $\|Tx/\alpha_0\|_n \leq r$. This means that $\|x/\alpha_0\|_n \leq r$, i.e. $q_r(x) \leq \alpha_0$. That is $q_r(x) \leq \inf \alpha_0 = \overline{q_r}(Tx)$. On the other hand, if β is an arbitrary number such that $x/\beta \in K_r$, then $\|x/\beta\|_n \leq r$. That is $\|Tx/\beta\|_n \leq r$. This means that $\mu_{\{V_n\}}(Tx/\beta) \leq r\}$, i.e., $d(Tx/\beta, 0) \leq r$. That is, $\overline{q_r}(Tx) \leq \inf \beta = q_r(x)$. Thus, (6) is proved.

Let F_1 be a linear space and μ be a nonnegative functional for which the sets $\{x \in F_1;\ \mu(x) \leq r\},\ r \in R^+$ are absolutely convex. Denote by q_r the Minkowski functional of this set. We say that a subspace $M \subset F_1$ is strongly proximal in F_1 with respect to μ, if for arbitrary $x \in F_1$ there exists a $h^* \in M$ such that $\inf\{\mu(x-h),\ h \in M\} = \mu(x - h^*) = r$ and if $r > 0$, then $\inf\{q_r(x-h),\ h \in M\} = q_r(x-h^*)$. We call such $h^* \in M$ the strong best approximate element for $x \in F_1$ in M. The definition of a strong proximality was introduced in [4].

Theorem 1. *Let $y \in I(F_1),\ T : F_1 \to E$ be the above mentioned linear operator and I be a nonadaptive information of a cardinality $m \in N$. Then there exists a generalized spline interpolating y, iff the subspace $KerI$ is strongly proximal in F_1 with respect to the functional $\mu_{\{V_n\}}$.*

Proof. First, we assume that $\operatorname{Ker} I$ is strongly proximal in F_1 with respect to the functional $\mu_{\{V_n\}}$. Let f be an arbitrary element belonging to the set $I^{-1}(y)$. Then we have

$$\inf\big\{\mu_{\{V_n\}}(f-h):\ h \in \operatorname{Ker} I\big\} = \mu_{\{V_n\}}(f - h^*) =: r$$

and, if $r > 0$, then

$$\inf\big\{q_r(f-h):\ h \in \operatorname{Ker} I\big\} = q_r(f - h^*)$$

for some $h^* \in \operatorname{Ker} I$. Denote $\sigma = f - h^*$. By the property (5) of the metric d, we have

$$\begin{aligned}\inf\big\{\mu_{\{V_n\}}(f-h):\ h \in \operatorname{Ker} I\big\} &= \mu_{\{V_n\}}(f-h^*) = r = \mu_{\{V_n\}}(\sigma) = \\ &= d(T\sigma, 0) = \inf\big\{d(Tz, 0);\ z \in F_1,\ I(z) = y\big\}.\end{aligned}$$

From the above and (6), we have

$$\begin{aligned}\inf\big\{q_r(f-h) : h \in \operatorname{Ker} I\big\} &= q_r(f-h^*) = \overline{q_r}(T\sigma) = \\ &= \inf \overline{q_r}(Tz);\ z \in F_1,\ I(z) = y\}.\end{aligned}$$

Conversely, let f be an element in $F_1,\ I(f) = y$ and σ be a generalized spline interpolating y. We represent an element $z \in I^{-1}(y)$ in the form $z = f - h$, where $h \in \operatorname{Ker} I$, and consider the element $h^* = f - \sigma \in \operatorname{Ker} I$.

It is clear that $\sigma = f - h^*$ satisfies (i)–(iii). Therefore, using (5) and (6), we can see that h^* is a strongly best approximate element to f in $\operatorname{Ker} I$. $\square$

In the sequel we will assume that F_1 is lcs with a decreasing sequence of absolutely convex closed neighborhoods $\{V_n\}$ such that $\cap_{n\in N} V_n = 0$. In particular, such a sequence of absolutely convex neighborhoods exists, if F_1 is a metrizable lcs. In this case, T is an identity operator and the space (E, d) is the linear metric lsc in which linear operations are continuous. The existence of such a metric on the strict (LF)-space is proved in [5]. The generalization of this result for strict inductive limits of lcs, on which there exists metrics, is proved by S. Dierolf and K. Floret [6]. In that case, T is a continuous imbedding from F_1 into E.

If, moreover, $\{V_n\}$ is a local basis of neighborhoods of zero for some topology, then $\mu_{\{V_n\}}(f - g) = d(f, g)$ is the continuous metric generating the topology defined by a sequence of $\{V_n\}$. This functional is quasiconvex, i.e., the sets $\{x : \mu_{\{V_n\}}(x) \le r\}$, $r \in R^+$ are absolutely convex and coincide with K_r. Topological boundary $\{x \in F_1;\ q_r(x) = 1\}$ of K_r coincides with the metric boundary $\{x \in F_1;\ \mu_{\{V_n\}}(x) = r\}$ for $r \in intI_n$ and they, in general, differ for $r = 2^{-n+1} (n \in N)$ [7].

Below, we will often replace an arbitrary translation invariant metric d by quasinorm $|\cdot|$ (i.e., we will replace $d(x, y)$ by $|x - y|$).

Let $\|\cdot\|_n$ be Minkowski's functional of $\{V_n\}$. The definition of the functional $\mu_{\{V_n\}}$ takes the following form [8]:

$$\mu_{\{V_n\}}(x) = \begin{cases} \|x\|_1 , & \text{when } \|x\|_1 \ge 1, \\ 2^{-n+1}, & \text{when } \|x\|_n \le 2^{-n+1} \text{ and} \\ & \qquad \|x\|_{n+1} \ge 2^{-n+1} \ (n \in N) , \\ \|x\|_{n+1}, & \text{when } 2^{-n} \le \|x\|_{n+1} < 2^{-n+1} \ (n \in N), \\ 0, & \text{when } x = 0 . \end{cases} \tag{7}$$

Since $q_r(\cdot) = r^{-1}\|\cdot\|_n$ for $r \in I_n$, we find that for the metric (6) in terms of the above notation, $\sigma = f - h^*$ is a generalized spline, iff $I(\sigma) = y$,

$$d(f, \operatorname{Ker} I) = d(f, h^*) = r \ne 2^{-n+1} \quad (n \in N)$$

and

$$E(f, \operatorname{Ker} I, V_n) := \inf\big\{\|f - h\|_n\ ;\ h \in \operatorname{Ker} I\big\} = \|f - h^*\|_n\ , \\ \text{when } r = 2^{-n+1} \quad (n \in N)\ .$$

For $V_1 = V_2 = \cdots = F$, we prove that $K_r = rF$, $|\cdot| = \mu_F(\cdot)$, and the generalized interpolation spline coincides with the classical one.

Proposition 1. *Let E be a lcs with a increasing sequence of seminorms $\{\|\cdot\|_n\}$ and with metric (7), let M be a convex subset of E, $x \in E\backslash M$ and $d(x, M) = r \in I_n$ (3). Then the following statements hold:*

(a) *If* $r \in \mathrm{int} I_n$ $(n \in N)$, *then the equalities* $d(x, M) = r = |x - h^*|$ *and* $\inf_{h \in M} \|x - h\|_n = \|x - h^*\|_n$, *where* h^* *is some element of* M, *are equivalent.*

(b) If $d(x, M) = r = 2^{-n+1}$ $(n \in N)$ and $\inf_{h \in M} \|x - h\|_n = \|x - h^*\|_n$ for some $h^* \in M$, then $d(x, M) = d(x, h^*)$.

