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A HIERARCHY OF SINGULAR INTEGRAL OPERATORS FOR MIXED BOUNDARY VALUE PROBLEMS

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Abstract. A class of integral operators having a hierarchy of polyharmonic kernels is introduced and some properties are derived. Iterated mixed boundary value problems for complex model equations and linear elliptic complex partial differential equations are discussed in the unit disc of the complex plane.

რეზიუმე. წარმოდგენილია ინტეგრალური ოპერატორების კლასი პოლიჰარმონიული გულებით და დადგენილია მათი ზოგიერთი თვისება. კომპლექსური სიბრტყის ერთეულოვან წრეზე განხილულია კომპლექსური ტიპის იტერაციული შერეული და წრფივი ელიფსური კომპლექსური კერძო წარმოებულებიანი დიფერენციალური განტოლებები.

1. Introduction

Complex analytic methods for partial differential equations were developed by N. Muskhelishvili, I. N. Vekua, L. Bers, and others in the 1930's and 1940's. These methods has been applied extensively to the mathematical physics, elasticity theory [31] and shell theory on the treatment of elliptic systems.

In complex analysis, there are two important boundary value problems namely, Riemann and Riemann-Hilbert boundary value problems. For analytic functions fundamental investigations were done by N. Muskhelishvili and F. D. Gakhov. The classical theory of these problems for generalized analytic functions are contained in the books of N. I. Muskhelishvili [31], F. D. Gakhov [27], I. N. Vekua [34] and L. Bers [24], see also [11, 28, 29, 30, 33]. For systems in several complex variables, see [13]. Riemann and Riemann-Hilbert boundary value problems are investigated for generalized Beltrami equation too, [11, 10, 8, 9, 12, 25, 26, 33, 34, 35]. Singular integral operators play important roles in the theory of generalized analytic functions. Their properties were extensively studied by Vekua [34]. H. Begehr and G. N.

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Hile investigated a hierarchy of integral operators and their applications to the boundary value problems in [14].

In recent years, particular cases of these problems, called as Dirichlet, Neumann, Schwarz and Robin problems, are considered for some complex model equations, including Cauchy-Riemann, Poisson, higher-order Poisson, polyanalytic equations, see [11, 16, 22, 23, 18, 17], and also some general complex linear elliptic partial differential equations [2, 3, 4, 5]. These problems are closely related with the theory of singular integral equations developed by N. I. Muskhelishvili [31, 32]. In last studies, there has been an increasing interest on the mixed boundary value problems. Begehr considered these type of problems for bi-Poisson equation, [19, 21]. In [6], some particular mixed problems are considered for higher order Poisson and generalized n-Poisson equations.

In this paper, we are concerned with the mixed type iterated boundary value problems, firstly for arbitrary higher-order model equations. In this case, the given equations may be reduced into a coupled system of equations with polyanalytic and polyharmonic operators as principal parts. Then, we investigate the conditions on solvabilities of iterated mixed type boundary value problems for linear elliptic complex partial differential equations of arbitrary order. For this purpose, we will introduce a hierarchy of polyharmonic Green-type functions defined by an iterative technique which was left as an open problem by H. Begehr [20], because of the complexity of the relevant combinatorial nature of the problem. Since it is difficult to obtain the explicit forms, we will give the polyharmonic kernel functions in an iterative way. Then, using these kernel functions, we will define and investigate the properties of a hierarchy of singular integral operators related to mixed boundary value problems.

In the following section, we will give a short preliminary related with the polyharmonic Green-type functions and their corresponding integral operators existing in the literature. Section 3 is devoted to the hierarchy of the iterated generalization of the polyharmonic kernel functions. In section 4, using the class of kernel functions defined in section 3, we give generalized integral representation formulas for suitable differentiable functions. These formulas will lead to the solutions of mixed boundary value problems that are the combinations of Dirichlet, Neumann and Robin type conditions. In section 5, mixed boundary value problems for higher order linear complex partial differential equations will be investigated by introducing a class of integral operators. We will use them to transform the original problem into a singular integral equation. Solvability of the problem will be studied by use of the Fredholm theory.

2. Preliminaries

2.1. Polyharmonic kernel functions. In the unit disc $\mathbb{D}$ harmonic Green, Neumann and Robin functions are defined as

$$
\begin{aligned}
G_1(z,\zeta) &= \log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2,\\
N_1(z,\zeta) &= -\log|(1-z\zeta)(\zeta-z)|^2,\\
R_1(z,\zeta) &= \log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2 - 2\left[\frac{\log(1-z\bar{\zeta})}{z\bar{\zeta}} + \frac{\log(1-\bar{z}\zeta)}{\bar{z}\zeta} + 1\right],
\end{aligned}
$$

respectively, together with their properties in [21, 19].

Polyharmonic Green, Neumann, Robin functions

$$
\begin{aligned}
G_n(z,\zeta) &= -\frac{1}{\pi}\iint_{\mathbb{D}} G_1(z,\tilde{\zeta})G_{n-1}(\tilde{\zeta},\zeta)d\tilde{\xi}\,d\tilde{\eta},\\
N_n(z,\zeta) &= -\frac{1}{\pi}\iint_{\mathbb{D}} N_1(z,\tilde{\zeta})N_{n-1}(\tilde{\zeta},\zeta)d\tilde{\xi}\,d\tilde{\eta},\\
R_n(z,\zeta) &= -\frac{1}{\pi}\iint_{\mathbb{D}} R_1(z,\tilde{\zeta})R_{n-1}(\tilde{\zeta},\zeta)d\tilde{\xi}\,d\tilde{\eta}
\end{aligned}
$$

are given by Begehr et al [22, 18, 17, 23].

