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FRACTIONAL ORDER HYPERBOLIC DIFFERENTIAL INCLUSIONS WITH INFINITE DELAY

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Abstract. This paper deals with the existence of solutions for the initial value problems (IVP for short), for fractional order hyperbolic and neutral hyperbolic functional differential inclusions with infinite delay. Our work will be considered by using the nonlinear alternative of Leray-Schauder type.

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1. Introduction

This paper deals with the initial value problems (IVP for short) for hyperbolic functional differential inclusions

$$({}^cD_0^r u)(x,y) \in F(x,y,u_{(x,y)}),\ \text{if } (x,y) \in J, \tag{1}$$

$$u(x,y) = \phi(x,y),\ \text{if } (x,y) \in \tilde{J}, \tag{2}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y),\ x \in [0,a],\ y \in [0,b], \tag{3}$$

where $J = [0,a] \times [0,b],\ a,b > 0,\ \tilde{J} = (-\infty,a] \times (-\infty,b] \backslash (0,a] \times (0,b],\ {}^cD_0^r$ is the standard Caputo's fractional derivative of order $r = (r_1, r_2) \in (0,1] \times (0,1]$, $F : J \times \mathcal{B} \to \mathcal{P}(\mathbb{R}^n)$ is a multivalued map with compact, convex values, $\mathcal{P}(\mathbb{R}^n)$ is the family of all subsets of $\mathbb{R}^n$, $\phi : \tilde{J} \to \mathbb{R}^n$ is a given continuous function, $\varphi : [0,a] \to \mathbb{R}^n$, $\psi : [0,b] \to \mathbb{R}^n$ are given absolutely continuous functions such that $\varphi(x) = \phi(x,0)$, $\psi(y) = \phi(0,y)$ for each $x \in [0,a]$, $y \in [0,b]$ and $\mathcal{B}$ is called a phase space that will be specified in Section 3.

We denote by $u_{(x,y)}$ the element of $\mathcal{B}$ defined by

$$u_{(x,y)}(s,t) = u(x+s, y+t);\ (s,t) \in (-\infty,0] \times (-\infty,0].$$

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Next we consider the following initial value problem for neutral hyperbolic functional differential inclusions

$$ {}^cD_0^r[u(x,y) - g(x,y,u_{(x,y)})] \in F(x,y,u_{(x,y)}), \text{ if } (x,y) \in J, \tag{4} $$

$$ u(x,y) = \phi(x,y), \text{ if } (x,y) \in \tilde{J}, \tag{5} $$

$$ u(x,0) = \varphi(x), \ u(0,y) = \psi(y), \quad x \in [0,a], \ y \in [0,b], \tag{6} $$

where F, ϕ, φ, ψ are as in problem (1)-(3) and $g : J \times \mathcal{B} \to \mathcal{P}(\mathbb{R}^n)$ is a given continuous function.

The problem of existence of solutions of Cauchy-type problems for ordinary differential equations of fractional order in spaces of integrable functions without delay was studies in numerous works (see [27, 40]), a similar problem in spaces of continuous functions was studies in [41]. We can find numerous applications of differential equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [15, 17, 18, 23, 34, 35]). There has been a significant development in ordinary and partial fractional differential equations in recent years; see the monographs of Kilbas citeKiSrJuTr, Lakshmikantham *et al.* [31], Miller and Ross [36], Samko *et al.* [39], the papers of Abbas and Benchohra [1, 2], Agarwal *et al.* [3], Benchohra *et al.* [5, 6, 7], Belarbi *et al.* [4], Diethelm [15, 16], Kilbas and Marzan [28], Mainardi [34], Podlubny [38], Vityuk and Golushkov [42], Zhou *et al.* [43, 44, 45] and the references therein.

In this paper, we present existence results for problems (1)-(3) and (4)-(6). Our approach here is based on the nonlinear alternative of Leray-Schauder type for multivalued operators. The present results extend those considered with integer order derivative [8, 9, 10, 13, 25, 26, 33, 37], and those with fractional derivative [1, 2, 28].

This paper initiates the study of fractional order differential inclusions with infinite delay involving the Caputo fractional derivative.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1(J, \mathbb{R}^n)$ we denote the space of Lebesgue-integrable functions $u : J \to \mathbb{R}^n$ with the norm

$$ \|u\|_{L^1} = \int_0^a \int_0^b \|u(x,y)\| dy dx, $$

where $\|.\|$ denotes a suitable complete norm on $\mathbb{R}^n$. $C(J, \mathbb{R}^n)$ is the Banach space of continuous functions on J normed by

$$ \|u\|_\infty = \sup_{(x,y)\in J} \|u(x,y)\|, $$

and $AC(J,\mathbb{R}^n)$ is the space of absolutely continuous functions from J into $\mathbb{R}^n$.

Definition 2.1 ([38])**.** Let $r_1,\ r_2 > 0$ and $r = (r_1, r_2)$. For $u \in L^1(J,\mathbb{R}^n)$, the expression

$$(I_0^r u)(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}u(s,t)dtds,$$

where $\Gamma(.)$ is the gamma function, is called the left-sided mixed Riemann-Liouville integral of order r.

