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# SOME APPROXIMATE PROCESSES FOR CAUCHY TYPE SINGULAR INTEGRALS

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Abstract. A quadrature process for Cauchy type singular integrals with Chebyshev weight function is indicated. The error estimate is obtained on some classes of densities.

**რეზიუმე.** განხილულია გარკვეული კვადრატურული პროცესი კოშის ტიპის სინგულარული ინტეგრალებისათვის ჩებიშევის წონითი ფუნქციით. დადგენილია ცდომილების შეფასებები სიმკვრივეთა ზოგიერთ კლასზე.

For certain classes of boundary value problems of the theory of functions of a complex variable and related to them problems, the most effective method is, as is known, the method of boundary integral equations. Meaningful is even the fact that this method can reduce by unity dimensionality of the initial problem. It should also be noted that in the existing literature under the above-mentioned method is understood application of Fredholm integral equations, that is, application of one or another algorithm for numerical solution of such equations. In this connection, we note that although the methods of solution of such equations in a general case are well known, in many cases the kernels and the right-hand sides of such boundary equations contain, as usual, the values difficult to define. Such fact occurs especially in the problems with experimental data, the case encountered commonly in practice. In analogous situations dealing with numerical solution of the initial problem, the more effective are calculation schemes which are based on the application of boundary integral equations with singular Cauchy type integrals (in a sense of their principal values). Special interest among the problems connected with the theory of application of such integrals deserve, in particular, contact problems of elasticity and related to them problems such, for example, as the problems of the theory of cracks and analogous to them problems (see [1], [2], [3], [4]).

Noteworthy are also some integral equations connected with certain problems of nuclear physics (see, e.g., [5], [6]). An important role in solving such

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kind of problems belongs to quadrature formulas for singular Cauchy type integrals. Speaking about the problems connected with consideration of singular integrals, much attention should be attached to classes of integral equations whose solutions are expressed effectively in terms of the same singular integrals ([7]). In this direction, primary attention receive singular integrals on unclosed contours of integration (on segments, inclusive) whose behavior near the ends of the lines of integration differs.

More or less intensive work in constructing and investigating quadrature formulas for approximate calculation of integrals with Cauchy kernel traces back to the middle of the past century (only some separate works in this direction were available previously). Seemingly, we can say that among the topics relating to the application of methods of approximation of singular integrals to numerical solution of singular integral equations, of special importance are the monograph [8] and also some separate works of another researchers. To somewhat later period belong the monographs [9], [10] and [11]. The most of the above-mentioned works deal with quadrature formulas for singular integrals of the type

$$\int_a^b \frac{\rho(t)\varphi(t)}{t-x}\,dt,$$

where $\rho(t)$ is a fixed, summable on $[a,b]$ function, and $\varphi(t)$ (density of a singular integral) is an arbitrary function from some class of functions satisfying usually certain conditions of smoothness.

Of a number of arguments, it is quite reasonable to base the construction of quadrature formulas for the above-mentioned singular integrals on the approximation of the function $\varphi(t)$ by interpolation polynomials which are constructed with respect to the nodes being zeros of polynomials, orthogonal on the segment $[a,b]$ with respect to the weight $\rho(t)$. Just such an approach is used in the works available at present [1]. In particular, in the case of important for applications weighted functions of the type $\rho(t)=(b-t)^p(t-a)^q$ $(p,q>-1)$, many authors in their works consider error estimates of quadrature formulas for singular integrals with the above-mentioned weighted functions, employing zeroes of the Jacobi polynomial corresponding to the given weight.

Meanwhile, we may point out the problems reducing to analogous singular integrals, but application of the above-mentioned quadrature formulas to them is, in a definite sense, less effective. To such problems belong, for example, applied problems with boundary integral equations whose kernels

[1]Note, however, that in [12], to approximate a singular integral on the segment $[-1,+1]$ with the constant weighted function, we have constructed and investigated the quadrature formula with Chebyshev nodes (corresponding, as is known, to the weight $\rho(t)=\sqrt{(1-t^2)}^{\,-1}$).