The higher order Poisson equation with Dirichlet conditions is investigated by Begehr and Vaitekhovich in [22] and with Neumann conditions is studied by Begehr and Vanegas in [18]. Particularly, for the inhomogeneous biharmonic equation, analogous results are presented in [21, 19]. Robin problem for inhomogeneous harmonic equation is treated in [21, 19, 23]. For the higher order Poisson operators the problem is studied by Begehr and Harutyunyan [17]. In the cases $n=1$ and $n=2$, the explicit solutions are given for the corresponding problems.

Apart from these, the iterations of harmonic Green, Neumann and Robin functions in a mixed way lead to different hybrid polyharmonic Green-type functions. In [5], convolution of the polyharmonic Green-Almansi function $\tilde{G}_n$,

$$
\begin{aligned}
\tilde{G}_n(z,\zeta) =& \frac{|\zeta-z|^{2(n-1)}}{(n-1)!^2}\log\left|\frac{1-z\bar{\zeta}}{\zeta-z}\right|^2 -\\
&-\sum_{\mu=1}^{n-1}\frac{1}{\mu(n-1)!^2}|\zeta-z|^{2(n-1-\mu)}(1-|z|^2)^{\mu}(1-|\zeta|^2)^{\mu} \qquad (1)
\end{aligned}
$$

and polyharmonic Green function $G_n(z,\zeta)$ is defined by Green-m-Green Almansi-n function $G_{m,n}(z,\zeta)$ for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ as

$$G_{m,n}(z,\zeta) = -\frac{1}{\pi}\iint_{\mathbb{D}} G_m(z,\tilde{\zeta})\tilde{G}_n(\tilde{\zeta},\zeta)d\tilde{\xi}\,d\tilde{\eta}\ . \tag{2}$$

Also, some other particular polyharmonic hybrid Green type functions are obtained by convoluting Green, Neumann and Robin functions iteratively [6]. One of them is

$$H_{m,n}(z,\zeta) = -\frac{1}{\pi}\iint_{\mathbb{D}} G_m(z,\tilde{\zeta})N_n(\tilde{\zeta},\zeta)d\tilde{\xi}\,d\tilde{\eta}.$$

2.2. **Integral operators.** To discuss the solvability of the equation

$$\frac{\partial^{2n}w}{\partial z^n\partial\bar{z}^n} + \sum_{\substack{k+l=2n\\ k\neq l}}\left(q_{kl}^{(1)}(z)\frac{\partial^{2n}w}{\partial z^k\partial\bar{z}^l} + q_{kl}^{(2)}(z)\frac{\partial^{2n}\overline{w}}{\partial z^k\partial\bar{z}^l}\right) +$$
$$+\sum_{0\le k+l<2n}\left[a_{kl}(z)\frac{\partial^{k+l}w}{\partial\bar{z}^k\partial z^l} + b_{kl}(z)\frac{\partial^{k+l}\overline{w}}{\partial z^k\partial\bar{z}^l}\right] = f(z) \text{ in } \mathbb{D} \tag{3}$$

with proper boundary conditions, we need the following integral operators:

(i) Using $G_{m,n}(z,\zeta)$ and its derivatives with respect to z and $\bar{z}$ as the kernels, we define a class of integral operators

$$G_{m,n}^{k,l}f(z) := -\frac{1}{\pi}\iint_{\mathbb{D}} \partial_z^k\partial_{\bar{z}}^l G_{m,n-m}(z,\zeta)f(\zeta)d\xi\,d\eta$$

for $m,k,l \in \mathbb{N}_0$, $n \in \mathbb{N}$ with $(k,l) \neq (n,n)$ and $k+l \le 2n$ (see [5]). These operators are related to the following (m,n)-type Dirichlet problems:

Find $w \in W^{2n,p}(\mathbb{D})$ as a solution to (3) *satisfying the Dirichlet conditions*

$$(\partial_z\partial_{\bar{z}})^\mu w = 0, \quad 0 \le \mu \le m-1 \qquad \text{on} \quad \partial\mathbb{D}, \tag{4}$$
$$(\partial_z\partial_{\bar{z}})^{\mu+m} w = 0, \quad 0 \le 2\mu \le n-m-1 \quad \text{on} \quad \partial\mathbb{D}, \tag{5}$$
$$\partial_{\nu_z}(\partial_z\partial_{\bar{z}})^{\mu+m} w = 0, \quad 0 \le 2\mu \le n-m-2 \quad \text{on} \quad \partial\mathbb{D}. \tag{6}$$

(ii) For $n \in \mathbb{N}$, $k,l \in \mathbb{N}_0$ with $(k,l) \neq (n,n)$ and $k+l \le 2n$, the operators given by

$$S_{n,k,l}f(z) = \frac{1}{\pi}\iint_{\mathbb{D}} \partial_z^k\partial_{\bar{z}}^l N_n(z,\zeta)f(\zeta)d\xi\,d\eta$$

for a suitable complex valued function f given in $\mathbb{D}$, are related to the following Neumann problem:

Find $w \in W^{2n,p}(\mathbb{D})$ as a solution to (3) *satisfying the Neumann conditions*

$$\partial_{\nu_z}(\partial_z\partial_{\bar{z}})^\mu w = \gamma_\mu \textit{ on } \mathbb{D} \quad \gamma_\mu \in C(\partial\mathbb{D}) \quad 0 \le \mu \le n-1,$$

see [4].