Proof. a) Let $d(x, M) = |x - h^*| = r \in \mathrm{int} I_n$ $(n \in N)$ for some $h^* \in M$. From the definition of the metric (7) it follows that $d(x, M) = |x - h^*| = \|x - h^*\|_n = r$. Let us prove that $\|x - h^*\|_n = \inf_{h \in M} \|x - h\|_n$. Assume the opposite that $s = \|x - h_1\|_n < \|x - h^*\|_n = r$ for some $h_1 \in M$. Then, according to the properties of the metric (7), we have $d(x, h_1) = \|x - h_1\|_n < r$ if $s \in I_n$ and $d(x, h_1) \leq 2^{-n+1} < r$ if $s < 2^{-n+1}$. Thus, $d(x, h_1) < r$, which is impossible. Analogously, we can show that the equality $\inf_{h \in M} \|x - h\|_n = \|x - h^*\|_n = r \in \mathrm{int} I_n$, $h^* \in M$, implies that $d(x, M) = |x - h^*| = r$. Thus, part (a) is proved. To prove (b), let $d(x, M) = r = 2^{-n+1}$ and $\inf_{h \in M} \{\|x - h\|_n,\ h \in M\} = \|x - h^*\|_n \leq 2^{-n+1}$, where $h^* \in M$. Let us show that $\|x - h^*\|_{n+1} = s \geq 2^{-n+1}$. If we assume that $s < 2^{-n+1}$, then according to (7), $d(x, h^*) \leq \max(s, 2^{-n}) < 2^{-n+1} = r$. But this is impossible. Thus, $\|x - h^*\|_n \leq 2^{-n+1}$ and $\|x - h^*\|_{n+1} \geq 2^{-n+1}$. By virtue of(7), this means that $d(x, h^*) = 2^{-n+1} = r$. $\square$

In the above notation, the element $\sigma = f - h^* \in F_1$ is a generalized interpolating $y \in \mathbb{R}^m$ spline, if $I(\sigma) = y$,

$$d(f, \mathrm{Ker}\, I) = d(f, h^*) = r = d(\sigma, 0) = |\sigma|$$

and, if $r > 0$,

$$\inf \big\{q_r(f - r);\ h \in \mathrm{Ker}\, I\big\} = q_r(f - h^*) = q_r(\sigma),$$

i.e., the generalized spline σ minimizes not only the metric, but also the corresponding Minkowski's functional. Here h^* is a strongly best approximation element of f in $\mathrm{Ker}\, I$. Such definition of the generalized interpolating splines is given in [2]. The number r mentioned in the definition of a generalized spline interpolating y, does not depend on the choice of $f \in F_1,\ If = y, d(f, \mathrm{Ker}\, I = r)$. Indeed, since if

$$I(f_1) = I(f_2) = y,\ f_2 - f_1 = z \in \mathrm{Ker}\, I,\ d(f_1, \mathrm{Ker}\, I) = d(f_1, h_1^*),$$

and

$$\inf\{q_r(f_1 - h);\ h \in \mathrm{Ker}\, I\} = q_r(f_1 - h^*),$$

therefore

$$d(f_2, \mathrm{Ker}\, I) = d(f_2,\ h_1^* + z) = d(f_1, h_1^*)$$

and

$$\inf \big\{q_r(f_2 - h);\ h \in \mathrm{Ker}\, I\big\} = q_r(f_2 - h_1^* + z) = q_r(f_1 - h_1^*).$$

Consider the definition of a generalized spline in the case of a normlike metric given by G.Albinus [9]. Assume that information is generated by continuous on F_1 linear functionals. Then $\operatorname{Ker} I$ is closed in F_1 and the distance $d(f, \operatorname{Ker} I) = r > 0$. From the properties of this metric it follows that $\inf\{q_r(f-h);\ h \in \operatorname{Ker} I\} = 1$ [9]. As is known, q_r is equivalent to some seminorm from the given sequence $\{\|\cdot\|_n\}$. The functional corresponding to this information is continuous with respect to this seminorm. Its kernel is closed, since the distance from f to this kernel with respect to the seminorm is positive, namely 1. If it were not closed, it would be everywhere dense and the distance will be zero. Thus, in the definition of a generalized spline we arrive always at such a seminorm with respect to which the functional, corresponding to this information is continuous. Thus, in the cases under consideration, the definition of a generalized spline needs the requirement of the existence of the best approximation only with respect only to the metric in $\operatorname{Ker} I$.

Consider the case in which $F_1 = E$ is a metrizable locally convex space whose topology is generated by a decreasing sequence of neighborhoods V_n of zero. Denote the Minkowski's functional of V_n by $\|\cdot\|_n$, i.e. $V_n = \{f \in E :\ \|f\|_n \le 1\}$. Let X_n be the normed space $X_n = (E/\operatorname{Ker}\|\cdot\|_n\ ,\ \widehat{\|\cdot\|_n})$, where $\widehat{\|\cdot\|_n}$ is the associated norm. If instead of F we consider the set V_n for each $n \in N$, then the canonical maps $K_n : F_1 \to X_n$ will be analogies of the operator $T : E \to X$ and $V_n = \{f \in E :\ \|\widehat{K_n(f)}\|_n \le 1\}$.

It should be noted that the existence of a generalized spline for any nonadaptive information of cardinality 1 in terms of the proximality of closed hypersubspaces was considered by many mathematicians and the final results were obtained in [10] (see also [2]).

Consider the set $F = \{f \in F_1 :\ \mu_{\{V_n\}}(f) = d(Tf, 0) \le 1\}$. If the generalized spline exists and is unique, then the generalized spline algorithm is defined analogously to [1] by means of the equality $\varphi^s(y) = S\sigma(y),\ y \in F$.

2. Generalized central algorithm and the condition of linearity and generalized centrality of generalized spline algorithms

Let us now define the notion of a generalized central algorithm for a solution operator $S : F_1 \to G$, where F_1 is a linear space with a decreasing sequence of absolutely convex subsets $\{V_n\}$ of the space F_1 and G is a lcs with metric d^*. Let $T : F_1 \to E$ be the operator introduced in §1 and consider the set $F = \{f \in F_1 :\ \mu_{\{V_n\}}(f) = d(Tf, 0) \le 1\}$. Let I be a nonadaptive information of cardinality $m \ge 1$ and $y = I(f),\ f \in F$. Then $\mu_{\{V_n\}}(f) = r \in I_n$, i.e. $f \in V_n$ for some $n \in N$. We call the value

$$e_n(\varphi, I, y) = \sup\left\{d(S(f), \varphi(y));\ f \in I^{-1}(y) \cap V_n\right\}$$

the local error of the algorithm φ at a point y. Denote by $r_n(I,y)$ the local radius of the nonadaptive information I at a point y defined by the equality

$$r_n(I,y) = \text{rad}\,\big(S(I^{-1}(y)\cap V_n)\big).$$

Here, the radius of the set $M \subset G$ is defined analogously to the case of a normed space by the equality $\text{rad}(M) = \inf\{\sup\{d(a,g);\ a \in M\};\ g \in G\}$. The Chebyshev center $c \in G$ of a set $M \subset G$ is defined by the equality $\text{rad}(M) = \sup\{d(a,c),\ a \in M\}$. It is a simple matter to verify that $r_n(I,y) = \inf\{e_n(\varphi,I,y) : \varphi \in \Phi\}$, where Φ is the set of all algorithms. The global radius $r_n(I)$ of nonadaptive information I is defined by the equality

$$r_n(I) = \sup\{r_n(I,y);\ y \in I(V_n)\}.$$

Let $y = I(F) \subset R^m$, i.e., $y \in I(V_n)$ for some $n \in N$. Assume that the sets $S(I^{-1}(y)\cap V_k)$ have a Chebychev center $c = c(y)$ for all $y \in I(F)$ and $k \le n$ if $y \in I(V_n)$. This inplies that for all $k \le n$,

$$\text{rad}\Big(S(I^{-1}(y)\cap V_k) = \inf\big\{\sup\{|S(f)-g|;\ f \in I^{-1}(y)\cap V_k\};\ g \in G\big\}\Big) =$$
$$= \sup\big\{|S(f)-c(y)|;\ f \in I^{-1}(y)\cap V_k\big\}.$$

In such cases we call the algorithm $\varphi^c(y) = c(y)$ a generalized central. If G is a normed space and $V_1 = V_2 = \cdots = F$, then $|\cdot| = q_F(\cdot)$, and the notion of generalized centrality coincides with the classical definition.

