Proceedings of A. Razmadze
Mathematical Institute
Vol. **146** (2008), 97–112

SOLUTION OF THE MIXED PROBLEM OF THE PLANE THEORY OF ELASTICITY FOR A MULTIPLY CONNECTED DOMAIN WITH PARTIALLY UNKNOWN BOUNDARY IN THE PRESENCE OF AXIAL SYMMETRY

N. T. ODISHELIDZE

Abstract. In the present paper we consider the problem of the plane theory of elasticity for the multiply connected domain S, a square, weakened by unknown equi-strong holes, four of which, the same ones, lie symmetrically with respect to the segments connecting midde points of the opposite square sides, while the fifth hole has a point of intersection of these segments and is symmetric to them and to the coordinate axes. The vertices of the square lie on the coordinate axes, and their vicinities are cut out by the equal smooth arcs, symmetric to the coordinate axes.

To every segment of the broken line of the elastic body are applied absolutely smooth, with rectilinear bases, stamps which are under action of the force P. Unknown equi-strong parts of the boundary are free from external forces. Using the method of the theory of functions of a complex variable the equi-strong parts of the boundary and a stressed state of the body are defined.

რეზიუმე. გამოკვლეულია ბრტყელი დრეკადობის ამოცანა მრავლად ბმული S არისათვის, კვადრატისათვის, შესუსტებულს თანაბრად მტკიცე ხვრელებით, რომელთაგან ოთხი ხვრელი ტოლია და სიმეტრიულია მოპირდაპირე გვერდების შუა წერტილების შემაერთებელი მონაკვეთების მიმართ. მეხუთე ხვრელი შეიცავს მათ გადაკვეთის წერტილს და სიმეტრიულია ამ მონაკვეთებისა და კოორდინატთა ღერძების მიმართ. კვადრატის წვეროები მდებარეობენ კოორდინატთა ღერძებზე და მათი მიდამოები ამოჭრილია კოორდინატთა ღერძების სიმეტრიული ტოლი სიდიდის გლუვი რკალებით.

საზღვრის წრფივ მონაკვეთებზე მოდებულია აბსოლუტურად გლუვი მყარი შტამპები სწორხაზოვანი ფუძეებით, რომლებზეც მოდებულია ძალა P. საძიებელი თანაბრადმტკიცე საზღვრის ნაწილები თავისუფალნი არიან გარეშე ზემოქმედებისაგან. კომპლექსური ცვლადის ფუნქციის თეორიის აპარატის გამოყენებით განისაზღვრება საზღვრის თანაბრად მტკიცე ნაწილები და სხეულის დაძაბული მდგომარეობა.

2000 *Mathematics Subject Classification.* 73C35.
Key words and phrases. Equi-strong, bending, Keldysh-Sedov problem.

The boundary value problems of the plane theory of elasticity for infinite plates with weakened unknown equi-strong holes when constant normal stresses act on the hole boundary and forces are applied at infinity, have been studied in [1,5,12,13], but analogous problems of bending were considered in [6].

The case dealing with a finite domain when some part of its boundary is unknown equi-strong, while the remaining portion of the boundary is a broken line, is investigated in [3,4].

In the present work we consider the problem of plane elasticity for a square which is weakened by five holes, and the vicinities of the angle vertices are cut out by equal unknown equi-strong arcs. The boundary conditions of the third problem are assigned on the linear segments.

The third problem for a polygon is solved in [10,11], but for a doubly connected domain bounded by a broken line it is considered in [2].

1. Statement of the Problem and the Method of Its Solution

Let a homogeneous isotropic elastic body on a complex plane $z = x + iy$ occupy a multiply connected domain S, the square, which is weakened by unknown equi-strong holes, four of which, the same ones, lie symmetrically with respect to the segments connecting middle points of opposite sides of the square, while the fifth hole has the point of intersection of these segments and is symmetric with respect to them and to the coordinate axes.