can be defined experimentally. Certain difficulties arise also for numerical solution of equations with insufficiently smooth kernels. In such cases, it is hard to substantiate theoretically computational processes of approximate solution, in particular, error estimation of quadrature scheme, and to argue about the exactness of the results, we have to compare all the results which are obtained at different steps of approximations. In conformity with the above, we can say that more effective for quadrature formulas with weighted functions $(b-t)^p(t-a)^q$ are quadrature formulas with nodes, being zeroes of the polynomial $J_n^{(p,q)}(x)J_{n+1}^{(p,q)}(x)$ of degree $2n+1$, where under $J_n^{(p,q)}(x)$ and $J_{n+1}^{(p,q)}(x)$ are meant Jacobi polynomials of degrees $n$ and $n+1$, respectively. In such cases, when passing from the given $n$ to $n+1$, a part of values $\varphi$ calculated at the previous step can be used again at the subsequent step. Considering the question on the approximation of such integrals, everywhere in the sequel it will, as usual, be assumed that $a=-1$, $b=1$.

Having next denoted by $\{x_{kn}\}$ and $\{x_{kn+1}\}$ the zeroes of the above-mentioned polynomials, under the quadrature formula under consideration will be understood approximation of the corresponding singular integral

$$\int\limits_{-1}^{+1} \frac{(1-t)^p(1+t)^q\varphi(t)}{t-x}dt$$

by the quadrature sum which is obtained after replacing the function $\varphi(t)$ by the interpolation polynomial constructed on the strength of the nodes $\{x_{kn}\}_{k=1}^n, \{x_{kn+1}\}_{k=1}^{n+1}$. Note, however, that the obtained quadrature sum is rather cumbersome and, hence, we will consider in detail a particular, but more important from the point of view of its applications, case $p=q=-1/2$ corresponding to the Chebyshev weight function $\rho(t)=(\sqrt{1-t^2})^{-1}$. Some of the obtained results can be extended to the cases involving another values $p,q>1$.

Thus, approximating the function $\varphi(t)$ by the interpolation polynomial, constructed by the above-mentioned system of nodes $\{x_{kn}\}_{k=1}^n \bigcup \{x_{kn+1}\}_{k=1}^{n+1}$, we obtain the quadrature formula

$$\int\limits_{-1}^{+1} \frac{\varphi(t)dt}{\sqrt{1-t^2}(t-x)} \approx S_n(\varphi;x) \quad (-1<x<1), \tag{1}$$

$$S_n(\varphi;x) = \frac{\pi}{n+1}\sum_{k=1}^{n+1}\frac{\lambda_n(x)-1}{x-x_{kn+1}}\varphi(x_{kn+1}) - \frac{\pi}{n}\sum_{k=1}^{n}\frac{\lambda_n(x)}{x-x_{kn}}\varphi(x_{kn}),$$
$$\lambda_n(x) = T_n(x)U_n(x),$$

where $T_n(x)$ and $U_n(x)$ are, respectively, the Chebyshev first and second kind polynomials:

$$T_n(x) = \cos(n \arccos x), \quad U_n(x) = \frac{\sin[(n+1)\arccos x]}{\sqrt{1-x^2}}.$$

Under the values of the given quadrature sum at the points $x_{km}$ $(m = n, n+1)$ we mean its corresponding limiting values.

We say that the function $\varphi(x)$ belongs to the class $\bigvee_r[-1,+1]$ if the derivative $\varphi(x)$ of the $r$-th $(r \geq 1)$ order has on the segment a bounded variation. The theorem below, relating to the estimation of exactness of the above quadrature formula, is valid.

**Theorem .** *If $\varphi(\tau) \in \bigvee_r[-1,+1]$, then at every point $x \in (-1,+1)$ the estimate*

$$\big|S(\varphi;x) - S_n(\varphi;x\big| \leq \frac{1}{r!}\big\{|\lambda_n(x)| \sup|G_n(\tau)| + |1-\lambda_n(x)| \sup|G_{n+1}(\tau)|\big\} \times \\ \times \bigvee_{-1}^{+1}((\varphi^{(r)})(u)), \tag{2}$$

*where*

$$G_m(\tau) = \int_\tau^1 \frac{(1-t)^{r-1}}{\sqrt{1-t^2}} dt - \frac{1}{m}\sum_{k=1}^m E_{r-1}(x_{km} - \tau) \quad (m = n, n+1),$$