Consider a metrizable locally convex space whose topology is defined by an increasing sequence $\|\cdot\|_n$ of seminorms. Below, by d^* will be denoted one of the following metrics: 1) the metric, defined by (7); 2) the normlike metric given by Albinus [9]; 3) the supremum metric defined by the formula $d(x,y) = \sup\limits_{n\in N} \frac{\|x-y\|_n}{2^n(1+\|x-y\|_n)}$; 4) the metric $d(x,y) = \sum\limits_{n=1}^{\infty} \frac{\|x-y\|_n}{2^n(1+\|x-y\|_n)}$ given by Mazur.

Proposition 2. *Let G be a metric space with the metric d^* and let the closure $\bar{A}$ of the set $A \subset G$ be symmetric with respect to some element $p \in G$. Then p is the Chebyshev center of A.*

Proof. Let us consider the case, where d^* is the metric given by (7) and assume that p is not a Chebychev center of the set A. Then there exists an element u from G, such that $\sup\{|a-u| : a \in A\{< \sup\{|a-p| : a \in A\}$, where $|\cdot|$ is the quasinorm of the metric (7). Take $x \in A$ such that

$$|a-u| < |x-p| \quad \text{for all} \quad a \in \bar{A}\ . \tag{8}$$

Let $|x-p| = r \in I_n$ for some $n \in N$. If $r \in \text{int}I_n$, then $|x-p| = \|x-p\|_n$. Denote $|a_0-u| = r_1$ for some $a_0 \in A$. Let $r_1 \in I_{n_1}$, $n_1 \ge n$. If $r_1 \in \text{int}I_{n_1}$, then $r_1 = \|a_0-u\|_{n_1}$. According to (8), in this case we have $\|a_0-u\|_{n_1} < \|x-p\|_n$ and $\|a_0-u\|_n \le \|a_0-u\|_{n_1} < \|x-p\|_n$. Thus,

$$\|a_0-u\|_n < \|x-p\|_n\ . \tag{9}$$

Let now $|a_0 - u| = r_1 = 2^{-n_1+1} \in I_{n_1}$, $n_1 \geq n$. From the properties of the metric (7) it follows that $\|a_0 - u\|_{n_1} \leq 2^{-n_1+1}$. Therefore, $\|a_0 - u\|_n \leq \|a_0 - u\|_{n_1} \leq r_1 = 2^{-n_1+1} < r = \|x - p\|_n$ and (9) is valid.

Consider now the case in which $|x - p| = r = 2^{-n+1} \in I_n$ and $r_1 \in \operatorname{int} I_{n_1}$, $n_1 > n$. Then $|a_0 - u| = r_1 = \|a_0 - u\|_{n_1} \geq \|a_0 - u\|_{n+1}$ and therefore $\|a_0 - u\|_{n+1} < 2^{-n+1}$. From the properties of the metric (7) it follows that $\|x - p\|_n \leq 2^{-n+1} \leq \|x - p\|_{n+1}$. Thus, we obtain $\|a_0 - u\|_{n+1} < 2^{-n+1} \leq \|x - p\|_{n+1}$, and hence (9) is valid for $n + 1$.

It remains to consider the case, where $r = 2^{-n+1} \in I_n$ and $r_1 = 2^{-n_1}$, $n_1 \geq n$. We have $\|a_0 - u\|_{n+1} \leq \|a_0 - u\|_{n_1+1} \leq r_1 = 2^{-n_1} < r = 2^{-n+1} \leq \|x - p\|_{n+1}$, and hence (9) is valid for $n + 1$.

Let now d^* be a normlike, supremal or Mazur's metric. Then (7) is likewise valid for some $n \in N$. Indeed, if we assume that (9) is not true , then $\|a - u\|_n \geq \|x - p\|_n$ for all n and $a \in A$. From the properties of the above-considered metrics it follows that $|a - u| \geq |x - p|$ for all $a \in A$, but this contradicts (8).

Let $x = p + h$. While $x \in A$ and $\bar{A}$ is symmetric with respect to p, in $\bar{A}$ there is also $p - h$. If $k = n$ or $k = n + 1$, we have

$$2\|h\|_k = \|2h\|_k = \big\|(p + h - u) - (p - h - u)\big\|_k \leq \big\|(p + h) - u\big\|_k + \\ + \big\|(p - h) - u\big\|_k < 2\big\|x - p\big\|_k = 2\|h\|_k \ ,$$

which is a contradiction. □

Assume that the topology of the Fréchet space E is given by a sequence of Hilbertian seminorms $\{\|\cdot\|_n\}$, i.e., each seminorm $\|\cdot\|_n$ is generated by the semiinner product $(x, y)_n$ and $V_n = \{x \in E;\ \|x\|_n \leq 1\}$. For such spaces, the notion of orthogonality is defined naturally as follows: the elements $x, y \in E$ are called orthogonal, if $(x, y)_n = 0$ for each $n \in N$. A subspace M possesses an orthogonal complement $M^\perp$ in E, if each element $x \in E$ is represented as the sum $x = y + z$, where $y \in M$, $z \in M^\perp$ and $(y, z)_n = 0$ for each $n \in N$. In other words, this implies that each element $x \in E$ in the subspaces M and $M^\perp$ possesses a unique best approximation y and z, respectively, with respect to all seminorms $\|\cdot\|_n$ generated by $(\cdot, \cdot)_n$.

Theorem 2. *Let E be a Fréchet space with an increasing sequence of Hilbertian seminorms $\{\|\cdot\|_n\}$, $V_n = \{x \in E : \|x\|_n \leq 1\}$ and with the metric* (7)*. Let $K_n : E \to E/\operatorname{Ker}\|\cdot\|_n$ be the canonical mapping, $X_n = (E/\operatorname{Ker}\|\cdot\|_n\ ,\ \widehat{\|\cdot\|_n})$ and G be a metrizable locally convex space, $S : E \to G$ be a linear solution operator and I be a nonadaptive information of cardinality $m \geq 1$. Then the following assertions are valid:*

a) *If $K_n(\operatorname{Ker} I)$ is closed in the Hilbert space X_n, $n \in N$, then $\operatorname{Ker} I$ is strongly proximal in E with respect to the metric* (7)*, and for any $y \in I(E)$ there exists a generalized spline σ interpolating y.*

b) *If, moreover, the subspace* $\operatorname{Ker} I$ *possesses an orthogonal complement in* E*, then for any* $y \in I(E)$ *there exists the unique generalized spline* σ *interpolating* y *such that* $(\sigma, h)_n = 0$ *for any* $n \in N$ *and* $h \in \operatorname{Ker} I$*. If* $y \in I(V_1)$*, then* σ *is a center of all sets* $I^{-1}(y) \cap V_k$*, for which this intersections are non-empty. The corresponding spline algorithm* $\varphi^s(y) = S(\sigma)$ *is linear and generalized central.*