Definition 2.2 ([38])**.** For $u \in L^1(J,\mathbb{R}^n)$, the Caputo fractional-order derivative of order r is defined by the expression

$$({}^cD_0^r u)(x,y) = (I_0^{1-r}\frac{\partial^2}{\partial x \partial y}u)(x,y).$$

We need also some properties of set-valued maps. Let $(X, \|\cdot\|)$ be a Banach space. let $P_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \,\text{closed}\}$, $P_b(X) = \{Y \in \mathcal{P}(X) : Y \,\text{bounded}\}$, $P_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$ and $P_{cp,c}(X) = \{Y \in \mathcal{P}(X) : Y \,\text{compact and convex}\}$.

Definition 2.3. A multivalued map $T : X \to P(X)$ is convex(closed) valued if $T(x)$ is convex (closed) for all $x \in X$.
A multivalued map $T : X \to P(X)$ is bounded on bounded sets if $T(B) = \cup_{x\in B}T(x)$ is bounded in X for all $B \in P_b(X)$ (i.e. $\sup\limits_{x\in B}\{\sup\{\|y\| : y \in T(x)\}\} < \infty$).
A multivalued map $T : X \to P(X)$ is called upper semi-continuous (u.s.c.) on X if for each $x_0 \in X$, the set $T(x_0)$ is a nonempty closed subset of X, and if for each open set N of X containing $T(x_0)$, there exists an open neighborhood N_0 of x_0 such that $T(N_0) \subseteq N$.
A multivalued map $T : X \to P(X)$ is said to be completely continuous if $T(B)$ is relatively compact for every $B \in P_b(X)$. A multivalued map $T : X \to P(X)$ has a fixed point if there is $x \in X$ such that $x \in T(x)$. The fixed point set of the multivalued operator T will be denoted by $FixT$.
A multivalued map $T : X \to P_{cl}(\mathbb{R}^n)$ is said to be measurable if for every $v \in \mathbb{R}^n$, the function

$$x \longmapsto d(v, T(x)) = \inf\{\|v - z\| : z \in T(x)\}$$

is measurable.

Definition 2.4. A multivalued map $F : J \times \mathcal{B} \to \mathcal{P}(\mathbb{R}^n)$ is said to be Carathéodory if

(i) $(x, y) \longmapsto F(x, y, u)$ is measurable for each $u \in \mathcal{B}$;
(ii) $u \longmapsto F(x, y, u)$ is upper semicontinuous for almost all $(x, y) \in J$.

For each $u \in C((-\infty, a] \times (-\infty, b], \mathbb{R}^n)$, define the set of selections of F by

$$S_{F,u} = \{f \in L^1(J, \mathbb{R}^n) : f(x, y) \in F(x, y, u_{(x,y)}) \ a.e. \ (x, y) \in J\}.$$

Let (X, d) be a metric space induced from the normed space $(X, \|\cdot\|)$. Consider $H_d : \mathcal{P}(X) \times \mathcal{P}(X) \longrightarrow \mathbb{R}_+ \cup \{\infty\}$ given by

$$H_d(A, B) = \max\left\{\sup_{a\in A} d(a, B), \sup_{b\in B} d(A, b)\right\},$$

where $d(A, b) = \inf_{a\in A} d(a, b)$, $d(a, B) = \inf_{b\in B} d(a, b)$. Then $(P_{b,cl}(X), H_d)$ is a metric space and $(P_{cl}(X), H_d)$ is a generalized metric space (see [29]).

Theorem 2.5 ([19]). (*Nonlinear alternative of Leray Schauder type*) *Let X be a Banach space and C a nonempty convex subset of X. Let U a nonempty open subset of C with $0 \in U$ and $T : \overline{U} \to \mathcal{P}(C)$ an upper semicontinuous and compact multivalued operator. Then either*

(a) *T has fixed points. Or*
(b) *There exist $u \in \partial U$ and $\lambda \in [0, 1]$ with $u \in \lambda T(u)$.*

3. The Phase Space $\mathcal{B}$

The notation of the phase space $\mathcal{B}$ plays an important role in the study of both qualitative and quantitative theory for functional differential equations. A usual choice is a semi-normed space satisfying suitable axioms, which was introduced by Hale and Kato (see [21, 24, 32]).

For any $(x, y) \in J$ denote $E_{(x,y)} := [0, x] \times \{0\} \cup \{0\} \times [0, y]$, furthermore in case $x = a$, $y = b$ we write simply E. Consider the space $(\mathcal{B}, \|(.,.)\|_{\mathcal{B}})$ is a seminormed linear space of functions mapping $(-\infty, 0] \times (-\infty, 0]$ into $\mathbb{R}^n$, and satisfying the following fundamental axioms which were adapted from those introduced by Hale and Kato for ordinary differential functional equations:

(A_1) If $z : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n$ continuous on J and $z_{(x,y)} \in \mathcal{B}$, for all $(x, y) \in E$, then there are constants $H, K, M > 0$ such that for any $(x, y) \in J$ the following conditions hold:
(i) $z_{(x,y)}$ is in $\mathcal{B}$;
(ii) $\|z(x, y)\| \le H\|z_{(x,y)}\|_{\mathcal{B}}$,
(iii) $\|z_{(x,y)}\|_{\mathcal{B}} \le K \sup_{(s,t)\in[0,x]\times[0,y]} \|z(s, t)\| + M \sup_{(s,t)\in E_{(x,y)}} \|z_{(s,t)}\|_{\mathcal{B}}$,
(A_2) For the function $z(.,.)$ in (A_1), $z_{(x,y)}$ is a $\mathcal{B}$-valued continuous function on J.
(A_3) The space $\mathcal{B}$ is complete.