*is valid.*

To prove the above-formulated statement, we have first to show that for the condition $\varphi(t) \in \bigvee_r[-1,+1]$ $(r \geq 1)$, the representation in (1) of the remainder term

$$S(\varphi;x) - S_n(\varphi;x) = \int_{-1}^x \bigg[(x-u)^r \int_{-1}^u \frac{K_{rn}(\tau,x)}{(x-r)^{r+1}} d\tau\bigg] d\varphi^{(r)}(u) - \\ - \int_x^1 \bigg[(x-u)^r \int_u^1 \frac{K_{rn}(\tau,x)}{(x-r)^{r+1}} d\tau\bigg] d\varphi^{(r)}(u) \tag{3}$$

is valid, where

$$K_m(\tau;x) = \frac{\pi}{(r-1)!}\bigg\{\int_\tau^1 \frac{(t-\tau)^{r-1}dt}{\sqrt{1-t^2}} + \\ + \frac{1}{n+1}\sum_{k=1}^n [\lambda_n(x) - 1] E_{r-1}(x_{kn+1} - \tau) -$$

$$-\frac{1}{n}\sum_{k=1}^{n}\lambda_n(x)E_{r-1}(x_{kn}-\tau)\Big\},$$

$$E_{r-1}(y)=\begin{cases} y^{r-1}, & y\geq 0;\\ 0, & y<0,\end{cases}$$

(the integration with respect to $u$ in (3) is considered in a Stieltjes sense). The above reprsentation is based on the application of formulas

$$\begin{aligned}\varphi(t) = {} & \varphi(x)+\\ & +(t-x)\sum_{\sigma=0}^{r-1}\frac{(t+1)^\sigma}{\sigma!}\frac{1}{(x+1)^{\sigma+1}}\int_{-1}^{x}(x-u)^\sigma\varphi^{(\sigma+1)}(u)du+\\ & +(t-x)\rho_n(\varphi;x),\end{aligned}$$

$$\begin{aligned}\rho_n(\varphi;x) = {} & \frac{1}{(r-1)!}\int_{-1}^{t}\left[\frac{(t-\tau)^{r-1}}{(x-\tau)^{r+1}}\int_{\tau}^{x}(x-u)^r d\varphi^{(r)}(u)\right]d\tau=\\ = {} & \frac{1}{(r-1)!}\int_{-1}^{+1}\left[\frac{E_{r+1}(t-\tau)}{(x-\tau)^{r+1}}\int_{\tau}^{x}(x-u)^r d\varphi^{(r)}(u)\right]d\tau,\end{aligned}$$

which are obtained by the subsequent integration by parts of the expression

$$\frac{1}{(r-1)!}\int_{-1}^{t}(t-\tau)^{r-1}\frac{d\tau}{(x-\tau)^{r+1}}\int_{r}^{x}(x-u)^r d\varphi^{(r)}(u),$$

having convinced that the relations

$$\frac{d}{d\tau}\left[\frac{1}{(x-\tau)^\mu}\int_{\tau}^{x}(x-u)^{\mu-1}\varphi^{(\mu)}(u)du\right]=\frac{1}{(x-\tau)^{\mu+1}}\int_{\tau}^{x}(x-u)^\mu\varphi^{(\mu+1)}(u)du,$$

$\tau\neq x$ ($\mu=1,2,\dots,r-1$) are true.

Inserting the above expansion of $\varphi(t)$ into the right-hand side of (3) and taking into account the way of constructing the sum $S_n$, the corresponding difference vanishes for arbitrary polynomials of degree $\leq 2n$, we obtain

$$S(\varphi;x)-S_n(\varphi;x)=\int_{-1}^{+1}\frac{K_{rn}(\tau;x)}{(x-\tau)^{r+1}}d\tau\int_{\tau}^{x}(x-u)^r d\varphi^{(r)}(u).$$