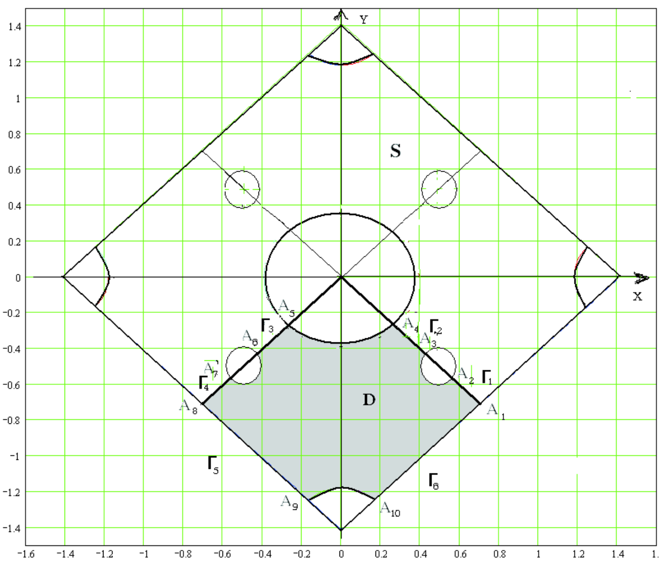

Fig. 1

Assume that the square vertices lie on the coordinate axes, and their vicinities are cut out by equal smooth arcs which are symmetric with respect to the coordinate axes (Fig. 1).

Assume that the side length of the square is $2a$.

Let to every link of the broken line, the outer boundary of the given elastic body, be applied absolutely smooth rigid stamps with rectilinear bases which are under action of the force P. Under the above assumptions, the normal displacement of every link of the broken line is $\nu_n = \nu = \text{const}$. The unknown parts of the boundary are free from external forces. On the boundary of the domain S the tangential stress $\tau_{ns} = 0$.

We formulate the following problem: Find a stressed state of the body and unknown parts of the boundary under the condition that tangential normal stress σ_s takes on them constant value $\sigma_s = K = \text{const}$.

Since the problem is axially symmetric, therefore on the segments $[A_1, A_2]$, $[A_3, A_4]$, $[A_5, A_6]$, $[A_7, A_8]$, the normal displacements and tangential stesses $\nu_n = \tau_{ns} = 0$. To study the above-posed problem it suffices to consider a curvilinear polygon $A_1A_2A_3A_4A_5A_6A_7A_8A_9A_{10}$ which we denote by D. Introduce the notation

$$\Gamma_1 = [A_1, A_2], \quad \Gamma_2 = [A_3, A_4], \quad \Gamma_3 = [A_5, A_6], \quad \Gamma_4 = [A_7, A_8],$$

$$\Gamma_5 = [A_9, A_{10}], \quad \Gamma_6 = [A_{10}, A_1], \quad \Gamma = \bigcup_{j=1}^{6} \Gamma_j,$$

$$\gamma_1 = A_2A_3, \quad \gamma_2 = A_3A_4, \quad \gamma_3 = A_6A_7, \quad \gamma_4 = A_9A_{10}, \quad \gamma = \bigcup_{j=1}^{4} \gamma_j.$$

By $\frac{P_1}{2}$ and $\frac{P_2}{2}$ we denote principal vectors of normal stresses applied on Γ_1 and Γ_2, respectively:

$$\int_{\Gamma_1} \sigma_n \, ds = \frac{P_1}{2}, \quad \int_{\Gamma_2} \sigma_n \, ds = \frac{P_2}{2}.$$

Since $\Gamma_5 \parallel \Gamma_1$ and $\Gamma_5 \parallel \Gamma_2$ the principal vector of the normal stress $\int_{\Gamma_2} \sigma_n \, ds = \frac{P_2}{2}$ acts on Γ_5, in view of equilibrium of the cut out body D, we have

$$\int_{\Gamma_1} \sigma_n \, ds + \int_{\Gamma_2} \sigma_n \, ds = \int_{\Gamma_5} \sigma_n \, ds = \frac{P}{2}.$$

Because of the symmetry of D, we obtain

$$\int_{\Gamma_2} \sigma_n \, ds = \int_{\Gamma_3} \sigma_n \, ds = \frac{P_2}{2}, \quad \int_{\Gamma_1} \sigma_n \, ds = \int_{\Gamma_4} \sigma_n \, ds = \frac{P_1}{2},$$

and analogously, just as above, $\frac{P_1}{2} + \frac{P_2}{2} = \frac{P}{2}$.

The boundary conditions of the problem can be written as follows:

$$\nu_n = \begin{cases} 0, & t \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ \nu, & t \in \Gamma_5 \cup \Gamma_6, \end{cases} \tag{1.1}$$

$$\tau_{ns} = 0, \quad t \in \Gamma \cup \gamma, \tag{1.2}$$

$$\sigma_n = \frac{P}{2}, \quad t \in \Gamma_5 \cup \Gamma_6, \quad \sigma_s = K, \quad t \in \gamma, \tag{1.3}$$

where σ_n is the normal stress.