Now, we present some examples of phase spaces.

Example 3.1. *Let B be the set of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$, $\alpha, \beta \geq 0$, with the seminorm*

$$\|\phi\|_B = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\|.$$

Then we have $H = K = M = 1$. The quotient space $\widehat{B} = B/\|.\|_B$ is isometric to the space $C([-\alpha, 0] \times [-\beta, 0], \mathbb{R}^n)$ of all continuous functions from $[-\alpha, 0] \times [-\beta, 0]$ into $\mathbb{R}^n$ with the supremum norm, this means that partial differential functional equations with finite delay are included in our axiomatic model.

Example 3.2. *Let C_γ be the set of all continuous functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ for which a limit $\lim_{\|(s,t)\|\to\infty} e^{\gamma(s+t)}\phi(s,t)$ exists, with the norm*

$$\|\phi\|_{C_\gamma} = \sup_{(s,t)\in(-\infty,0]\times(-\infty,0]} e^{\gamma(s+t)}\|\phi(s,t)\|.$$

Then we have $H = K = M = 1$.

Example 3.3. *Let $\alpha, \beta, \gamma \geq 0$ and let*

$$\|\phi\|_{CL_\gamma} = \sup_{(s,t)\in[-\alpha,0]\times[-\beta,0]} \|\phi(s,t)\| + \int_{-\infty}^{0}\int_{-\infty}^{0} e^{\gamma(s+t)}\|\phi(s,t)\|dtds.$$

be the seminorm for the space CL_γ of all functions $\phi : (-\infty, 0] \times (-\infty, 0] \to \mathbb{R}^n$ which are continuous on $[-\alpha, 0] \times [-\beta, 0]$ measurable on $(-\infty, -\alpha] \times (-\infty, 0] \cup (-\infty, 0] \times (-\infty, -\beta]$, and such that $\|\phi\|_{CL_\gamma} < \infty$. Then

$$H = 1,\ K = \int_{-\alpha}^{0}\int_{-\beta}^{0} e^{\gamma(s+t)}dtds,\ M = 2.$$

4. Main Results

Let us start by defining what we mean by a solution of the problem (1)-(3). Let the space

$$\Omega := \{u : (-\infty, a] \times (-\infty, b] \to \mathbb{R}^n :\ u_{(x,y)} \in \mathcal{B}$$
$$\text{for } (x,y) \in E \text{ and } u|_J \text{ is continuous}\}.$$

Definition 4.1. A function $u \in \Omega$ is said to be a solution of (1)-(3) if there exists a function $f \in L^1(J, \mathbb{R}^n)$ with $f(x,y) \in F(x,y,u_{(x,y)})$ such that $({}^cD_0^r u)(x,y) = f(x,y)$ and u satisfies (3) on J and the condition (2) on $\tilde{J}$.

Let $f \in L^1(J, \mathbb{R}^n)$ and consider the following problem

$$\begin{cases} ({}^cD_0^r u)(x,y) = f(x,y),\ (x,y) \in J, \\ u(x,0) = \varphi(x),\ u(0,y) = \psi(y),\ \varphi(0) = \psi(0). \end{cases} \tag{7}$$

For the existence of solutions for the problem (1)-(3), we need the following lemma:

Lemma 4.2 ([1, 2]). *A function $u \in AC(J, \mathbb{R}^n)$ is a solution of problem (7) if and only if $u(x,y)$ satisfies*

$$u(x,y) = \mu(x,y) + (I_0^r f)(x,y); (x,y) \in J, \tag{8}$$

where

$$\mu(x,y) = \varphi(x) + \psi(y) - \varphi(0).$$

Remark 4.3. *A function $u \in \Omega$ is said to be a solution of (2) and (7) if and only if $u(x,y)$ satisfies (8) on J and the condition (2) on $\tilde{J}$.*

In the sequel we will make use of the following generalization of Gronwall's lemma for two independent variables and singular kernel.

Lemma 4.4 ([22]). *Let $\upsilon : J \to [0,\infty)$ be a real function and $\omega(.,.)$ be a nonnegative, locally integrable function on J. If there are constants $c > 0$ and $0 < r_1, r_2 < 1$ such that*

$$\upsilon(x,y) \leq \omega(x,y) + c\int_0^x\int_0^y \frac{\upsilon(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

then there exists a constant $k = k(r_1, r_2)$ such that

$$\upsilon(x,y) \leq \omega(x,y) + kc\int_0^x\int_0^y \frac{\omega(s,t)}{(x-s)^{r_1}(y-t)^{r_2}}dtds,$$

for every $(x,y) \in J$.

Theorem 4.5. *Assume*
$(H1)$ $F : J \times \mathcal{B} \to \mathcal{P}_{cp,c}(\mathbb{R}^n)$ is a Carathéodory multi-valued map;
$(H2)$ there exists $l \in L^\infty(J, \mathbb{R})$ such that

$$H_d(F(x,y,u), F(x,y,\overline{u})) \leq l(x,y)\|u - \overline{u}\|_B \text{ for every } u, \overline{u} \in \mathcal{B},$$

and

$$d(0, F(x,y,0)) \leq l(x,y), \text{ a.e. } (x,y) \in J.$$

Then the IVP $(1)-(3)$ has at least one solution on $(-\infty, a] \times (-\infty, b]$.