(iii) The operators $R^{k,l}_{n,\alpha,1}$ are defined by

$$R^{k,l}_{n,\alpha,1}f(z) := \frac{1}{\pi}\iint\limits_{\mathbb{D}} \partial_z^k \partial_{\bar z}^l R_{n,\alpha,1}(z,\zeta) f(\zeta) d\xi\, d\eta$$

for $n \in \mathbb{N}$, $\alpha \neq 0$, $k, l \in \mathbb{N}_0$ with $(k,l) \neq (n,n)$ and $k + l \leq 2n$, for a suitable complex valued function f given in $\mathbb{D}$, see [7]. They are related to the following problem:

Find $w \in W^{2n,p}(\mathbb{D})$ as a solution to (3) *satisfying the Robin conditions*

$$\alpha(\partial_z\partial_{\bar z})^\sigma w + \partial_\nu(\partial_z\partial_{\bar z})^\sigma w = \gamma_\sigma \text{ on } \partial\mathbb{D}\ ,\ \gamma_\sigma \in C(\partial\mathbb{D}) \text{ for } 0 \leq \sigma \leq n-1\ .$$

Remark 1. For a general linear elliptic complex partial differential equation whose leading term is the polyanalytic operator, Schwarz problem is given by (see [2]):

Find $w \in W^{k,p}(\mathbb{D})$ as a solution to the k-th order complex differential equation

$$\frac{\partial^k w}{\partial \bar z^k} + \sum_{j=1}^{k} q_{1j}(z)\frac{\partial^k w}{\partial \bar z^{k-j}\partial z^j} + \sum_{j=1}^{k} q_{2j}(z)\frac{\partial^k \overline{w}}{\partial z^{k-j}\partial \bar z^j} +$$
$$+\sum_{l=0}^{k-1}\sum_{m=0}^{l}\left[a_{ml}(z)\frac{\partial^l w}{\partial \bar z^{l-m}\partial z^m} + b_{ml}(z)\frac{\partial^l \overline{w}}{\partial z^{l-m}\partial \bar z^m}\right] = f(z) \text{ in } \mathbb{D} \qquad (7)$$

satisfying the nonhomogeneous Schwarz boundary conditions

$$\operatorname{Re}\frac{\partial^l w}{\partial \bar z^l} = \gamma_l \text{ on } \partial\mathbb{D}, \qquad \operatorname{Im}\frac{\partial^l w}{\partial \bar z^l}(0) = c_l\ ,\ \ 0 \leq l \leq k-1, \qquad (8)$$

where $\gamma_l \in C(\partial\mathbb{D};\mathbb{R})$, $c_l \in \mathbb{R}$, $0 \leq l \leq k-1$.

To discuss the solvability of the above problem, $\widetilde{T}_k$ operators defined by

$$\widetilde{T}_k f(z) := \frac{(-1)^k}{2\pi(k-1)!}\iint\limits_{\mathbb{D}}(\overline{\zeta - z} + \zeta - z)^{k-1}\left[\frac{f(\zeta)}{\zeta}\frac{\zeta+z}{\zeta-z} + \frac{\overline{f(\zeta)}}{\bar\zeta}\frac{1+z\bar\zeta}{1-z\bar\zeta}\right] d\xi\, d\eta$$

for $k \in \mathbb{N}$ with $\widetilde{T}_0 f(z) = f(z)$ are used. For the properties of such operators, see [11, 15]. $\partial_z^l \widetilde{T}_k$ is a weakly singular integral operator for $0 \leq l \leq k-1$, while

$$\Pi_k f(z) := \frac{\partial^k}{\partial z^k}\widetilde{T}_k f(z) = \frac{(-1)^k k}{\pi}\iint\limits_{\mathbb{D}}\left[\left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{k-1}\frac{f(\zeta)}{(\zeta - z)^2} + \right.$$
$$\left. + \left(\frac{\zeta - z + \overline{\zeta - z}}{1 - z\bar\zeta}\bar\zeta - 1\right)^{k-1}\frac{\overline{f(\zeta)}}{(1 - z\bar\zeta)^2}\right] d\xi\, d\eta \qquad (9)$$

is a Calderon-Zygmund type strongly singular integral operator. Π_k are shown to be bounded in the space L^p for $1 < p < \infty$ and in particular their

L^2 norms are estimated in [1]. These operators are studied by decomposing them into two parts as $\Pi_k = T_{-k,k} + P_k$, where

$$T_{-k,k}f(z) = \frac{(-1)^k k}{\pi} \iint\limits_{\mathbb{D}} \left(\frac{\overline{\zeta - z}}{\zeta - z}\right)^{k-1} \frac{f(\zeta)}{(\zeta - z)^2}\, d\xi\, d\eta, \tag{10}$$

which is investigated extensively in [14].