The affixes of the points A_k, $k = \overline{1,10}$ we denote by the same letters.

Owing to the well-known Kolosov-Muskhelishvili's formulas [8], the solution of the above-formulated problem is reduced to finding two holomorphic functions ψ and φ in the domain D with the following boundary conditions:

$$\mathrm{Re}\, e^{-ia(t)}\big(\chi\,\varphi(t) - t\overline{\varphi'(t)} - \overline{\psi(t)}\big) = 2\mu\,\nu_n(t), \quad t \in \Gamma, \tag{1.4}$$

$$\mathrm{Re}\, e^{-ia(t)}\big(\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)}\big) = C(t), \quad t \in \Gamma, \tag{1.5}$$

$$\varphi(t) + t\overline{\varphi'(t)} + \overline{\psi(t)} = B(t), \quad t \in \gamma, \tag{1.6}$$

$$\mathrm{Re}\, \varphi'(t) = \frac{\sigma_n + \sigma_s}{4} = \frac{K}{4}, \quad t \in \gamma, \tag{1.7}$$

where χ and μ are the elastic constants, $\alpha(t)$ is the angle made by the outer normal to Γ and the Ox-axis. The coordinate abscissa of the point t counted from the point A_1 we denote by s

$$\begin{gathered} \alpha(t) = \alpha_k, \quad t \in \Gamma_k, \quad k = \overline{1,6}, \quad \alpha_1 = \alpha_2 = \frac{\pi}{4}, \\ \alpha_3 = \alpha_4 = \frac{3\pi}{4}, \quad \alpha_5 = \frac{5\pi}{4}, \quad \alpha_6 = \frac{7\pi}{4}, \end{gathered} \tag{1.8}$$

$$C(t) = \mathrm{Re}\,\big(e^{-i\alpha(t)}B(t)\big), \quad B(t) = i\left(\int_{A_1}^{t} \sigma_n(t_0)\, e^{i\alpha(t_0)} ds_0 - \frac{P}{2}\, e^{\frac{\pi}{4}\, i}\right).$$

Taking into account (1.8), we obtain

$$\begin{gathered} B(t) = \begin{cases} -\dfrac{P_2}{2}\, e^{\frac{\pi}{4}\, i}\, i, & t \in \gamma_1, \\ 0, & t \in \gamma_2, \\ -\dfrac{P_2}{2}\, e^{\frac{\pi}{4}\, i}, & t \in \gamma_3, \\ -\dfrac{P_2}{2}\, e^{\frac{\pi}{4}\, i}\,(i+1), & t \in \gamma_4, \end{cases} \\ C(t) = \begin{cases} 0, & t \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ \dfrac{P}{2}, & t \in \Gamma_5 \cup \Gamma_6. \end{cases} \end{gathered} \tag{1.9}$$

Let $t \in \Gamma_k$, $k = \overline{1,6}$, then $t - A_k = i|t - A_k|e^{i\alpha_k}$. Thus we have $\mathrm{Re}\, t\, e^{-i\alpha(t)} = \mathrm{Re}\, e^{-i\alpha(t)} A(t)$, where $A(t)$ is the piecewise constant function, $A(t) = A_k$, $t \in \Gamma_k$, $k = 1,2,3,4,5,6$.

The functions $\varphi'(z)$ and $\psi(z)$ are continuously extendable on the boundary of the domain D everywhere, except possibly of the vertices of the broken lines $A_2, A_3, A_4, A_5, A_6, A_7, A_9, A_{10}$, and in their vicinities are fulfilled the conditions of the type

$$|\varphi'(z)| < M|z - A_k|^{-\delta_k}, \quad |\psi(z)| < M|z - A_k|^{-\delta_k}, \tag{1.10}$$

where $0 \le \delta_k < \frac{1}{2}$, $k = 2,3,4,5,6,7,9,10$, $M > 0$.