Proof. Let $l^* = \|l\|_{L^\infty}$. Transform the problem (1)-(3) into a fixed point problem. Consider the multivalued operator $N : \Omega \to \mathcal{P}(\Omega)$ defined by

$$N(x,y) = \{h \in \Omega\},$$

such that

$$h(x,y) = \begin{cases} \phi(x,y), & (x,y) \in \tilde{J}, \\ \mu(x,y) + \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x \times & \\ \times \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)dtds, \; f \in S_{F,u}, & (x,y) \in J. \end{cases}$$

Let $v(.,.):(-\infty,a]\times(-\infty,b]\to\mathbb{R}^n$ be a function defined by,

$$v(x,y)=\begin{cases}\phi(x,y), & (x,y)\in\tilde{J},\\ \mu(x,y), & (x,y)\in J.\end{cases}$$

Then $v_{(x,y)}=\phi$ for all $(x,y)\in E$. For each $w\in C(J,\mathbb{R}^n)$ with $w(0,0)=0$, we denote by $\overline{w}$ the function defined by

$$\overline{w}(x,y)=\begin{cases}0, & (x,y)\in\tilde{J},\\ w(x,y) & (x,y)\in J.\end{cases}$$

If $u(.,.)$ satisfies the integral equation,

$$u(x,y)=\mu(x,y)+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)dtds,$$

we can decompose $u(.,.)$ as $u(x,y)=\overline{w}(x,y)+v(x,y)$; $(x,y)\in J$, which implies $u_{(x,y)}=\overline{w}_{(x,y)}+v_{(x,y)}$, for every $(x,y)\in J$, and the function $w(.,.)$ satisfies

$$w(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)dtds,$$

where $f\in S_{F,\overline{w}_{(s,t)}+v_{(s,t)}}$. Set

$$C_0=\{w\in C(J,\mathbb{R}^n):\ w(x,y)=0\ \text{ for } (x,y)\in E\},$$

and let $\|.\|_{(a,b)}$ be the seminorm in C_0 defined by

$$\|w\|_{(a,b)}=\sup_{(x,y)\in E}\|w_{(x,y)}\|_B+\sup_{(x,y)\in J}\|w(x,y)\|=\sup_{(x,y)\in J}\|w(x,y)\|,\ w\in C_0.$$

C_0 is a Banach space with norm $\|.\|_{(a,b)}$. Let the operator $P:\ C_0\to\mathcal{P}(C_0)$ be defined by

$$(Pw)(x,y)=\{h\in C_0\},$$

such that

$$h(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)dtds,\ (x,y)\in J,$$

where $f\in S_{F,\overline{w}_{(s,t)}+v_{(s,t)}}$. Obviously, that the operator N has a fixed point is equivalent to P has a fixed point.

Step 1: *$P(u)$ is convex for each $u\in C_0$.*

Indeed, if $h_1,\ h_2$ belong to $P(u)$, then there exist $f_1,\ f_2\in S_{F,\overline{w}_{(s,t)}+v_{(s,t)}}$ such that for each $(x,y)\in J$ we have

$$h_i(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f_i(s,t)dtds,\quad i=1,2.$$

Let $0 \leq \xi \leq 1$. Then, for each $(x,y) \in J$, we have

$$\begin{aligned}(\xi h_1 + (1-\xi)h_2)(x,y) &= \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \\ &\quad \times [\xi f_1(s,t) + (1-\xi) f_2(s,t)] dtds.\end{aligned}$$

Since $\tilde{S}_{F,\overline{w}_{(s,t)}+v_{(s,t)}}$ is convex (because F has convex values), we have

$$\xi h_1 + (1-\xi)h_2 \in P(u).$$

Step 2: *P maps bounded sets into bounded sets in C_0:* Indeed, it is enough to show that there exists a positive constant ℓ such that, for each $z \in B_\rho = \{u \in C_0 : \|z\| \leq \rho\}$, one has $\|P(z)\| \leq \ell$. Let $z \in B_\rho$ and $h \in P(z)$ Then there exists $f \in S_{F,\overline{w}_{(s,t)}+v_{(s,t)}}$, such that, for each $(x,y) \in J$, we have

$$h(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t) dtds.$$

Then, for each $(x,y) \in J$,

$$\begin{aligned}\|h(x,y)\| &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} \|f(s,t)\| dtds \leq \\ &\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} l(s,t)(1+ \\ &\quad + \|\overline{w}_{(s,t)} + v_{(s,t)}\|_{\mathcal{B}}) dtds \leq \\ &\leq \frac{l^*(1+\rho^*)}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} dtds \leq \\ &\leq \frac{a^{r_1} b^{r_2} l^*(1+\rho^*)}{\Gamma(r_1+1)\Gamma(r_2+1)},\end{aligned}$$

where

$$\begin{aligned}\|\overline{w}_{(s,t)} + v_{(s,t)}\|_{\mathcal{B}} &\leq \|\overline{w}_{(s,t)}\|_{\mathcal{B}} + \|v_{(s,t)}\|_{\mathcal{B}} \leq \\ &\leq K\rho + M\|\phi\|_{\mathcal{B}} = \rho^*.\end{aligned}$$