Proof. a) For $y \in I(E)$, there exists $f \in E$ such that $I(f) = y$. The subspace $\operatorname{Ker} I$ is strongly proximal in E and for this f there exists a strongly best approximation element h^* in $\operatorname{Ker} I$ ([2], Theorem 4). Then $\sigma = f - h^*$ is a generalized spline interpolating y.

b) If $y = 0$, then likewise $\sigma = 0$ and item b) is trivial. For any nontrivial $y \in I(E)$ and information I we take f such that $I(f) = y$. Since the subspace $\operatorname{Ker} I$ possesses an orthogonal complement in E, there exists the unique representation $f = h^* + \sigma$ and $(h^*, \sigma)_n = 0$ for any $n \in N$, where $h^* \in \operatorname{Ker} I$ and $\sigma \in \operatorname{Ker} I^{\perp}$. This implies that $< K_n h^*, K_n \sigma >_n = 0$ for any $n \in N$, where $< \cdot, \cdot >_n$ is the inner product in the space X_n, generating the associated norm $\widehat{\|\cdot\|}_n$. $K_n(\sigma)$ is orthogonal to $K_n(\operatorname{Ker} I)$ in X_n for any $n \in N$ and σ is a best approximation element for f in $\operatorname{Ker} I^{\perp}$ with respect to the $\|\cdot\|_n$ for any $n \in N$. It is clear that $I(\sigma) = y$. Let us prove that σ is a generalized spline interpolating y. Let $d(f, \operatorname{Ker} I) = r \in I_n$ for some $n \in N$, then

$$\inf \left\{ \left\| K_n f - K_n h \right\|_n ,\ h \in \operatorname{Ker} I \right\} = \left\| K_n f - K_n h^* \right\|_n = \|\sigma\|_n := \lambda.$$

If $r \in \operatorname{int} I_n$, then according to Proposition 1, $\lambda = r = d(f, h^*)$ and σ is a generalized spline interpolating y. If $r = 2^{-n+1}$ ($n \in N$), then again by Proposition 1, $d(f, h^*) = r$ and σ is a generalized spline interpolating y. If $r = 2^{-n+1}$ and $\lambda = 0$, then $f - h_0 \in V_n$ for some $h_0 \in \operatorname{Ker} I$. Indeed, in this case there exists a minimized sequence $\{h_k\}$ such that $\lim_{k\to\infty} \|K_n f - K_n h_k\|_n = \lambda = 0$. Since $K_n(\operatorname{Ker} I)$ is closed in X_n, therefore $K_n f \in K_n(\operatorname{Ker} I)$ i.e., $f \in \operatorname{Ker} I$. But this is out of the question and hence $\lambda = 0$ is impossible.

We obtain $f - h_0 \in V_n$. Assuming now that $f - h_0 \in 2\operatorname{int} V_{n+1}$, we will have $d(f, h_0) \le 2 \cdot 2^{-n} \|f - h_0\|_{n+1} < 2^{-n+1} = r$, but this is impossible. Therefore, $\|f - h_0\|_n \le 2^{-n+1}$ and $\|f - h_0\|_{n+1} \ge 2^{-n+1}$. This implies that $|f - h_0| = r = 2^{-n+1}$.

From the above-said it follows that if for some element σ belonging to E the equalities $I(\sigma) = y$ and$(\sigma, h)_n = 0$ are valid for any $n \in N$ and $h \in \operatorname{Ker} I$, then σ is a generalized spline interpolating y.

Build now a linear spline algorithm. Towards this end, we apply the method considered in [1]. Let σ_i be the unique generalized spline interpolating $e_i = \{0, \ldots, 1, \ldots, 0\}$ for the information $I(f) = [L_1(f), \ldots, L_m(f)]$

with linearly independent linear functionals $L_i(f)$, such that $K_n(\sigma_i)$ is orthogonal to the $K_n(\operatorname{Ker} I)$ in X_n for any $n \in N$, i.e., $(h, \sigma_i)_n = 0$ for all $n \in N$. Consider the expression $\sigma = \sum\limits_{i=1}^{m} L_i(f)\sigma_i$. Then $K_n(\sigma)$ will be orthogonal to the $K_n(\operatorname{Ker} I)$ in X_n for all $n \in N$ and σ will be a generalized spline interpolating y. It is clear that $\varphi^s(I(f)) = \sum\limits_{i=1}^{m} L_i(f) S\sigma_i$ will be a linear algorithm. It should also be noted that the operator $y \to \sigma$, acting from the finite dimensional space $I(E)$ to the finite dimensional space $(\operatorname{Ker} I)^{\perp}$, is linear. It remains to prove that φ^s is generalized central, i.e., that the center of the set $S(I^{-1}(y) \cap V_{n_0})$ for each $y \in I(V_{n_0})$ is $S(\sigma)$, where σ is the above-mentioned unique generalized spline interpolating y. The existence of such spline σ has been proved above. We have now to prove that if g is an arbitrary element of $I^{-1}(y) \cap V_{n_0}$, then $2\sigma - g \in I^{-1}(y) \cap V_{n_0}$. This fact may be proved just in the same way as in ([1], p. 97). Really, for $h = \sigma - g \in \operatorname{Ker} I$ we have

$$\big\|\widehat{K_{n_0}(2\sigma - g)}\big\|_{n_0} = \big\|\widehat{K_{n_0}(\sigma + g)}\big\|_{n_0} =$$
$$= \sqrt{\big\|\widehat{K_{n_0}(\sigma)}\big\|_{n_0}^2 + \big\|\widehat{K_{n_0}(h)}\big\|_{n_0}^2} = \big\|\widehat{K_{n_0}(h)}\big\|_{n_0} = \big\|g\big\|_{n_0} \le 1 \ ,$$

i.e., $2\sigma - g \in I^{-1}(y) \cap V_{n_0}$. Therefore, the set $S(I^{-1}(y) \cap V_{n_0})$ is symmetric with respect to $\varphi^s(y) = S(\sigma)$, i.e., $\operatorname{rad}(S(I^{-1}(y) \cap V_k)) = \inf\{\sup\{|S(f) - y|;\ f \in I^{-1}(y) \cap V_k\};\ y \in G\} = \sup\{|S(f) - S(\sigma)|;\ f \in I^{-1}(y) \cap V_k\}$, for all $k \le n$. From the above and Proposition 1 it follows that the generalized spline σ interpolating y is a center for the set $I^{-1}(y) \cap V_{n_0}$, i.e., $\mathrm{rad}(\mathrm{I}^{-1}(\mathrm{y}) \cap \mathrm{V_k}) = \inf\{\sup\{|\mathrm{f} - \mathrm{q}|;\ \mathrm{f} \in \mathrm{I}^{-1}(\mathrm{y}) \cap \mathrm{V_k}\};\ \mathrm{q} \in \mathrm{E}\}$ for all $k \le n_0$. □

3. Construction of generalized central linear spline algorithms for direct problems

Let H be a Hilbert space with a inner product $(\cdot, \cdot)$, A be a selfadjoint positive definite operator from H into H. The topology of the well-known countable-Hilbert space $D(A^{\infty}) = \cap_{k=1}^{\infty} D(A^{k-1})$ can be given by the following sequence of Hilbertian norms:

$$\|x\|_n = \big(\|x\|^2 + \|Ax\|^2 + \cdots + \|A^{n-1}x\|^2\big)^{1/2},\ x \in D(A^{\infty}),\ n \in N,$$

which are generated by the inner products

$$(x, y)_n = (x, y) + (Ax, Ay) + \cdots + \big(A^{n-1}x, A^{n-1}y\big),\ x, y \in D(A^{\infty}).$$