Summing up equations (1.4) and (1.5), differentiating with respect to the coordinate abscissa s, and taking into account that $\nu_n(t)$ is the piecewise constant function, we obtain

$$\mathrm{Im}\, \varphi'(t) = 0, \quad t \in \Gamma_0. \tag{1.11}$$

The conditions (1.7) and (.11) is, in fact, the Keldysh-Sedov's problem [7]:

$$\mathrm{Re}\left(\varphi'(t) - \frac{K}{4}\right) = 0, \quad t \in \gamma,$$
$$\mathrm{Im}\left(\varphi'(t) - \frac{K}{4}\right) = 0, \quad t \in \Gamma.$$

On the basis of the conditions (1.10) it is proved that the Keldysh-Sedov's problem has a unique solution [7]

$$\varphi'(z) = \frac{K}{4}\,,$$

whence

$$\varphi(z) = \frac{K}{4}\, z. \tag{1.12}$$

Here we neglect the constant summand.

Substituting the values $B(t)$, $C(t)$, $\varphi(t)$ defined by formulas (1.9) and (1.12) into the boundary conditions (1.5)–(1.6), we get

$$\mathrm{Re}\left[e^{-\alpha(t)}\left(\frac{k}{2}\, t + \overline{\psi(t)}\right)\right] = \begin{cases} 0, & t \in \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4, \\ \dfrac{P}{2}, & t \in \Gamma_5 \cup \Gamma_6, \end{cases} \tag{1.13}$$

$$e^{-\frac{\pi}{4}\, i}\left(\frac{k}{2}\, t + \overline{\psi(t)}\right) = B(t) = \begin{cases} -\dfrac{P_2}{2}\, i, & t \in \gamma_1, \\ 0, & t \in \gamma_2, \\ -\dfrac{P_2}{2}, & t \in \gamma_3, \\ -\dfrac{P}{2}(i+1), & t \in \gamma_4, \end{cases} \tag{1.14}$$

$$\mathrm{Re}\left[e^{-\alpha_j t} t\right] = \begin{cases} 0, & t \in \Gamma_j, \quad j = 1,2,3,4, \\ -a, & t \in \Gamma_5 \cup \Gamma_6. \end{cases} \tag{1.15}$$

Taking into account (1.8), we rewrite the conditions (1.13), (1.14) and (1.15) in the form

$$\operatorname{Re}\Big[\frac{k}{2}\,e^{-\frac{\pi}{4}\,i}\,t+e^{\frac{\pi}{4}\,i}\,\psi(t)\Big]=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2\cup\gamma_1\cup\gamma_2,\\ -\dfrac{P}{2}, & t\in\Gamma_5\cup\gamma_4,\\ -\dfrac{P_2}{2}, & t\in\gamma_3,\end{cases}\tag{1.16}$$

$$\operatorname{Im}\Big[e^{-\frac{\pi}{4}\,i}\,\frac{k}{2}\,t-e^{\frac{\pi}{4}\,i}\,\psi(t)\Big]=\begin{cases}0, & t\in\Gamma_3\cup\Gamma_4\cup\gamma_2\cup\gamma_3,\\ -\dfrac{P}{2}, & t\in\Gamma_6\cup\gamma_4.\\ -\dfrac{P_2}{2}, & t\in\gamma_1,\end{cases}\tag{1.17}$$

$$\operatorname{Re}\big[e^{-\frac{\pi}{4}\,i}\,t\big]=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2,\\ -a, & t\in\Gamma_5,\end{cases}\tag{1.18}$$

$$\operatorname{Im}\big[e^{-\frac{\pi}{4}\,i}\,t\big]=\begin{cases}0, & t\in\Gamma_3\cup\Gamma_4,\\ -a, & t\in\Gamma_6.\end{cases}\tag{1.19}$$

Adding equality (1.19) multiplied by k, and equality (1.17) with the changed sign, we find that

$$\operatorname{Im}\Big[\frac{k}{2}\,e^{-\frac{\pi}{4}\,i}\,t+\psi(t)\,e^{\frac{\pi}{4}\,i}\Big]=\begin{cases}0, & t\in\Gamma_3\cup\Gamma_4,\\ \dfrac{P}{2}-ak, & t\in\Gamma_6.\end{cases}\tag{1.20}$$

Analogously, equalities (1.18) and (1.16) yield

$$\operatorname{Re}\Big[\frac{k}{2}\,e^{-\frac{\pi}{4}\,i}\,t-\psi(t)\,e^{\frac{\pi}{4}\,i}\Big]=\begin{cases}0, & t\in\Gamma_1\cup\Gamma_2,\\ \dfrac{P}{2}-ak, & t\in\Gamma_5.\end{cases}\tag{1.21}$$

Let the function $z=\omega(\zeta)$, $\zeta=\xi+i\eta$ map conformally the domain D onto the upper half-plane $\operatorname{Im}\zeta>0$. By a_j we denote the image of the point A_j, $j=\overline{1,10}$. Assume that $a_9=-1$, $a_{10}=1$, and the middle point of the arc A_4A_5 turns into $\zeta=\infty$.