Step 3: *$P(B_\rho)$ is equicontinuous.* Let $P(B_\rho)$ as in Step 2 and let $(x_1,y_1),(x_2,y_2) \in J$, $x_1 < x_2$ and $y_1 < y_2$, let $u \in B_\rho$ and $h \in P(u)$, then there exists $f \in S_{F,\overline{w}_{(s,t)}+v_{(s,t)}}$ such that for each $(x,y)\in J$ we have

$$\|h(x_2,y_2)-h(x_1,y_1)\| =$$

$$= \Big\| \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_0^{y_1} [(x_2-s)^{r_1-1}(y_2-t)^{r_2-1} - (x_1-s)^{r_1-1}(y_1-t)^{r_2-1}] \times$$

$$\times f(s,t)dtds + \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} f(s,t)dtds +$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} f(s,t)dtds +$$

$$+ \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} f(s,t)dtds \Big\| \leq$$

$$\leq \frac{l^*(1+\rho^*)}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_0^{y_1} [(x_1-s)^{r_1-1}(y_1-t)^{r_2-1} - (x_2-s)^{r_1-1}(y_2-t)^{r_2-1}]dtds +$$

$$+ \frac{l^*(1+\rho^*)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} dtds +$$

$$+ \frac{l^*(1+\rho^*)}{\Gamma(r_1)\Gamma(r_2)} \int_0^{x_1}\int_{y_1}^{y_2} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} dtds +$$

$$+ \frac{l^*(1+\rho^*)}{\Gamma(r_1)\Gamma(r_2)} \int_{x_1}^{x_2}\int_0^{y_1} (x_2-s)^{r_1-1}(y_2-t)^{r_2-1} dtds \leq$$

$$\leq \frac{l^*(1+\rho^*)}{\Gamma(r_1+1)\Gamma(r_2+1)} [2y_2^{r_2}(x_2-x_1)^{r_1} + 2x_2^{r_1}(y_2-y_1)^{r_2} +$$
$$+ x_1^{r_1}y_1^{r_2} - x_2^{r_1}y_2^{r_2} - 2(x_2-x_1)^{r_1}(y_2-y_1)^{r_2}].$$

As $x_1 \longrightarrow x_2$ and $y_1 \longrightarrow y_2$, the right-hand side of the above inequality tends to zero. As a consequence of Steps 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that $N : C_0 \longrightarrow \mathcal{P}_{cp}(C_0)$ is completely continuous.

Step 4: *P has a closed graph.*

Let $u_n \to u_*$, $h_n \in P(u_n)$ and $h_n \to h_*$. We need to show that $h_* \in P(u_*)$.
$h_n \in P(u_n)$ means that there exists $f_n \in S_{F,\overline{w}_{(s,t)}+v_{(s,t)}}$ such that, for each

$(x,y)\in J$,

$$h_n(x,y)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f_n(s,t)dtds.$$

We must show that there exists $f_*\in \tilde{S}_{F,\overline{w}_{(s,t)}+v_{(s,t)})}$ such that, for each $(x,y)\in J$,

$$h_*(x,y) \quad = \quad \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f_*(s,t)dtds.$$

Since $F(x,y,\cdot)$ is upper semicontinuous, then for every $\varepsilon>0$, there exist $n_0(\epsilon)\geq 0$ such that for every $n\geq n_0$, we have

$$\begin{gathered} f_n(x,y)\in F(x,y,\overline{w}_n(x,y)+v_{(x,y)})\subset \\ \subset F(x,y,\overline{w}*_{(x,y)})+\varepsilon B(0,1),\ \text{a.e. } (x,y)\in J. \end{gathered}$$

Since $F(.,.,.)$ has compact values, then there exists a subsequence f_{n_m} such that

$$f_{n_m}(\cdot,\cdot)\to f_*(\cdot,\cdot) \text{ as } m\to\infty$$

and

$$f_*(x,y)\in F(x,y,\overline{w}_*(x,y)+v_{(x,y)})(x,y),\ \text{a.e. } (x,y)\in J.$$

Then for every $w\in F(x,y,\overline{w}(x,y)+v_{(x,y)})$, we have

$$\|f_{n_m}(x,y)-f_*(x,y)\|\leq \|f_{n_m}(x,y)-w\|+\|w-f_*(x,y)\|.$$

Then

$$\|f_{n_m}(x,y)-f_*(x,y)\|\leq d(f_{n_m}(x,y),F(x,y,\overline{w}_*(x,y)+v_{(x,y)}).$$

By an analogous relation, obtained by interchanging the roles of f_{n_m} and f_*, it follows that

$$\begin{gathered} \|f_{n_m}(x,y)-f_*(x,y)\|\leq \\ \leq H_d(F(x,y,\overline{w}_n(x,y)+v_{(x,y)}),F(x,y,\overline{w}_*(x,y)+v_{(x,y)})\leq \\ \leq l(x,y)\|\overline{w}_n-\overline{w}_*\|_B. \end{gathered}$$

Then

$$\|h_{n_m}(x,y)-h_*(x,y)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|\overline{w}_{n_m}(s,t)-$$

$$- \overline{w}_*(s,t)\|dtds \leq \frac{\|\overline{w}_{n_m}-\overline{w}_*\|_{(a,b)}}{\Gamma(r_1)\Gamma(r_2)}\times$$

$$\times \int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}l(s,t)dtds \leq$$

$$\leq \frac{a^{r_1}b^{r_2}l^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\|\overline{w}_{n_m}-\overline{w}_*\|_{(a,b)}.$$