3. A Hierarchy of Polyharmonic Kernel Functions

Iterating the harmonic Green, Neumann and Robin functions, we introduce the following hybrid Green type polyharmonic kernel functions:

Definition 1. Iterated polyharmonic kernel functions $K_{e_1,e_2,\ldots,e_n}(z,\zeta)$ are defined by

$$K_{e_1,e_2,\ldots,e_n}(z,\zeta) = \\ = \left(-\frac{1}{\pi}\right)^{n-1} \iint\limits_{\mathbb{D}} \iint\limits_{\mathbb{D}} \cdots \iint\limits_{\mathbb{D}} F_{e_1}(z,\zeta_1)\ldots F_{e_n}(\zeta_{n-1},\zeta) d\xi_1 d\eta_1 \ldots d\xi_{n-1}\, d\eta_{n-1}$$

for $z, \zeta \in \mathbb{D}$ where $\zeta_j = \xi_j + \eta_j$, $e_j \in \{1,2,3\}$ for $j = 1,2,\ldots,n$, $n \geq 2$ and $F_1 := G_1$, $F_2 := N_1$, $F_3 := R_1$.

3.1. Properties of Polyharmonic Kernel Functions.

The function $K_{e_1 e_2 \ldots e_n}(z,\zeta)$ is polyharmonic of order n in $\mathbb{D}\backslash\{\zeta\}$ for any $\zeta \in \mathbb{D}$. For any t, we can write

$$K_{e_1,e_2,\ldots,e_n}(z,\zeta) = \left(-\frac{1}{\pi}\right)^{n-t} \times \\ \times \iint\limits_{\mathbb{D}} \cdots \iint\limits_{\mathbb{D}} K_{e_1,\ldots,e_t}(z,\zeta_1) F_{e_{t+1}}(\zeta_1,\zeta_2) \ldots F_{e_n}(\zeta_{t-1},\zeta) d\xi_1 d\eta_1 \ldots d\xi_{t-1}\, d\eta_{t-1}.$$

For any t, since the functions G_1, N_1 and R_1 are fundamental solutions of $\partial_z \partial_{\bar z}$

- $(\partial_z \partial_{\bar z})^t K_{e_1,e_2,\ldots,e_n}(z,\zeta) = K_{e_{t+1},e_{t+2},\ldots,e_n}(z,\zeta)$ in $\mathbb{D}$,

in $\mathbb{D}$ for any $\zeta \in \mathbb{D}$. Moreover,

- $(\partial_z \partial_{\bar z})^\gamma K_{e_1,e_2,\ldots,e_n}(z,\zeta) = 0, \quad \gamma = j-1 \;$ for $\; e_j = 1 \;$ on $\; \partial\mathbb{D}$.
- $\partial_\nu (\partial_z \partial_{\bar z})^\gamma K_{e_1,e_2,\ldots,e_n}(z,\zeta) = H_\gamma(z,\zeta), \gamma = j-1 \;$ for $\; e_j = 2 \;$ on $\; \partial\mathbb{D}$.
- $\displaystyle\int\limits_{\partial\mathbb{D}} (\partial_z \partial_{\bar z})^\gamma K_{e_1,e_2,\ldots,e_n}(z,\zeta) \frac{dz}{z} = 0 \quad \gamma = j-1 \;$ for $\; e_j = 2 \;$ on $\; \partial\mathbb{D}$.
- $(I + \partial_\nu)(\partial_z \partial_{\bar z})^\gamma K_{e_1,e_2,\ldots,e_n}(z,\zeta) = 0, \; \gamma = j-1$ for $e_j = 3$ on $\partial\mathbb{D}$

holds for $z \in \partial\mathbb{D}$, $\zeta \in \mathbb{D}$.

For any t, since the functions G_1, N_1 and R_1 are fundamental solutions of $\partial_\zeta \partial_{\bar\zeta}$

- $(\partial_\zeta \partial_{\bar\zeta})^t K_{e_1,e_2,\dots,e_n}(z,\zeta) = K_{e_1,e_2,\dots,e_t}(z,\zeta)$ in $\mathbb{D}$,

in $\mathbb{D}$ for any $z \in \mathbb{D}$. Also

- $(\partial_\zeta \partial_{\bar\zeta})^\gamma K_{e_1,e_2,\dots,e_n}(z,\zeta) = 0, \quad \gamma = j-1$ for $e_j = 1$ on $\partial\mathbb{D}$.
- $\partial_\nu(\partial_\zeta \partial_{\bar\zeta})^\gamma K_{e_1,e_2,\dots,e_n}(z,\zeta) = H_\gamma(z,\zeta), \ \gamma = j-1$ for $e_j = 2$ on $\partial\mathbb{D}$, where $H_\gamma(z,\zeta)$ can be found as in [18].
- $\int\limits_{\partial\mathbb{D}} (\partial_\zeta \partial_{\bar\zeta})^\gamma K_{e_1,e_2,\dots,e_n}(z,\zeta)\frac{d\zeta}{\zeta} = 0 \quad \gamma = j-1$ for $e_j = 2$ on $\partial\mathbb{D}$.
- $(I+\partial_\nu)(\partial_\zeta \partial_{\bar\zeta})^\gamma K_{e_1,e_2,\dots,e_n}(z,\zeta) = 0, \ \gamma = j-1$ for $e_j = 3$ on $\partial\mathbb{D}$

holds for $\zeta \in \partial\mathbb{D}, z \in \mathbb{D}$.