Since the domain is symmetric with respect to the Oy-axis, $-a_1=a_8$, $-a_2=a_7$, $-a_3=a_6$, $-a_4=a_5$.

Denote by

$$\Phi(\zeta)=\frac{k}{2}\,e^{-\frac{\pi}{4}\,i}\,\omega(\zeta)+e^{\frac{\pi}{4}\,i}\,\psi(\omega(\zeta)),\tag{1.22}$$

$$\Psi(\zeta)=\frac{k}{2}\,e^{-\frac{\pi}{4}\,i}\,\omega(\zeta)-e^{\frac{\pi}{4}\,i}\,\psi(\omega(\zeta)).\tag{1.23}$$

Taking into account (1.22) and (1.23), the boundary conditions (1.16), (1.17), (1.1.20) and (1.21) take the form

$$\Phi(\xi)+\overline{\Phi(\xi)}=\begin{cases}0, & \xi\in(-\infty,-a_4)\cup(a_1,\infty),\\ -P & \xi\in(-a_1,1),\\ -P_2 & \xi\in(-a_3,-a_2),\end{cases}$$
$$\Phi(\xi)-\overline{\Phi(\xi)}=\begin{cases}0, & \xi\in(-a_4,-a_3)\cup(-a_2,-a_1),\\ (P-2ak)i, & \xi\in(1,a_1),\end{cases} \tag{1.24}$$

$$\Psi(\xi)+\overline{\Psi(\xi)}=\begin{cases}0, & \xi\in(a_1,a_2)\cup(a_3,a_4),\\ P-2ak, & \xi\in(-a_1,-1),\end{cases}$$
$$\Psi(\xi)-\overline{\Psi(\xi)}=\begin{cases}0, & \xi\in(-\infty,-a_1)\cup(a_4,\infty),\\ -Pi & \xi\in(-1,a_1),\\ -P_2 i & \xi\in(a_2,a_3).\end{cases} \tag{1.25}$$

The solution of the boundary problems (1.24), (1.25) is given by the Keldysh-Sedov's formula ([12],§95)

$$\Phi(\zeta)=\frac{X_1(\zeta)}{2\pi i}\Bigg[\int\limits_{-a_3}^{-a_2}\frac{-P_2\,d\xi}{X_1(\xi)(\xi-\zeta)}-\int\limits_{-a_1}^{1}\frac{P\,d\xi}{X_1(\xi)(\xi-\zeta)}+ + \int\limits_{1}^{a_1}\frac{(P-2ak)i\,d\xi}{X_1(\xi)(\xi-\zeta)}-C\Bigg], \tag{1.26}$$

$$\Psi(\zeta)=\frac{X_2(\zeta)}{2\pi i}\Bigg[\int\limits_{-a_1}^{-1}\frac{(P-2ak)i\,d\xi}{X_2(\xi)(\xi-\zeta)}-\int\limits_{-1}^{a_1}\frac{Pi\,d\xi}{X_2(\xi)(\xi-\zeta)}+ + \int\limits_{a_2}^{a_1}\frac{-P_2 i\,d\xi}{X_2(\xi)(\xi-\zeta)}+Ci\Bigg],\quad \operatorname{Im}\zeta>0, \tag{1.27}$$

where

$$X_1(\zeta)=\sqrt{\frac{(\zeta+a_4)(\zeta+a_2)(\zeta-1)}{(\zeta+a_3)(\zeta+a_1)(\zeta-a_1)}},\quad \operatorname{Im}\zeta>0, \tag{1.28}$$

$$X_2(\zeta)=\sqrt{\frac{(\zeta+1)(\zeta-a_2)(\zeta-a_4)}{(\zeta+a_1)(\zeta-a_1)(\zeta-a_3)}},\quad \operatorname{Im}\zeta>0. \tag{1.29}$$