Hence

$$\|h_{n_m}-h_*\|_{(a,b)} \leq \frac{a^{r_1}b^{r_2}l^*}{\Gamma(r_1+1)\Gamma(r_2+1)}\|\overline{w}_{n_m}-\overline{w}_*\|_{(a,b)} \to 0 \text{ as } m\to\infty.$$

Step 5: (A priori bounds)

We now show there exists an open set $U \subseteq C_0$ with $w \in \lambda P(w)$, for $\lambda \in (0,1)$ and $w \in \partial U$. Let $w \in C_0$ and $w \in \lambda P(w)$ for some $0 < \lambda < 1$. Thus there exists $f \in S_{F,\overline{w}_{(s,t)}+v_{(s,t)}}$ such that, for each $(x,y) \in J$,

$$w(x,y) = \frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)dtds.$$

This implies by (H_2) that, for each $(x,y) \in J$, we have

$$\|w(x,y)\| \leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\times$$

$$\times\, l(s,t)(1+\|\overline{w}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}})dtds \leq$$

$$\leq \frac{l^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}+$$

$$+\frac{l^*}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1}\|\overline{w}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}}dtds.$$

But

$$\|\overline{w}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}} \leq \|\overline{w}_{(s,t)}\|_{\mathcal{B}}+\|v_{(s,t)}\|_{\mathcal{B}} \leq$$

$$\leq K\sup\{w(\tilde{s},\tilde{t}) : (\tilde{s},\tilde{t}) \in [0,s]\times[0,t]\} + M\|\phi\|_{\mathcal{B}}. \quad (9)$$

If we name $z(s,t)$ the right hand side of (4), then we have

$$\|\overline{w}_{(s,t)}+v_{(s,t)}\|_{\mathcal{B}} \leq z(x,y),$$

and therefore, for each $(x,y)\in J$ we obtain

$$\|w(x,y)\| \leq \frac{l^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \\ + \frac{l^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} z(s,t)dtds. \tag{10}$$

Using the above inequality and the definition of z we have that

$$z(x,y) \leq M\|\phi\|_{\mathcal{B}} + \frac{Kl^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} + \\ + \frac{Kl^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} z(s,t)dtds,$$

for each $(x,y)\in J$. Then, Lemma 4.4 implies there exists $\delta=\delta(r_1,r_2)$

$$\|z(x,y)\| \leq R + \delta \frac{Kl^*}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} R dtds,$$

where

$$R = M\|\phi\|_{\mathcal{B}} + \frac{Kl^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}.$$

Hence

$$\|z\|_\infty \leq R + \frac{R\delta K l^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)} := \widetilde{M}.$$

Then, (4) implies that

$$\|w\|_\infty \leq \frac{l^* a^{r_1} b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}(1+\widetilde{M}) := M^*.$$

Set

$$U = \{w \in C_0 : \|w\|_{(a,b)} < M^* + 1\}.$$

$P:\overline{U}\to C_0$ is continuous and completely continuous. By Theorem 2.5 and our choice of U, there is no $w\in\partial U$ such that $w\in\lambda P(w)$, for $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [19], we deduce that N has a fixed point which is a solution to problem (1)-(3). □

Now we present a similar existence result for the problem (4)-(6).

Definition 4.6. A function $u\in\Omega$ is said to be a solution of (4)-(6) if there exists $f\in F(x,y,u_{(x,y)})$ such that u satisfies the equations ${}^cD_0^r[u(x,y)-g(x,y,u_{(x,y)})]=f(x,y)$ and (6) on J and the condition (5) on $\tilde{J}$.

Let $f \in L^1(J, \mathbb{R}^n)$ and $g \in AC(J, \mathbb{R}^n)$ and consider the following linear problem

$$^cD_0^r\Big(u(x,y) - g(x,y)\Big) = f(x,y);\ (x,y) \in J, \tag{11}$$

$$u(x,0) = \varphi(x),\ u(0,y) = \psi(y);\ (x,y) \in J, \tag{12}$$

with $\varphi(0) = \psi(0)$.

For the existence of solutions for the problem $(4) - (6)$, we need the following lemma:

Lemma 4.7. *A function $u \in AC(J, \mathbb{R}^n)$ is a solution of problem* $(11) - (12)$ *if and only if $u(x,y)$ satisfies*

$$u(x,y) = \mu(x,y) + g(x,y) - g(x,0) - \\ -g(0,y) + g(0,0) + I_0^r(f)(x,y),\ \ (x,y) \in J. \tag{13}$$

Proof. Let $u(x,y)$ be a solution of problem (11)-(12). Then, taking into account the definition of the fractional Caputo derivative, we have

$$I_0^{1-r}\Big(D_{xy}^2(u(x,y) - g(x,y))\Big) = f(x,y).$$

Hence, we obtain

$$I_0^r(I_0^{1-r}D_{xy}^2)\Big(u(x,y) - g(x,y)\Big) = (I_0^r f)(x,y),$$

then

$$I_0^1 D_{xy}^2\Big(u(x,y) - g(x,y)\Big) = (I_0^r f)(x,y).$$

Since

$$I_0^1(D_{xy}^2)\Big(u(x,y) - g(x,y)\Big) = u(x,y) - u(x,0) - u(0,y) + u(0,0) - \\ - [g(x,y) - g(x,0) - g(0,y) + g(0,0)],$$

we have

$$u(x,y) = \mu(x,y) + g(x,y) - g(x,0) - g(0,y) + g(0,0) + I_0^r(f)(x,y).$$

Now let $u(x,y)$ satisfy (4.7). It is clear that $u(x,y)$ satisfies (11)-(12). □

As a consequence of Lemma 4.7 we have the following auxiliary result

Corollary 4.8. *The function $u \in \Omega$ is a solution of problem* $(4) - (6)$ *if and only if there exists $f \in F(x,y,u_{(x,y)})$ such that u satisfies the equation*

$$u(x,y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1}(y-t)^{r_2-1} f(s,t)\,dtds + \\ + \mu(x,y) + g(x,y,u_{(x,y)}) - g(x,0,u_{(x,0)}) - \\ - g(0,y,u_{(0,y)}) + g(0,0,u_{(0,0)}),$$

for all $(x,y) \in J$ and the condition (5) *on* $\tilde{J}$.