The space $D(A^{\infty})$ is isomorphic to a subspace M of the Fréchet-Hilbert space H^N considered with the product topology. This isomorphism is obtained by the mapping

$$D(A^{\infty}) \ni x \to Orb(A, x) : = \{x, Ax, \ldots, A^{n-1}x, \ldots\} \in M \subset H^N.$$

Using this representation, we can define the operator $A^\infty: D(A^\infty) \to D(A^\infty)$ by the equality

$$A^\infty: A^\infty\{x, Ax, A^2x, \dots\} = \{Ax, A^2x, \dots\}.$$

A^∞ coincides with the restriction of A^N from the Fréchet-Hilbert space H^N to $D(A^\infty)$. (Due to this notation, the space $D(A^\infty)$ acquires a new meaning, different from the classical case, where $D(A^\infty)$ is a whole symbol, and A^∞ taken separately, means nothing). To obtain an approximate solution of the equation

$$A^\infty u = f \tag{10}$$

in the Fréchet space $D(A^\infty)$, we apply the extended Ritz's method. For that we consider $D(A^\infty)$ with the following energetic norms:

$$[x]_n^2 = (A^\infty x, x)_n = (Ax, x) + (A^2x, Ax) + \dots + (A^nx, A^{n-1}x). \tag{11}$$

As basis functions, we choose an orthogonal sequence of eigenfunctions $\{\varphi_j\}$ of the operators A and A^∞ (We suppose that φ_j are embedded in $D(A^\infty) \in H$ and identified with the $(\varphi_j, A\varphi_j, \dots) \in D(A^\infty)$). For $l \in N$, a system of equations for defining the coefficients of an approximate solution is written in the form

$$\sum_{k=1}^{m} (A^\infty \varphi_k, \varphi_j)_l \ a_k = (f, \varphi_j)_l \ , \ j = 1, \dots, m \ .$$

Performing calculations, we prove that an approximate solution for the equation (10), which is obtained by the Ritz's extended method, takes the form

$$u_m = \sum_{j=1}^{m} \frac{(f, \varphi_j)}{\lambda_j \|\varphi_j\|^2} \ \varphi_j \ , \tag{12}$$

where λ_j are the corresponding to φ_j eigenvalues. u_m do not depend on l [11]. Let $I(f) = [L_1(f), L_2(f), \dots, L_m(f)]$ be nonadaptive information of cardinality m, where $L_i(f) = (f, \varphi_i)$. $\operatorname{Ker} I$ is a finite-codimensional subspace in the energetic space E_{A^∞}. Therefore, (12) implies that $\operatorname{Ker} I$ admits an orthogonal complement $\operatorname{Ker} I^\perp = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ in E_{A^∞}. If $e_i = (0, \dots, 1, \dots, 0)$, where 1 lies on the i-th place, then $\varphi_i \in I^{-1}(e_i)$ and the best approximation element for f in $\operatorname{Ker} I^\perp$ coincides with φ_i. This means that interpolating e_i, the generalized spline is φ_i, and interpolating $y = I(f)$, the spline σ has the form $\sigma = \sum_{i=1}^m (f, \varphi_i)\|\varphi_i\|^{-2}\varphi_i$. The solution operator for the equation (10) is $S = (A^\infty)^{-1}$ and it is an isomorphism of the space E_{A^∞} onto itself. We have that $S\sigma = \sum_{i=1}^m (f, \varphi_i)\|\varphi_i\|^{-2} S\varphi_i = \sum_{i=1}^m (f, \varphi_i)\|\varphi_i\|^{-2}(A^\infty)^{-1}\varphi_i = \sum_{i=1}^m \lambda_i^{-1}(f, \varphi_i)\|\varphi_i\|^{-2}\varphi_i = u_m$, where u_m is approximate solution of equation (10), constructed by the Ritz's extending method. $S\sigma = u_m$ is also the best approximation element for $Sf = (A^\infty)^{-1}f$ in the subspace $\operatorname{Ker} I^\perp$ with respect to energetic norms (11) of the energetic space E_{A^∞} for the operator A^∞. The subspace $\operatorname{Ker} I$

admits an orthogonal complement $\operatorname{Ker} I^{\perp}$ in a Frechet space E_{A^∞}. According to Theorem 2, the generalized spline algorithm φ^s, defined by the equality $\varphi^s(I(f)) = u_m$, is generalized central.

From the above-mentioned reasoning follows the following

Theorem 3. *Let A be a selfadjoint positive definite operator in the Hilbert space H with an orthogonal sequence of eigenfunctions $\{\varphi_j\}$. Let λ_j be eigenvalue which corresponds to the eigenfunction φ_j and u_m is defined by* (12). *The algorithm $\varphi^s(I(f)) = u_m$ is the linear generalized spline and generalized central for the solution operator $S = (A^\infty)^{-1}$ and information $I(f) = [(f, \varphi_1), (f, \varphi_2), \dots, (f, \varphi_m)]$. Moreover, the sequence of approximate solutions $\{u_m\}$ converges to a solution of equation* (10) *in the space $D(A^\infty)$.*

We will now give a few examples of selfadjoint and positive definite differential operators in Hilbert spaces; these operators satisfy the conditions of Theorem 3. These examples are taken mainly from [12].

1. Strongly degenerate elliptic differential operators. For an arbitrary domain $\Omega \subset R^l$ we, as usual, denote by $C^\infty(\Omega)$ the space of all infinitely differentiable functions defined in Ω. Further, let $\rho(x) \in C^\infty(\Omega)$ be a positive function such that

a) For any multi-indices γ, there exists $C_\gamma > 0$ such that $|D^\gamma \rho(x)| \leq C_\gamma \rho^{1+|\gamma|}(x)$, for all $x \in \Omega$.

b) For any $k > 0$, there exist numbers $\varepsilon_k > 0$ and $r_k > 0$ such that $\rho(x) > k$ if $d(x) \leq \varepsilon_k$ or $|x| \geq r_k$ when $x \in \Omega$ ($d(x)$ is a distance from x to the boundary $\partial(\Omega)$).

Denote by $S_{\rho(x)}(\Omega)$ the metrizable, locally convex space

$$\begin{aligned} S_{\rho(x)}(\Omega) = \big\{ f \in C^\infty(\Omega);\ \|f\|_{n,\alpha} = \sup \rho^n(x) |D^\alpha f(x)| < \infty, \\ \text{for all } n = 0, 1, \dots \text{and all multi-indices } \alpha \big\}. \end{aligned} \tag{13}$$

Note that for each bounded domain Ω there exists a function $\rho(x)$ for which ρ^{-1} actually coincides with $d(x)$. $S_{\rho(x)}(\Omega)$ is a nuclear Fréchet space isomorphic to the space s of fast decreasing sequences. The well-known Schwartz space $S(R)$ is a particular case of such spaces.