In the adopted for $\varphi$ assumptions, the obtained representation of the remainder term can be transformed into

$$S(\varphi;x)-S_n(\varphi;x)=\int_{-1}^{x}\left[(x-u)^r\int_{-1}^{u}\frac{K_{rn}(\tau;x)d\tau}{(x-\tau)^{r+1}}\right]d\varphi^{(u)}(\tau)-$$

$$-\int_x^1 \left[(x-u)^r \int_u^1 \frac{K_{rn}(\tau;x)d\tau}{(x-\tau)^{r+1}}\right] d\varphi^{(r)}(u).$$

Thus, we obtain the estimate of the type

$$\begin{aligned}
&\big|S(\varphi;x)-S_n(\varphi;x)\big| \le \\
&\quad \le \frac{1}{r} \sup_{-1\le\tau\le1} \big|K_{rn}(\tau;x)\big| \bigg\{ \int_{-1}^{x} \left[1-\left(\frac{x-u}{x+1}\right)^r\right] \big|d\varphi^{(r)}(\tau)\big| + \\
&\quad + \int_x^1 \left[1-\left(\frac{x-u}{x+1}\right)^r\right] \big|d\varphi^{(r)}(\tau)\big| \bigg\} \le \frac{1}{r} \sup_{-1\le\tau\le1} \big|K_{rn}(\tau;x)\big| \bigvee_{-1}^{+1} \big(\varphi^{(r)}(u)\big).
\end{aligned}$$

Next, having noticed that since

$$K_{rn}(\tau;x) = \frac{\pi}{(r-1)!} \big\{\lambda_n(x) G_{rn}(\tau) + [1-\lambda_n(x)]\, G_{rn+1}(\tau)\big\},$$

where

$$G_{rm}(\tau) = \int_\tau^1 \frac{(t-\tau)^{r-1}}{\sqrt{1-t^2}} dt - \frac{1}{m}\sum_{k=1}^m E_{r-1}(x_{km}-\tau) \quad (m=n,n+1),$$

the previous estimate yields

$$\begin{aligned}
|S(\varphi;x)-S_n(\varphi;x)| \le \frac{\pi}{r!} &\big\{|\lambda_n(x)| \sup_\tau |G_{rn}(\tau)| + |1-\lambda_n(x)| \sup_\tau |G_{rn+1}(\tau)|\big\} \times \\
&\times \bigvee_{-1}^{+1} (\varphi^{(r)}(u)),
\end{aligned}$$

which is, in fact, the estimate of type (2).

In the commonly adopted terminology, the obtained estimate belongs to a number of efficient estimates in a sense that for any given $x \in (-1,1)$ the expression (a constant) for $\bigvee_{-1}^{+1}(\varphi^{(r)}(u)$ can be calculated exactly or approximately with any preassigned accuracy. Such kind of estimates present certain interest, taking into account that the literature dealt with the problems of approximation of singular integrals considers, as usual, ordered estimates of the type $O\left(\frac{\ln n}{n^{r+\alpha}}\right)$ $(n > 1)$. Such estimates imply, as a rule, classes of densities $\varphi(t)$ having on the given segment $[-1,+1]$ a derivative satisfying the Hölder condition $r$ $(r \ge 1)$ with the exponent $\alpha$ $(0 < \alpha \le 1)$. In such situations, the values of constants contained in $O$ are not defined.

Getting back to the previous reasoning, we note that the singular integral $S(\varphi;x)$ among the integrals of the type

$$\int_{-1}^{+1} \frac{\rho(t)\varphi(t)}{t-x}dt \ \ (-1<x<1)$$

possesses the following noteworthy property: according to the well-known result [7], in case $\rho(t)=(\sqrt{1-t^2})^{-1}$, under the assumption that the function $\varphi(t)$ satisfies on $[-1,+1]$ Hölder's condition $H_\alpha$ with exponent $\alpha>\frac{1}{2}$, there exist finite limits, as $x\to\pm1$, of the corresponding singular integral. Thus, taking these limits as values of the given integral at the points $x=\pm1$, the corresponding integral may be assumed to be defined on the segment $[-1,1]$, including the ends $\pm1$ and, hence, to put the question on its approximation on the whole segment $[-1,+1]$. According to the result obtained above, such an approximation is realized by the sums $S_n(\varphi;x)$, and thus we have the estimate which is performed uniformly on $[-1,+1]$.

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