In the rest of the article, using these properties, integral representation formulas will be obtained. These formulas are important for investigating the corresponding boundary value problems.

4. Iterated Mixed Problems for Higher Order Poisson Equations

The following integral representation formulas are the generalizations of the Cauchy-Pompeiu formulas.

4.1. **Integral representation formulas.**

Theorem 1. *Let $e_j \in \{1,2,3\}$ for $j = 1,2,\dots,n$. Any $w \in C^{2n}(\mathbb{D}) \cap C^{2n-1}(\overline{\mathbb{D}})$, $n \in \mathbb{N}$ can be represented as*

$$\begin{aligned} w(z) = \sum_{j=1}^{n} \frac{1}{4\pi i} \int\limits_{\partial\mathbb{D}} &(\beta_j I - \alpha_j \partial_\nu)(\partial_\zeta \partial_{\bar\zeta})^{n-j} \times \\ &\times K_{e_1,e_2,\dots,e_n}(z,\zeta)(\alpha_j I + \beta_j \partial_\nu)(\partial_\zeta \partial_{\bar\zeta})^{j-1} w(\zeta)\frac{d\zeta}{\zeta} - \\ &- \sum_{j=1}^{n} t_j \frac{1}{4\pi i} \int\limits_{\partial\mathbb{D}} \partial_\nu (\partial_\zeta \partial_{\bar\zeta})^{n-j} K_{e_1,e_2,\dots,e_n}(z,\zeta)(\partial_\zeta \partial_{\bar\zeta})^{j-1} w(\zeta)\frac{d\zeta}{\zeta} - \\ &- \frac{1}{\pi} \iint\limits_{\mathbb{D}} K_{e_1,e_2,\dots,e_n}(z,\zeta)(\partial_\zeta \partial_{\bar\zeta})^n w(\zeta) d\xi\, d\eta \end{aligned}$$

for

$$(\alpha_j,\beta_j) = \begin{cases} (1,0) & \text{if } e_j = 1 \text{ for some } j, \\ (0,1) & \text{if } e_j = 2 \text{ for some } j, \\ (1,1) & \text{if } e_j = 3 \text{ for some } j \end{cases}$$

and

$$t_j = \begin{cases} 1 & \text{if } e_j = 2 \text{ for some } j, \\ 0 & \text{if } e_j \neq 2 \text{ for some } j. \end{cases}$$

Proof. Applying the Gauss theorem iteratively to the integral

$$-\frac{1}{\pi}\iint\limits_{\mathbb{D}} K_{e_1,e_2,\ldots,e_n}(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^n w(\zeta)\, d\xi\, d\eta$$

the result follows. Because of lack of space, let us consider the case $n=2$, $e_1=1$ and $e_2=1$. Now we write

$$-\frac{1}{\pi}\iint\limits_{\mathbb{D}} K_{1,1}(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^2 w(\zeta) d\xi\, d\eta = -\frac{1}{2\pi}\iint\limits_{\mathbb{D}} \big(\partial_{\bar{\zeta}}\left[K_{1,1}(z,\zeta)\partial_\zeta^2\partial_{\bar{\zeta}} w(\zeta)\right] +$$

$$+\partial_\zeta\left[K_{1,1}(z,\zeta)\partial_\zeta\partial_{\bar{\zeta}}^2 w(\zeta)\right] - \partial_{\bar{\zeta}}\left[\partial_\zeta K_{1,1}(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})w(\zeta)\right] -$$

$$-\partial_\zeta\left[\partial_\zeta K_{1,1}(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})w(\zeta)\right] + 2\partial_\zeta\partial_{\bar{\zeta}} K_{1,1}(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})w(\zeta)\big)\, d\xi\, d\eta.$$

Applying the Gauss theorem and using the fact that $\bar{\zeta}d\zeta = -\zeta d\bar{\zeta}$ on $\partial\mathbb{D}$ with the boundary behavior of $K_{1,1}(z,\zeta)$, we obtain

$$-\frac{1}{\pi}\iint\limits_{\mathbb{D}} K_{1,1}(z,\zeta)(\partial_\zeta\partial_{\overline{\zeta}})^n w(\zeta) d\xi\, d\eta =$$

$$= \frac{1}{4\pi}\int\limits_{\partial\mathbb{D}} \partial_\nu K_{1,1}(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})^{n-1} w(\zeta) - \frac{1}{\pi}\iint\limits_{\mathbb{D}} G_1(z,\zeta)(\partial_\zeta\partial_{\bar{\zeta}})w(\zeta) d\xi\, d\eta.$$

Using the solution of the Dirichlet problem for Poisson equation (see [19]), we get the result. □

4.2. **Iterated mixed problems.** The next result gives the unique solution to the n-Poisson equation satisfying homogeneous mixed-type boundary conditions. Let $e_j \in \{1,2,3\}$ for $j=1,\ldots,n$.