The class $\mathfrak{R}^r_{\mu,\nu}(\Omega, \rho(x))$ considered in [12] is a quite wide class of degenerating elliptical differential operators. We will give an example of an operator from that class. The operator A given by the relations

$$Au = -\Delta u + \rho^\nu(x) u,\ \nu > 2,\ D(A) = C_0^\infty(\Omega) \tag{14}$$

is essentially selfadjoint in $L^2(\Omega)$, i.e., its closure $\overline{A}$ is a selfadjoint operator in $L^2(\Omega)$, $D(\overline{A}) = W_2^2(\Omega, 1, \rho^{2\nu})$ ([12], 6.4.3) and A has a purely pointwise spectrum. Moreover, A is positive definite. A sequence of eigenfunctions $\{\varphi_j\}$ of the operator A belongs to the space $S_{\rho(x)}(\Omega)$ ([12], 6.4.2). It is also proved that $D(\overline{A}^j) = W_2^j(\Omega, 1, \rho^{2\nu j})$ ([12], 6.4.3) and the space $S_{\rho(x)}(\Omega)$ is isomorphic to the space $D(\overline{A}^{\infty})$ whose topology is given by the sequence of Hilberian norms (11), where A is defined by (14). In connection with the the above-said, we note that the topology of the space $S_{\rho(x)}(\Omega)$ is given by the sequence $\{\|\cdot\|_{n,\alpha}\}$. Therefore, if we consider the equation

$$-\Delta u + \rho^{\nu}(x)u = f \tag{15}$$

in the Fréchet space $S_{\rho(x)}(\Omega)$ with the sequence of norms (9), then by virtue of proposition b) of Theorem 2 (see also ([12], 6.4.3)), it has a unique solution for each $f \in S_{\rho(x)}(\Omega)$. If the sequence of eigenfunctions $\{\varphi_j\}$ is orthogonal in the space $L^2(\Omega)$, then for the sequence of approximate solutions $\{u_m\}$, which is given by (12), Theorem 3 in the space $S_{\rho(x)}(\Omega)$ is valid.

Let us now give a concrete definition of the result in the one-dimensional case for the Hermitean operator, i.e., for the harmonic oscillator

$$Au = -u''(t) + t^2 u \tag{16}$$

with the boundary conditions $u(\pm\infty) = 0$. This is a selfadjoint and positive definite operator in the Hilbert space $L^2(R)$. According to ([12], 6.2.3.), the Schwartz space $S(R)$ serves as the space $D(A^{\infty})$ for the operator A. The eigenfunctions of the operator A are the Hermitean functions (the wave functions of the harmonic oscillator) [13]:

$$\varphi_j(t) = \big(2^{j-1}(j-1)!\big)^{-1/2}(-1)^{j-1}\pi^{-1/4}e^{t^2/2}\frac{d^{j-1}e^{-t^2}}{dt^{j-1}}, \quad j \in N. \tag{17}$$

The eigenvalues of A are $\lambda_j = 2j+1, j = 1, 2, \dots$. The sequence φ_j is an orthonormal basis of the space $L^2(R)$ and, by virtue of the nuclearity of the space $S(R)$, it is also an absolute basis in the latter space. Let us consider the space $S(R)$ with the sequence of Hilbertian norms (11), where A is given by (16) and $\|\cdot\|$ is the norm of the space $L^2(R)$.

Let the operator A^{∞} be the restriction of A on the space $S(R) \subset D(A)$ when the topology of the space $S(R) = D(A^{\infty})$ is taken into account. By virtue of (17), an approximative solution u_m of equation (10) has the form (12), where $\{\varphi_j\}$ are defined by (17) and $\lambda_j = 2j+1$. For such sequence of approximate solutions $\{u_m\}$ given by (12), Theorem 3 in the space $S(R)$ with the norms (7) is valid.

The obtained results can be applied to essentially selfadjoint and positive definite Legendre operators $A_{m,k}(2k \le m)$ ([12], 7.4.1) and Tricomi operators $B_{n,k}$ ([12], 7.6.3). These works also give the representations of the spaces $D(\overline{A}^{\infty}_{m,k})$ ([12], 7.4.4) and $D(\overline{B}^{\infty}_{m,k})$ ([12], 7.6.3).

2. Let us consider the differential operator ([14], Ch.5, § 9)

$$Bu = -\frac{1}{t}\left[\frac{d}{dt}\Big(t\frac{du}{dt}\Big) - \frac{\nu^2}{t}u\right], \quad \nu = \text{const} > 1/2, \quad 0 < t < 1.$$

in the space $H = L_2(t; 0, 1)$ of the quadratically summable on $]0, 1[$ with the weight functions t . The domain of definition $D(B)$ consists of the functions u for which: $u(t)$ and $u'(t)$ are absolutely integrable on any interval $[\varepsilon, 1]$ $(0 < \varepsilon < 1)$; $\sqrt{t}u'(t)$ is continuous on $[0, 1]$ and vanish at $t = 0$; $Bu \in H$ and $u(1) = 0$. In ([14], Ch.5, § 9) it is proved that $D(B)$ is dense in H, B is symmetric and positive definite in H and has a discrete spectrum. Eigen-values of the operator B are

$$\lambda_k = j^2_{\nu,k}, \quad k = 1, 2, \dots, \tag{18}$$

where $j_{\nu,k}$ is the k-th positive root of Bessel function $J_\nu(t)$; the corresponding orthonormal eigenfunctions are

$$\varphi_k(t) = \frac{\sqrt{2}}{J_{\nu+1}(j_{\nu,k})} J_\nu(j_{\nu,k}t), \quad k = 1, 2, \dots. \tag{19}$$

The approximate solutions u_m of the equation $B^\infty u = f$ in the Fréchet space $D(B^\infty)$ have the following form:

$$u_m(t) = \sum_{k=1}^{m} \lambda_k^{-1} \int_0^1 sf(s)\varphi_k(s)ds \ \varphi_k(t),$$

where λ_k and φ_k are defined by (18) and (19). The sequence u_m converges in the space $D(B^\infty)$ to the solution of the equation $B^\infty u = f$ if $f \in D(B^\infty)$. For such sequence of approximative solutions $\{u_m\}$, Theorem 3 is valid in the space $D(B^\infty)$ with the norms (11), in which A is replaced by B.

3. The Laplace-Beltrami operator $\boldsymbol{\delta}$. Let S be the unit sphere in the l-dimensional Euclidean space R^l, $\vartheta_1, \vartheta_2, \dots, \vartheta_{l-1}$ be the spherical coordinates of the point $\theta \in S$ and $\Sigma = \{x : \ \rho_1 \le |t| \le \rho_2, \ t \in R^l\}$, where ρ_1 and ρ_2 are arbitrary fixed positive numbers such that $\rho_1 < 1 < \rho_2$, so that $S \subset \Sigma$. Consider the function f defined on S and let $f^*(t) = f(t/|t|)$ be an extension of f on the Σ. We shall say that the function f belongs to the class $C^{(2)}(\Sigma)$, if all second order derivatives of f^* are continuous in Σ. The operator δ is defined on the $C^{(2)}(S)$ as

$$\delta = -\sum_{j=1}^{l-1} \frac{1}{q_j \sin^{l-j-1}\vartheta_j} \frac{\partial}{\partial\vartheta j}\left(\sin^{l-j-1}\vartheta_j \frac{\partial}{\partial\vartheta j}\right),$$

where $q_1 = 1, \ q_j = (\sin\vartheta_1 \sin\vartheta_2 \dots \Sigma\vartheta_{j-1})^2, \ j \ge 2$. This operator is symmetric in the space $H = L_2(S)$ and its eigen-values $\lambda_n = n(n+l-2), n \in N$, have the multiplicity $k_{n,l} = (2n + l - 2)(l + n - 3)!((l - 2)!n!)^{-1}$. The corresponding to the eigen-value λ_n eigen-functions are spherical functions $Y^{(k)}_{n,l}(\theta)$, $1 \le k \le k_{n,l}$ ([14], Ch.13, §2). They represent a whole orthonormal system in $L_2(S)$. Since all eigenvalues are positive, δ is the positive definite operator and its spectrum is discrete. We number the spherical functions

$Y_{n,l}^{(k)}$ in the following way. It is assumed that $l \geq 2$. If $1 \leq k \leq k_{1,l} = l$, then we take $\lambda_k = l(l-1)$; $\varphi_k(\theta) = Y_{1,l}^{(k)}(\theta)$ and if $k_{1,l} + \cdots + k_{j,l} < k \leq k_{1,l} + \cdots + k_{j+1,l}$, then $\lambda_k = (j+1)(j+l-1)$; $\varphi_k(\theta) = Y_{j+1,l}^{(k-(k_{1,l}+\cdots+k_{j,l}))}(\theta)$. If we substitute these λ_k and φ_k in (12), then we will obtain a sequence $\{u_m\}$ for the approximative solution of the equation $\delta^\infty(u) = f(\theta)$. For such sequence, Theorem 3 is valid in the space $D(\delta^\infty)$ with the norms (11), in which A is replaced by δ.