Theorem 4.9. *Assume* $(H1)-(H2)$ *and the following condition holds*
$(H3)$ *the function* g *is continuous and completely continuous, and for any bounded set* B *in* Ω, *the set* $\{(x,y)\to g(x,y,u):\ u\in\mathcal{B}\}$, *is equicontinuous in* $C(J,\mathbb{R}^n)$, *and there exist constants* $0\le d_1K<\frac{1}{4},\ d_2\ge 0$ *such that*

$$\|g(x,y,u)\|\le d_1\|u\|_B+d_2,\ (x,y)\in J,\ u\in\mathcal{B}.$$

Then the IVP $(4)-(6)$ *has at least one solution on* $(-\infty,a]\times(-\infty,b]$.

Proof. Consider the operator $N_1:\Omega\to\mathcal{P}(\Omega)$ defined by

$$(N_1u)(x,y)=$$

$$=\left\{\begin{array}{l} h\in\Omega: h(x,y)=\left\{\begin{array}{ll}\phi(x,y), & (x,y)\in\tilde{J},\\ \mu(x,y)+g(x,y,u_{(x,y)})-g(x,0,u_{(x,0)}) & \\ -g(0,y,u_{(0,y)})+g(0,0,u_{(0,0)}) & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1} & \\ f(s,t)dtds, & (x,y)\in J,\end{array}\right.\end{array}\right.$$

where $f\in S_{F,u}$.

In analogy to Theorem 4.5, we consider the operator $P_1:C_0\to\mathcal{P}(C_0)$ defined by

$$(P_1u)(x,y)=$$

$$=\left\{\begin{array}{l} h\in\Omega: h(x,y)=\left\{\begin{array}{ll}0, & (x,y)\in\tilde{J},\\ g(x,y,\overline{w}_{(x,y)}+v_{(x,y)}) & \\ -g(x,0,\overline{w}_{(x,0)}+v_{(x,0)}) & \\ -g(0,y,\overline{w}_{(0,y)} & \\ +v_{(0,y)})+g(0,0,\overline{w}_{(0,0)}+v_{(0,0)}) & \\ +\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1} & \\ f(s,t)dtds, & (x,y)\in J,\end{array}\right.\end{array}\right.$$

where $f\in S_{F,\overline{w}+v}$. We shall show that the operator P_1 is continuous and completely continuous. Using $(H3)$ it suffices to show that the operator $P_2:C_0\to\mathcal{P}(C_0)$ defined by,

$$(P_2u)(x,y)=$$

$$=\left\{\begin{array}{l} h\in\Omega: h(x,y)=\left\{\begin{array}{ll}0, & (x,y)\in\tilde{J},\\ \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1} & \\ f(s,t)dtds, & (x,y)\in J,\end{array}\right.\end{array}\right.$$

is continuous and completely continuous. This was proved in Theorem 4.5. We now show there exists an open set $U\subseteq C_0$ with $w\in\lambda P_1(w)$, for $\lambda\in(0,1)$ and $w\in\partial U$. Let $w\in C_0$ and $w\in\lambda P_1(w)$ for some $0<\lambda<1$. Thus for each $(x,y)\in J$,

$$\begin{aligned} w(x,y) =&\lambda[g(x,y,\overline{w}_{(x,y)}+v_{(x,y)})-g(x,0,\overline{w}_{(x,0)}+v_{(x,0)})-\\ &-g(0,y,\overline{w}_{(0,y)}+v_{(0,y)})+g(0,0,\overline{w}_{(0,0)}+v_{(0,0)})]+\\ &+\frac{\lambda}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}f(s,t)dtds, \end{aligned}$$

where $f\in F(x,y,\overline{w}+v)$. Then

$$\|w(x,y)\|\leq 4d_1\|\overline{w}_{(x,y)}+v_{(x,y)}\|_B+\frac{l^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}+$$

$$+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^x\int_0^y(x-s)^{r_1-1}(y-t)^{r_2-1}l(s,t)\|\overline{w}_{(s,t)}+v_{(s,t)}\|_Bdtds.$$

Using the above inequality and the definition of z we have that

$$\|z\|_\infty\leq R_1+\frac{R_1\delta(r_1,r_2)Kl^{**}a^{r_1}b^{r_2}}{(1-4d_1K)\Gamma(r_1+1)\Gamma(r_2+1)}:=L,$$

where

$$R_1=\frac{1}{1-4d_1K}\Big[8d_2K+\frac{Kl^*a^{r_1}b^{r_2}}{\Gamma(r_1+1)\Gamma(r_2+1)}\Big],$$

and

$$l^{**}=\frac{l^*}{1-4d_1K}.$$

Then

$$\|w\|_\infty\leq 4d_1\|\phi\|_B+8d_2+4Ld_1+\frac{a^{r_1}b^{r_2}l^*(1+L)}{\Gamma(r_1+1)\Gamma(r_2+1)}:=L^*.$$