Theorem 2. *Let the mixed type* $(e_1e_2\ldots e_n)$*-problem*

$$(\partial_z\partial_{\bar{z}})^n w = f \quad \text{in} \quad \mathbb{D},$$

$$(\alpha_j I + \beta_j\partial_\nu)(\partial_z\partial_{\bar{z}})^{j-1} w = 0 \quad \text{on} \quad \partial\mathbb{D}\ ,$$

where $j=1,\ldots,n$ *for*

$$(\alpha_j,\beta_j) = \begin{cases} (1,0) & \textit{if} \ \ e_j = 1 \ \ \text{for some } j \ \ (\textbf{Dirichlet condition}),\\ (0,1) & \textit{if} \ \ e_j = 2 \ \ \text{for some } j \ \ (\textbf{Neumann condition}),\\ (1,1) & \text{if} \ \ \mathrm{e_j} = 3 \ \ \text{for some } j \ \ (\textbf{Robin condition}),\end{cases}$$

with the normalization conditions

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} t_j(\partial_z\partial_{\bar{z}})^{j-1} w(\zeta)\frac{d\zeta}{\zeta} = 0 \tag{11}$$

for

$$t_j = \begin{cases} 1 & \textit{if} \ \ e_j = 2 \ \ \text{for some } j,\\ 0 & \textit{if} \ \ e_j \neq 2 \ \ \text{for some } j.\end{cases}$$

This problem is uniquely solvable for $f \in L^p(\mathbb{D})$ *if*

$$\frac{1}{\pi}\iint\limits_{\mathbb{D}} \partial_{\nu_z} N_{n-\gamma}(z,\zeta) f(\zeta) d\xi\, d\eta = 0 \quad \gamma = j-1, \ \textit{for } e_j = 2 \tag{12}$$

is satisfied. The solution is

$$w(z) = -\frac{1}{\pi}\iint\limits_{\mathbb{D}} K_{e_1,e_2,\ldots,e_n}(z,\zeta) f(\zeta) d\xi\, d\eta.$$

Proof. The case $n = 1$ corresponds to the Dirichlet, Neumann and Robin boundary value problems for Poisson equation. Such problems are studied in [19]. In the case $n = 2$, H. Begehr stated nine mixed problems (such as Dirichlet-Dirichlet, Neumann-Robin, Robin- Dirichlet...) in [21] and gave the explicit solutions. Therefore, we will consider only the proof of the case $n > 2$. We need to discuss only the existence of the solution, since the uniqueness of the solution comes from the integral representation given in Theorem 1. Using the polyharmonic property of $K_{e_1,e_2,\ldots,e_n}(z,\zeta)$, one can easily see that the representation for w in Theorem 1 satisfies the equation. To show that it satisfies the boundary condition, boundary behaviors of $K_{e_1,e_2,\ldots,e_n}(z,\zeta)$ given in Section 3 are used. □

The next result gives the unique solution to an equation of arbitrary order satisfying mixed-type boundary conditions. Let $e_j \in \{1, 2, 3\}$ for $j = 1, \ldots, n$.

Theorem 3. *The mixed type* $(e_1 e_2 \ldots e_n)$*-Schwarz problem given by*

$$\begin{gathered} \partial_z^m \partial_{\bar z}^n w = f \quad \text{in} \quad \mathbb{D}, \\ \operatorname{Re} \frac{\partial^l w}{\partial \bar z^l} = 0 \ \ \textit{on} \ \ \partial\mathbb{D}, \qquad \operatorname{Im} \frac{\partial^l w}{\partial \bar z^l}(0) = 0, \ \ 0 \le l \le n-m-1 \\ (\alpha I + \beta \partial_\nu) \partial_z^\gamma \partial_{\bar z}^{\gamma+n-m} w = 0 \quad \text{on} \quad \partial\mathbb{D}\ , \end{gathered} \tag{13}$$

where $\gamma = j-1$, $j = 1, \ldots, m$ *for*

$$(\alpha_j, \beta_j) = \begin{cases} (1,0) & \textit{if} \ \ e_j = 1 \ \ \text{for some } j \ \ (\textbf{Dirichlet condition}), \\ (0,1) & \textit{if} \ \ e_j = 2 \ \ \text{for some } j \ \ (\textbf{Neumann condition}), \\ (1,1) & \text{if} \ \ \mathrm{e_j} = 3 \ \ \text{for some } j \ \ (\textbf{Robin condition}), \end{cases}$$

$$\frac{1}{2\pi i}\int\limits_{\partial\mathbb{D}} t_j (\partial_z \partial_{\bar z})^{j-1} w(\zeta) \frac{d\zeta}{\zeta} = 0 \tag{14}$$

for

$$t_j = \begin{cases} 1 & \textit{if} \ \ e_j = 2 \ \ \text{for some } j, \\ 0 & \textit{if} \ \ e_j \ne 2 \ \ \text{for some } j \end{cases}$$

is uniquely solvable for $f \in L^p(\mathbb{D})$ *if*

$$\frac{1}{\pi}\iint_{\mathbb{D}} \partial_{\nu_z} N_{m-\gamma}(z,\zeta)f(\zeta)d\xi d\eta = 0 \ \gamma = j-1 \ \textit{ for } \ e_j = 2 \tag{15}$$

is satisfied. The solution is

$$w(z) = \tilde{T}_{n-m}\Big(- \frac{1}{\pi}\iint_{\mathbb{D}} K_{e_1,e_2,\dots,e_m}(z,\zeta)f(\zeta)d\xi\, d\eta\Big) \ .$$