4. On the Stability and Construction of Central Linear Spline Algorithms for Ill-Posed Problems

Let K be a selfadjoint, positive, compact and one-to-one operator in the Hilbert space H with a dense image. Let $\{\varphi_k\}$ be an orthonormal sequence of eigenfunctions of K corresponding to a sequence of eigenvalues λ_k. Then K has the form $K(u) = \sum\limits_{k=1}^{\infty} \lambda_k (u, \varphi_k)\varphi_k$, $\lambda_k \to 0$, $\lambda_k > 0$. In [11], we have introduced the Fréchet space $D(K^{-\infty}) = \cap_{n=1}^{\infty} D(K^{-n+1})$, where K^{-1} is the inverse to the operator K and $K^{-n} = K^{-1}(K^{-n+1})$, $n \in N$. In the same place is introduced the operator $K^{-\infty} : D(K^{-\infty}) \to D(K^{-\infty})$ as

$$K^{-\infty}(x) = \big\{K^{-1}x, K^{-2}x, \ldots, K^{-n}x, \ldots\big\}.$$

The topology of the Fréchet space $D(K^{-\infty}) = \cap_{n=1}^{\infty} D(K^{-n})$ is given by the following sequence of norms

$$\|x\|_n^2 = \|x\|^2 + \|K^{-1}x\|^2 + \cdots + \|K^{-n+1}x\|^2$$

which are generated by the inner product

$$(x,y)_n = (x,y) + \big(K^{-1}x, K^{-1}y\big) + \cdots + \big(K^{-n+1}x, K^{-n+1}y\big), \quad x, y \in D(K^{-\infty}).$$

The topology of the energetic space $E_{K^{-\infty}}$ of the operator $K^{-\infty}$ is given by the sequence of norms

$$[x]_n =: (K^{-\infty}x, x) = (K^{-1}x, x) + (K^{-2}x, K^{-1}x) + \cdots + (K^{-n}x, K^{-n+1}x).$$

It is known [11] that the operator $K^{-\infty}$ is continuous, positive definite, selfadjoint and admits the inverse one $(K^{-\infty})^{-1}$ which is selfadjoint and continuous. Therefore, the operator $K^{-\infty}$ is an isomorphism of the Fréchet space $D(K^{-\infty})$ onto itself. Let us denote the operator $(K^{-\infty})^{-1}$ by K_∞. We have also

$$K_\infty u = (K^{-\infty})^{-1} u = \big\{Ku, KK^{-1}u, \ldots, KK^{-n+1}u, \ldots\big\}$$

and therefore

$$K^{-\infty}K_\infty u = K_\infty(K^{-\infty}u) = K_\infty\big\{K^{-1}u, K^{-2}u, \ldots, K^{-n}u, \ldots\big\} = \\ = \big\{u, K^{-1}u, \ldots, K^{-n}u, \ldots\big\} = u.$$

We transfer the equation $Ku = f$ from the Hilbert space H to the Fréchet space $D(K^{-\infty})$ in which the restriction K_∞ of the operator K is a selfadjoint operator. Moreover, K_∞ is an onto isomorphism of $D(K^{-\infty})$ and therefore, the operator equation

$$K_\infty u = f \tag{20}$$

has in $D(K^{-\infty})$ a unique and stable solution in the Fréchet space $D(K^{-\infty})$. Consider the energetic space E_{K_∞} of K_∞ whose norms are of the form

$$\begin{aligned}[x]'_n = (K_\infty x, x)_n^{1/2} = \big((Kx, x) + (KK^{-1}x, K^{-1}x) + \cdots +\\ +(K^{-n+2}x, K^{-n+1}x)\big)^{1/2},\ n \in N.\end{aligned} \tag{21}$$

For an approximative solution of equation (16) we use the Ritz's extended method in the space E_{K_∞}. The coefficients of an approximative solution $u_m = \sum_{k=1}^m a_k \varphi_k$ are defined from the following system of equations

$$\sum_{k=1}^m a_i [\varphi_k, \varphi_i]'_r = (f, \varphi_k)_r,\ i = 1, 2, \dots, m,\ r \in N.$$

By calculations we find that u_m has the form

$$u_m = \sum_{k=1}^m (f, \varphi_k)\big((\varphi_k, \varphi_k)\lambda_k\big)^{-1} \varphi_k. \tag{22}$$

This means that the subspace $\operatorname{Ker} I$ admits an orthogonal complement subspace in the Fréchet space $D(K^{-\infty})$.

Let $I(f) = [L_1(f),\ L_2(f), \dots, L_m(f)]$ be a nonadaptive information of the cardinality m on $D(K^{-\infty})$, where $L_i(f) = (f, \varphi_i)$, $\operatorname{Ker} I$ is a finite-codimensional subspace in $D(K^{-\infty})$, $\operatorname{Ker} I^\perp = \operatorname{span}\{\varphi_1, \varphi_2, \dots, \varphi_m\}$ and $y = I(f)$. The generalized spline σ interpolating y has the form $\sigma = \sum_{k=1}^m (f, \varphi_k)\varphi_k$. The solution operator for the equation $K_\infty u = f$ is $S = (K_\infty)^{-1} = K^{-\infty}$ and it realizes isomorphism of the space $D(K^{-\infty})$ onto itself. In addition,

$$\begin{aligned}S(\sigma) = \sum_{k=1}^m (f, \varphi_k) S\varphi_k = \sum_{k=1}^m (f, \varphi_k)(K_\infty)^{-1}\varphi_k = \sum_{k=1}^m (f, \varphi_k)(K^{-\infty})\varphi_k =\\ = \sum_{k=1}^m \lambda_k^{-1} (f, \varphi_k)\varphi_k = u_m,\end{aligned}$$

since

$$\begin{aligned}K^{-\infty}(\varphi_k) = \big\{K^{-1}\varphi_k, K^{-2}\varphi_k, \dots, K^{-n}\varphi_k, \dots\big\} =\\ = \lambda_k^{-1}\big\{\varphi_k, K^{-1}\varphi_k, \dots, K^{-1}\varphi_k, \dots\big\} = \lambda_k^{-1}\varphi_k.\end{aligned}$$

$S(\sigma) = u_m$ is also the best approximation of $Sf = (K_\infty)^{-1} f$ in the subspace $\operatorname{Ker} I^\perp$ with respect to the energetic norms $[\cdot]'_n$ of the energetic space E_{K_∞} of the operator K_∞, i.e., the subspace $\operatorname{Ker} I$ admits an orthogonal complement subspace in the Fréchet space E_{K_∞}. According to part b) of Theorem 2, this generalized spline algorithm is linear and generalized central. From the above reasoning follows

Theorem 4. *Let K be a selfadjoint positive operator in the Hilbert space H with an orthogonal sequence of eigenfunctions φ_j. Let λ_j be the eigenvalues corresponding to the eigenfunctions φ_j and u_m be defined by* (22)*. Then the algorithm $\varphi^s(I(f)) = u_m$ is a linear generalized spline and generalized central for the solution operator $S = K_\infty^{-1}$ and information $I(f) = [(f, \varphi_1), (f, \varphi_2), \ldots, (f, \varphi_m)]$. Moreover, the sequence of approximate solutions $\{u_m\}$ converges to the solution of equation* (20) *in the energetic space E_{K_∞} of the operator K_∞.*

We will now give a few examples of selfadjoint and positive definite operators in the Hilbert space for which the operator $K^{-\infty}$ satisfies the conditions of Theorem 2.