Set

$$U_1=\{w\in C_0:\|w\|_{(a,b)}<L^*+1\}.$$

By Theorem 2.5 and our choice of U_1, there is no $w\in\partial U$ such that $w\in\lambda P_2(w)$, for $\lambda\in(0,1)$. As a consequence of the nonlinear alternative of Leray-Schauder type [19], we deduce that N_1 has a fixed point which is a solution to problem (4)-(6). □

5. An Example

As an application of our results we consider the following partial hyperbolic functional differential inclusion of the form

$$({}^cD_0^ru)(x,y)\in F(x,y,u_{(x,y)}),\quad \text{if } (x,y)\in J:=[0,1]\times[0,1], \tag{14}$$

$$u(x,y)=\phi(x,y),\ (x,y)\in\tilde{J}, \tag{15}$$

where $\tilde{J} := (-\infty, 1] \times (-\infty, 1] \backslash J$. Let $\gamma \geq 0$ and consider the space

$$B_\gamma = \{u \in C((-\infty, 0] \times (-\infty, 0], \mathbb{R}) : \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u(\theta, \eta) \text{ exists in } \mathbb{R}\}.$$

The norm of B_γ is given by

$$\|u\|_\gamma = \sup_{(\theta,\eta) \in (-\infty,0] \times (-\infty,0]} e^{\gamma(\theta+\eta)} |u(\theta, \eta)|.$$

Let $u : (-\infty, 1] \times (-\infty, 1] \to \mathbb{R}$ such that $u_{(x,y)} \in B_\gamma$ for $(x, y) \in E := [0, 1] \times \{0\} \cup \{0\} \times [0, 1]$, then

$$\begin{aligned} \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u_{(x,y)}(\theta, \eta) &= \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta-x+\eta-y)} u(\theta, \eta) = \\ &= e^{-\gamma(x+y)} \lim_{\|(\theta,\eta)\| \to \infty} e^{\gamma(\theta+\eta)} u(\theta, \eta) \\ &< \infty. \end{aligned}$$

Hence $u_{(x,y)} \in B_\gamma$. Finally we prove that

$$\begin{aligned} \|u_{(x,y)}\|_\gamma = K \sup\{|u(s, t)| : (s, t) \in [0, x] \times [0, y]\}+ \\ +M \sup\{\|u_{(s,t)}\|_\gamma : (s, t) \in E_{(x,y)}\}, \end{aligned}$$

where $K = M = 1$ and $H = 1$.
If $x + \theta \leq 0,\ y + \eta \leq 0$ we get

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s, t)| : (s, t) \in (-\infty, 0] \times (-\infty, 0]\},$$

and if $x + \theta \geq 0,\ y + \eta \geq 0$ then we have

$$\|u_{(x,y)}\|_\gamma = \sup\{|u(s, t)| : (s, t) \in [0, x] \times [0, y]\}.$$

Thus for all $(x + \theta, y + \eta) \in [0, 1] \times [0, 1]$, we get

$$\begin{aligned} \|u_{(x,y)}\|_\gamma = \sup\{|u(s, t)| : (s, t) \in (-\infty, 0] \times (-\infty, 0]\}+ \\ + \sup\{|u(s, t)| : (s, t) \in [0, x] \times [0, y]\}. \end{aligned}$$

Then

$$\begin{aligned} \|u_{(x,y)}\|_\gamma = \sup\{\|u_{(s,t)}\|_\gamma : (s, t) \in E\}+ \\ + \sup\{|u(s, t)| : (s, t) \in [0, x] \times [0, y]\}. \end{aligned}$$

$(B_\gamma, \|.\|_\gamma)$ is a Banach space. We conclude that B_γ is a phase space.
Set

$$F(x, y, u_{(x,y)}) = \{u \in \mathbb{R} : f_1(x, y, u_{(x,y)}) \leq u \leq f_2(x, y, u_{(x,y)})\},$$

where $f_1, f_2 : [0, 1] \times [0, 1] \times B_\gamma \to \mathbb{R}$. We assume that for each $(x, y) \in J$, $f_1(x, y, .)$ is lower semi-continuous (i.e, the set $\{z \in B_\gamma : f_1(x, y, z) > \nu\}$ is open for each $\nu \in \mathbb{R}$), and assume that for each $(x, y) \in J$, $f_2(x, y, .)$ is upper semi-continuous (i.e the set $\{z \in B_\gamma : f_2(x, y, z) < \nu\}$ is open for each $\nu \in \mathbb{R}$). Assume that there are $l \in L^\infty(J, \mathbb{R}_+)$ and $\Psi : [0, \infty) \to (0, \infty)$ continuous and nondecreasing such that

$$\max(|f_1(x, y, z)|, \ |f_2(x, y, z)|) \leq l(x, y)\Psi(|z|),$$
$$\text{for a.e. } (x, y) \in J \text{ and all } z \in B_\gamma.$$

It is clear that F is compact and convex valued, and it is upper semicontinuous (see [14]). Since all the conditions of Theorem 4.5 are satisfied, problem (14)-(15) has at least one solution defined on $(-\infty, 1] \times (-\infty, 1]$.

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