Proof. The solution is obtained by decomposing the above problem as

$$\partial_{\bar{z}}^{n-m} w = w_1,$$
$$\operatorname{Re}\frac{\partial^l w}{\partial \bar{z}^l} = 0 \ \text{ on } \ \partial\mathbb{D}, \qquad \operatorname{Im}\frac{\partial^l w}{\partial \bar{z}^l}(0) = 0, \ \ 0 \le l \le n-m-1$$

and

$$(\partial_z\partial_{\bar{z}})^m w_1 = f,$$
$$(\alpha I + \beta\partial_\nu)(\partial_z\partial_{\bar{z}})^\gamma w_1 = 0 \ \text{ on } \ \partial\mathbb{D} \ ,$$

where $\gamma = j-1$, $j = 1,\dots,m$ and using the representations of the solutions of mixed type $(e_1e_2\dots e_n)$ and Schwarz problems iteratively. □

Remark 2.

a) If the Schwarz conditions are given for $m \le l \le n-m-1$, then the solution can be found by decomposing the problem as

$$(\partial_z\partial_{\bar{z}}(\partial_z\partial_{\bar{z}})^m w = w_1,$$
$$(\alpha I + \beta\partial_\nu)(\partial_z\partial_{\bar{z}})^\gamma w = 0 \ \text{ on } \ \partial\mathbb{D} \ ,$$

where $\gamma = j-1$, $j = 1,\dots,m$ and

$$\partial_{\bar{z}}^{n-m} w_1 = f,$$
$$\operatorname{Re}\frac{\partial^l w_1}{\partial \bar{z}^l} = 0 \ \text{ on } \partial\mathbb{D}, \qquad \operatorname{Im}\frac{\partial^l w_1}{\partial \bar{z}^l}(0) = 0, \ \ m \le l \le n-1.$$

b) If the first m_1 conditions are related to the arbitrary iteration of Neumann, Dirichlet, Robin conditions, the second set of $n-m$ conditions are the Schwarz conditions and the third $m - m_1$ set of conditions are again the arbitrary iteration of Neumann, Dirichlet, Robin conditions, as in the above cases, this problem can be decomposed into a system of coupled equations which can be treated easily.

5. Iterated Mixed Problems for Linear Complex Partial Differential Equations

5.1. **Operators Related to Mixed-Type Boundary Value Problems.** In the following, using a hierarchy of polyharmonic kernels of the harmonic Green, Neumann and Robin problems in an arbitrary order, a class of integral operators are given.

Definition 2.

$$I^{k,l}_{e_1,e_2,\dots,e_n} f(z) = -\frac{1}{\pi} \iint\limits_{\mathbb{D}} \partial_z^k \partial_{\bar{z}}^l K_{e_1,e_2,\dots,e_n}(z,\zeta) f(\zeta) d\xi\, d\eta\ .$$

for a suitable complex valued f.

The operator $I^{k,l}_{e_1,e_2,\dots,e_n}$ may be considered as an iteration of the operators $S_{1,k,l}$, $G^{k,l}_{n,n-1}$ and $R_{1,k,l}$. Thus, the following theorems can be proved using the boundedness and continuity properties given for these operators, see [4, 5, 7].

Theorem 4.

(a) *For* $z \in \mathbb{D}$, $f \in L^p(\mathbb{D})$ *with* $p > 1$ *and* $k + l < 2n$,

$$\left| I^{k,l}_{e_1,e_2,\dots,e_n} f(z) \right| \le C \|f\|_{L^p(\mathbb{D})}. \tag{16}$$

(b) *For* z_1, $z_2 \in \mathbb{D}$, $f \in L^p(\mathbb{D})$ *with* $p > 2$

$$\begin{aligned} &\left| I^{k,l}_{e_1,e_2,\dots,e_n} f(z_1) - I^{k,l}_{e_1,e_2,\dots,e_n} f(z_2) \right| \le \\ &\le C \|f\|_{L^p(\mathbb{D})} \begin{cases} |z_1 - z_2|^{(p-2)/p} & \text{if} \quad k + l = 2n - 1, \\ |z_1 - z_2| & \text{if} \quad k + l < 2n - 1. \end{cases} \end{aligned}$$

(c) $I^{k,l}_{e_1,e_2,\dots,e_n} f \in L^p(\mathbb{D})$ *for* $f \in L^p(\mathbb{D})$ *with* $p > 1$ *where* $k + l = 2n$ *and*

$$\|I^{k,l}_{e_1,e_2,\dots,e_n} f\|_{L^p(\mathbb{D})} \le C_p \|f\|_{L^p(\mathbb{D})}\ . \tag{17}$$

Proof. Since the kernels $K_{e_1,e_2,\dots,e_n}(z,\zeta)$ are defined as an arbitrary iterations of harmonic Green, Neumann and Robin functions, the operator $I^{k,l}_{e_1,e_2,\dots,e_n}$ is simply a convolution of the operators $G^{0,0}_{0,1} f$, $S_{n,k,l}$ and $R_{n,1,1}$ defined in preliminaries. Since the statements of the theorem are valid for these operators, it is trivial how to prove them for the operator $I^{k,l}_{e_1,e_2,\dots,e_n}$. □

5.2. **Iterated mixed problems.** Now, we consider the mixed problems for the generalized higher-order Poisson equation. For simplicity, homogeneous boundary conditions are discussed.