Under $X_1(\zeta)$, $X_2(\zeta)$ we mean the branch of the holomorphic function in the upper half-plane which at infinity takes the value

$$X_1(\infty)=X_2(\infty)=1.$$

It can be easily shown that

$$X_1(\xi)=\begin{cases} |X_1(\xi)|, & \xi\in(-\infty,-a_4)\cup(-a_3,-a_2)\cup(-a_1,1),\\ -i|X_1(\xi)|, & \xi\in(-a_4,-a_3)\cup(-a_2,-a_1)\cup(1,a_1),\end{cases} \tag{1.30}$$

$$X_2(\xi)=\begin{cases} |X_2(\xi)|, & \xi\in(-\infty,-a_1)\cup(-1,a_1)\cup(a_2,a_3)\cup\\ & \qquad\cup(a_4,\infty),\\ i|X_2(\xi)|, & \xi\in(-a_1,-1)\cup(a_1,a_2)\cup(a_3,a_4),\end{cases} \tag{1.31}$$

$$|X_1(\xi)|=|X_2(-\xi)|. \tag{1.32}$$

Taking into account equalities (1.30) and (1.31), we rewrite formulas (1.26) and (1.27) in the form

$$\begin{aligned}\Phi(\zeta)=&\frac{X_1(\zeta)i}{2\pi}\Bigg[\int\limits_{-a_3}^{-a_2}\frac{P_2\,d\xi}{|X_1(\xi)|(\xi-\zeta)}+\int\limits_{-a_1}^{a_1}\frac{P\,d\xi}{|X_1(\xi)|(\xi-\zeta)}-\\ &-2ak\int\limits_{1}^{a_1}\frac{d\xi}{|X_1(\xi)|(\xi-\zeta)}+C\Bigg],\end{aligned} \tag{1.33}$$

$$\begin{aligned}\Psi(\zeta)=&-\frac{X_2(\zeta)}{2\pi i}\Bigg[\int\limits_{-a_1}^{a_1}\frac{P\,d\xi}{|X_2(\xi)|(\xi-\zeta)}+\int\limits_{a_2}^{a_3}\frac{P_2\,d\xi}{|X_2(\xi)|(\xi-\zeta)}-\\ &-2ak\int\limits_{-a_1}^{-1}\frac{d\xi}{|X_2(\xi)|(\xi-\zeta)}-C\Bigg],\quad \operatorname{Im}\zeta>0.\end{aligned} \tag{1.34}$$

2. Investigation of the Solution of the Problem and Construction of Graphs of the Unknown Part of the Boundary

Since the functions $X_1(\zeta)$, $X_2(\zeta)$ have singularities of 0.5 order at the points $\xi=\pm a_1,\pm a_3$, therefore for the functions $\Phi(\zeta)$, $\Psi(\zeta)$ to be bounded in the vicinity of the points $-a_1$, a_1, $-a_3$, a_3, it is necessary and sufficient that the following conditions be fulfilled:

$$\begin{cases} \int\limits_{-a_3}^{-a_2} \frac{P_2\,d\xi}{|X_1(\xi)|(\xi+a_1)} + \int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi+a_1)} - \\ \qquad -2ak \int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi+a_1)} + C = 0, \\ \int\limits_{-a_3}^{-a_2} \frac{P_2\,d\xi}{|X_1(\xi)|(\xi-a_1)} + \int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi-a_1)} - \\ \qquad -2ak \int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi-a_1)} + C = 0, \\ \int\limits_{-a_3}^{-a_2} \frac{P_2\,d\xi}{|X_1(\xi)|(\xi+a_3)} + \int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi+a_3)} - \\ \qquad -2ak \int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi+a_3)} + C = 0 \end{cases} \tag{2.1}$$

and

$$\begin{cases} \int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_2(\xi)|(\xi-a_1)} + \int\limits_{a_2}^{a_3} \frac{P_2\,d\xi}{|X_2(\xi)|(\xi-a_1)} - \\ \qquad -2ak \int\limits_{-a_1}^{-1} \frac{d\xi}{|X_2(\xi)|(\xi-a_1)} - C = 0, \\ \int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_2(\xi)|(\xi+a_1)} + \int\limits_{a_2}^{a_3} \frac{P_2\,d\xi}{|X_2(\xi)|(\xi+a_1)} - \\ \qquad -2ak \int\limits_{-a_1}^{-1} \frac{d\xi}{|X_2(\xi)|(\xi+a_1)} - C = 0, \\ \int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_2(\xi)|(\xi-a_3)} + \int\limits_{a_2}^{a_3} \frac{P_2\,d\xi}{|X_2(\xi)|(\xi-a_3)} - \\ \qquad -2ak \int\limits_{-a_1}^{-1} \frac{d\xi}{|X_2(\xi)|(\xi-a_3)} - C = 0. \end{cases} \tag{2.2}$$

If in the condition (2.2) we replace ξ by $-\xi$ and take into account (1.32), then we will obtain the condition coinciding with (2.1) which is, in fact, a system of three equations with unknown parameters a_1, a_2, a_3, C, k, P_2.