1. The inverse of the harmonic oscillator operator. For the Hermitian operator (12), the selfadjoint and positive inverse operator $K = A^{-1}$ in $L^2]-\infty, \infty[$ has the form

$$K(u) = \sum_{k=1}^{\infty} (2k+1)^{-1} (f, \varphi_k) \varphi_k.$$

For this operator K, we consider the equation $K_\infty(u) = f$ in the space $D(K^{-\infty}) = D(A^\infty) = S(R)$. In this case, the energetic space E_{K_∞} of K_∞ is $S(R)$. For the generalized spline σ interpolating y the generalized spline algorithm $\sum_{k=1}^{m} (2k+1)(f, \varphi_k)\varphi_k = u_m$. According to part b) of Theorem 2, this algorithm is linear and generalized central in the space E_{K_∞} with the sequence of energetic norms (17).

2. Integral equations of the first kind.

2.1. Consider the following integral equation of the first kind:

$$K(u) = \int_a^b K(s,t) u(s) ds = f(t), \tag{23}$$

where

$$K(s,t) = \begin{cases} (s-a)(t-b)(a-b)^{-1}, & a \le s \le t \le b, \\ (t-a)(s-b)(a-b)^{-1}, & a \le t \le s \le b. \end{cases}$$

It is well-known,what $K(s,t)$ is the Green's function for the symmetric and positive definite operator $A = -d^2/dt^2$ in the Hilbert space $L^2[a,b]$ with the boundary conditions $u(a) = u(b) = 0$. $D(A)$ is the set of functions having absolutely continuous first order derivatives and second order quadratically summable derivatives on [a,b]. $D(A^\infty)$ consists of the functions having infinite order quadratically summable derivatives on $[a,b]$. This space contains a countable normed infinite order Sobolev space $W^\infty[a,b]$ [15]. The operator K_∞, i.e., the restriction of the integral operator K on the space $D(A^\infty)=D(K^{-\infty})$, is a topological isomorphism onto and the equation (20) has a unique and stable solution. The eigenvalues and the corresponding orthonormal eigenfunctions of A are $\lambda_k=k^2\pi^2/(b-a)^2$ and $\varphi_k(t)=\sqrt{\frac{2}{b-a}}\sin\frac{\pi k(t-a)}{b-a}$, $k\in N$. An approximate solution of the equation (20) has the following form:

$$u_m(t)=\sum_{k=1}^m \frac{2k^2\pi^2}{(b-a)^3}\sin\frac{\pi k(t-a)}{b-a}\int_a^b f(s)\sin\frac{\pi k(s-a)}{b-a}ds.$$

The sequence $\{u_m\}$ converges in the space $E_{K_\infty}=D(K^{-\infty})$ to the solution of the equation (20). For that sequence the above reasoning is valid, and according to Theorem 4, this generalized spline algorithm is linear and generalized central one.

2.2. Consider the integral equation of the first kind (23), where

$$K(s,t)=\begin{cases}(e^s+e^{2a-s})(e^t+e^{2b-t})2^{-1}(e^{2b}-e^{2a})^{-1}, & a\le s\le t\le b,\\ (e^t+e^{2a-t})(e^s+e^{2b-s})2^{-1}(e^{2b}-e^{2a})^{-1}, & a\le t\le s\le b.\end{cases}$$

It is well-known what $K(s,t)$ is the Green's function for the symmetric and positive operator $Au=-d^2u/dt^2+u$ in the Hilbert space $L^2[a,b]$ with the boundary conditions $u'(a)=u'(b)=0$. $D(A)$ is the set of functions having absolutely continuous first order derivatives and second order quadratically summable derivatives on [a,b]. $D(A^\infty)$ consists of functions having infinite order quadratically summable derivatives on $[a,b]$. This space contains countable normed infinite order Sobolev space $W^\infty[a,b]$ [15]. The operator K_∞, i.e., the restriction of the integral operator K on the space $D(A^\infty)=D(K^{-\infty})$, is a topological isomorphism onto and the equation (20) has a unique and stable solution. The eigenvalues and the corresponding orthonormal eigenfunctions of A are $\lambda_k=1+k^2\pi^2/(b-a)^2$ and $\varphi_k(t)=\sqrt{\frac{2}{b-a}}\cos\frac{\pi k(t-a)}{b-a}$, $k\in N$. An approximate solution of the equation (23) has the following form

$$u_m(t)=\sum_{k=1}^m\left(1+\frac{k^2\pi^2}{(b-a)^2}\right)\frac{2}{b-a}\cos\frac{\pi k(t-a)}{b-a}\int_a^b f(s)\cos\frac{\pi k(s-a)}{b-a}ds.$$

The sequence $\{u_m\}$ converges in the space E_{K_∞} to the solution of the equation (20). For this sequence the above reasoning is valid, and according to Theorem 4, this generalized spline algorithm is linear and generalized central one.

2.3. Consider the integral equation (23), where $a = -\infty,\ b = +\infty$ and

$$K(s,t) = \begin{cases} -\pi^{-1/2} I(-\infty, s) I(t, \infty) \exp \dfrac{s^2+t^2}{2}, & s \le t, \\ -\pi^{-1/2} I(s, \infty) I(-\infty, t) \exp \dfrac{s^2+t^2}{2}, & s \ge t, \end{cases}$$

where $I(u,v) = \int_{-\infty}^{\infty} e^{-t^2} dt$. It is well-known what $K(s,t)$ is the Green's function for the symmetric and positive definite degenerate hypergeometrical operator $Au(t) = -d^2u/dt^2 + (t^2+1)u$ in the Hilbert space $L^2[a,b]$ with the boundary conditions $u(-\infty) = u(\infty) = 0$. $D(A)$ is the set of functions having absolutely continuous first order derivatives and second order quadratically summable derivatives on $(-\infty, \infty)$. $D(A^\infty)$ consists of the functions having infinite order quadratically summable derivatives on $R =]-\infty, \infty[$. This space contains countable normed infinite order Sobolev space $W_0^\infty(R)$ [15]. The operator K_∞, i.e., the restriction of the integral operator K on the space $D(A^\infty) = D(K^{-\infty})$, is a topological isomorphism onto and the equation (20) has a unique and stable solution. The eigenvalues and the corresponding orthonormal eigen-functions of A are $\lambda_k = 2k$ and $\varphi_k(t) = (-1)^{k-1}(k-1)^{-1/4}((k-1)!)^{-1/2}\pi^{-1/4}2^{1-k}e^{t^2/2}\frac{d^{k-1}e^{-t^2}}{dt^{k-1}}$, $k \in N$. Using these functions φ_k, we can construct an approximate solution of the equation (23) of the form

$$u_m(t) = 2\sum_{k=1}^{m} k \int_{-\infty}^{\infty} f(s)\varphi_k(s)ds\varphi_k(t).$$

The sequence $\{u_m\}$ converges in the space E_{K_∞} to the solution of the equation (20). For that sequence the above reasoning is also valid, and according to Theorem 2, this generalized spline algorithm is linear and generalized central one.

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(Received 26.04.2012)

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