Mixed $(e_1e_2\dots e_n)$-Problem. Find $w \in W^{2n,p}(\mathbb{D})$ as a solution to equation

$$\frac{\partial^{2n} w}{\partial z^n \partial \bar{z}^n} + \sum_{\substack{k+l=2n\\(k,l)\neq(n,n)}} \left(q_{kl}^{(1)}(z) \frac{\partial^{2n} w}{\partial z^k \partial \bar{z}^l} + q_{kl}^{(2)}(z) \frac{\partial^{2n} \overline{w}}{\partial \bar{z}^k \partial z^l} \right) + \\ + \sum_{0\leq k+l<2n} \left[a_{kl}(z) \frac{\partial^{k+l} w}{\partial z^k \partial \bar{z}^l} + b_{kl}(z) \frac{\partial^{k+l} \overline{w}}{\partial \bar{z}^k \partial z^l} \right] = f(z) \tag{18}$$

satisfying

$$(\alpha_j I + \beta_j \partial_\nu)(\partial_z \partial_{\bar{z}})^{j-1} w = 0 \quad \text{on} \quad \partial\mathbb{D},$$

where $j = 1, \dots, n$ for

$$(\alpha_j, \beta_j) = \begin{cases} (1,0) & \text{if } \; e_j = 1 \; \text{ for some } \; \mathrm{j}\,, \\ (0,1) & \text{if } \; e_j = 2 \; \text{ for some } \; \mathrm{j}\,, \\ (1,1) & \text{if } \; \mathrm{e_j} = 3 \; \text{ for some } \; \mathrm{j}\,, \end{cases}$$

$$\frac{1}{2\pi i} \int_{\partial\mathbb{D}} t_j (\partial_z \partial_{\bar{z}})^{j-1} w(\zeta) \frac{d\zeta}{\zeta} = 0 \tag{19}$$

in which

$$a_{kl}, b_{kl}, f \in L^p(\mathbb{D}), \tag{20}$$

and $q_{kl}^{(1)}$ and $q_{kl}^{(2)}$, are measurable bounded functions subject to

$$\sum_{\substack{k+l=2n\\k\neq l}} (|q_{kl}^{(1)}(z)| + |q_{kl}^{(2)}(z)|) \leq q_0 < 1, \tag{21}$$

To derive the solutions of the iterated mixed boundary value problem, we start with transforming the equation (18) to a singular integral equation.

Lemma 1. *The $(e_1e_2\dots e_n)$-problem is equivalent to the singular integral equation*

$$(I + \hat{M} + \hat{K})g = f \tag{22}$$

if

$$w = I^{0,0}_{e_1,e_2,\dots,e_n} g\ ,$$

where

$$\hat{M}g = \sum_{\substack{k+l=2n\\k\neq l}} (q_{kl}^{(1)} I^{k,l}_{e_1,e_2,\dots,e_n} g + q_{kl}^{(2)} \overline{I^{k,l}_{e_1,e_2,\dots,e_n} g})\ ,$$

$$\hat{K}g = \sum_{k+l<2n} \left(a_{kl} I^{k,l}_{e_1,e_2,\dots,e_n} g + b_{kl} \overline{I^{k,l}_{e_1,e_2,\dots,e_n} g} \right)\ .$$

For the proof, we use the ideas given in [4], for example.

5.3. **Solvability of the Mixed $(e_1e_2\ldots e_n)$-Problem.** Lastly, we will give solvability result for such boundary value problems.

Theorem 5. *If*

$$q_0 \max_{k+l=2n} \|I^{k,l}_{e_1,e_2,\ldots,e_n}\|_{L^p(\mathbb{D})} \leq 1 \tag{23}$$

is satisfied, then the Fredholm alternative applies the equation (22) *and the* $(e_1e_2\ldots e_n)$*-problem has a solution of the form*

$$w = I^{0,0}_{e_1,e_2,\ldots,e_n} g \tag{24}$$

where $g \in L^p(\mathbb{D})$*,* $p > 2$*, is a solution of the singular integral equation* (22) *satisfying the conditions*

$$\frac{1}{\pi}\iint\limits_{\mathbb{D}} \partial_{\nu_z} N_{n-\gamma}(z,\zeta) g(\zeta) d\xi d\eta = 0 \quad \gamma = j-1, \ \textit{for } e_j = 2 \ .$$

Proof. As an outline of the proof, we can state that, $\hat{K}$ is compact by Theorem 4 parts (a) and (b), $\hat{M}$ is L^p bounded by Theorem 4 part (c). $I+\hat{M}$ is invertible if (23) is satisfied, which leads that $I+\hat{M}+\hat{K}$ is a Fredholm operator with index zero. Thus the representation (24) is valid. □

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