Solving the system with respect to k and C, P_2 for fixed a_1, a_2, a_3, we obtain the solutions of the problem and hence the equations of unknown parts of the boundary.

From equalities (1.22), (1.23) we have

$$\omega(\zeta) = \frac{\Phi(\zeta) + \Psi(\zeta)}{k}\, e^{\frac{\pi}{4}\, i}, \quad \operatorname{Im}\zeta > 0. \tag{2.3}$$

If we pass in formulas (1.33) and (1.34) to the limit $\zeta \to \xi_0 \in (-1,1)$ and use the Sokhotskii-Plemelj formulas, then we will get

$$\Phi(\xi_0) = i\,\Phi_0(\xi_0) - \frac{P}{2}\,, \quad \Psi(\xi_0) = \Psi_0(\xi_0) - \frac{P}{2}\, i, \tag{2.4}$$

where

$$\begin{aligned}\Phi_0(\xi_0) =& \frac{X_1(\xi_0)}{2\pi}\Bigg[\int\limits_{-a_3}^{-a_2} \frac{P_2\, d\xi}{|X_1(\xi)|(\xi-\xi_0)} + \int\limits_{-a_1}^{a_1} \frac{P\, d\xi}{|X_1(\xi)|(\xi-\xi_0)} - \\ & - 2ak\int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi-\xi_0)} + C\Bigg], \quad \xi_0 \in (-1,1),\end{aligned} \tag{2.5}$$

$$\begin{aligned}\Psi_0(\xi_0) =& -\frac{X_2(\xi_0)}{2\pi}\Bigg[\int\limits_{a_2}^{a_3} \frac{P_2\, d\xi}{|X_2(\xi)|(\xi-\xi_0)} + \int\limits_{-a_1}^{a_1} \frac{P\, d\xi}{|X_2(\xi)|(\xi-\xi_0)} - \\ & - 2ak\int\limits_{-a_1}^{-1} \frac{d\xi}{|X_2(\xi)|(\xi-\xi_0)} - C\Bigg], \quad \xi_0 \in (-1,1).\end{aligned} \tag{2.6}$$

The integrals $\int\limits_{-1}^{1} \frac{P\, d\xi}{|X_1(\xi)|(\xi-\xi_0)}$ and $\int\limits_{-1}^{1} \frac{P\, d\xi}{|X_2(\xi)|(\xi-\xi_0)}$ appearing in the second integrals of equalities (2.5) and (2.6) are singular and they exist in the sense of the Cauchy principal value.

If in equality (2.5) we replace ξ_0 by $-\xi_0$ and ξ by $-\xi$, then using formula (1.32) we will obtain

$$\Phi_0(-\xi_0) = \Psi_0(\xi_0). \tag{2.7}$$

Equation of the arc γ_4 is given by the formula

$$t = \omega(\xi_0) = \frac{\Phi(\xi_0) + \Psi(\xi_0)}{k}\, e^{\frac{\pi}{4}\, i}, \quad \xi_0 \in (-1,1).$$

Inserting in the above formula the values $\Phi(\xi_0)$ and $\Psi(\xi_0)$ defined by equalities (2.4), and taking into account (2.7), we obtain

$$t = \omega(\xi_0) = \frac{\Phi(-\xi_0) - \Psi(\xi_0)}{2k}\sqrt{2} + i\,\frac{\Phi(\xi_0) + \Psi(-\xi_0) - P}{2k}\sqrt{2}, \quad \xi_0 \in (-1,1). \tag{2.8}$$

Obviously, the arc γ_2 is symmetric with respect to the Oy-axis.

Analogously we can obtain equations for the arcs γ_1 and γ_3, if in formulas (1.33) and (1.34) we pass to the limits $\zeta \to \xi_0 \in (a_2, a_3)$ and $\zeta \to \xi_0 \in (-a_3, -a_2)$ and make use of Sokhotskii-Plemelj formulas. Thus we get

$$\Phi(\xi_0) = i\,\Phi_0(\xi_0), \quad \Psi(\xi_0) = \Psi_0(\xi_0) - \frac{P_2}{2}\,i, \quad \text{where } \xi_0 \in (a_2, a_3) \text{ for } \gamma_1,$$

$$\Phi(\xi_0) = i\,\Phi_0(\xi_0) - \frac{P_2}{2}, \quad \Psi(\xi_0) = \Psi_0(\xi_0), \quad \text{where } \xi_0 \in (-a_3, -a_2) \text{ for } \gamma_3,$$

$$\Phi_0(\xi_0) = \frac{X_1(\xi_0)}{2\pi}\Bigg[\int\limits_{-a_3}^{-a_2} \frac{P_2\,d\xi}{|X_1(\xi)|(\xi - \xi_0)} + \int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi - \xi_0)} - {}$$
$$- 2ak \int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi - \xi_0)} + C\Bigg], \tag{2.9}$$
$$\xi_0 \in (a_2, a_3) \quad \text{and} \quad \xi_0 \in (-a_3, -a_2),$$

$$\Psi_0(\xi_0) = -\frac{X_2(\xi_0)}{2\pi}\Bigg[\int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_2(\xi)|(\xi - \xi_0)} + \int\limits_{a_2}^{a_3} \frac{P_2\,d\xi}{|X_2(\xi)|(\xi - \xi_0)} - {}$$
$$- 2ak \int\limits_{-a_1}^{-1} \frac{d\xi}{|X_2(\xi)|(\xi - \xi_0)} - C\Bigg], \tag{2.10}$$
$$\xi_0 \in (a_2, a_3), \quad \text{and} \quad \xi_0 \in (-a_3, -a_2).$$

The equation of the arcs γ_1 and γ_3 is given by the formula

$$t = \omega(\xi_0) = \frac{\Phi(\xi_0) + \Psi(\xi_0)}{k}\, e^{\frac{\pi}{4} i}$$

for $\xi_0 \in (a_2, a_3)$ for the arc γ_1, and for $\xi_0 \in (-a_3, -a_2)$ for the arc γ_3, that is,

$$t = \omega(\xi_0) = \frac{\Phi_0(-\xi_0) - \Phi_0(\xi_0) + \frac{P_2}{2}}{2k}\sqrt{2} + {}$$
$$+ i\frac{\Phi_0(\xi_0) + \Phi_0(-\xi_0) - \frac{P_2}{2}}{2k}\sqrt{2}, \quad \xi_0 \in (a_2, a_3), \tag{2.11}$$

$$t = \omega(\xi_0) = \frac{\Phi_0(-\xi_0) - \Phi_0(\xi_0) - \frac{P_2}{2}}{2k}\sqrt{2} + {}$$
$$+ i\frac{\Phi_0(\xi_0) + \Phi_0(-\xi_0) - \frac{P_2}{2}}{2k}\sqrt{2}, \quad \xi_0 \in (-a_3, -a_2). \tag{2.12}$$

Similarly, the equation of the arc γ_2 is given by the formula

$$t = \omega(\xi_0) = \frac{\Phi_0(-\xi_0) - \Phi_0(\xi_0)}{2k}\sqrt{2} + i\frac{\Phi_0(\xi_0) + \Phi_0(-\xi_0)}{2k}\sqrt{2}, \tag{2.13}$$
$$\xi_0 \in (-\infty, -a_4) \cup (a_4, \infty),$$

$$\Phi_0(\xi_0) = \frac{X_1(\xi_0)}{2\pi}\Bigg[\int\limits_{-a_3}^{-a_2} \frac{P_2\,d\xi}{|X_1(\xi)|(\xi-\xi_0)} + \int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi-\xi_0)} - $$
$$- 2ak\int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi-\xi_0)} + C\Bigg],$$
$$\xi_0 \in (-\infty,-a_4)\cup(a_4,\infty).$$

Let us consider a particular case when the square is weakened only by one equi-strong central hole, i.e., $a_2 = a_3$.

In this case the solution of the problem takes the form

$$\Phi(\zeta) = \frac{X_1(\zeta)i}{2\pi}\Bigg[\int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi-\zeta)} - 2ak\int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi-\zeta)} + C\Bigg], \quad (2.14)$$

$$\Psi(\zeta) = -\frac{X_2(\zeta)}{2\pi i}\Bigg[\int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_2(\xi)|(\xi-\zeta)} - 2ak\int\limits_{-a_1}^{-1} \frac{d\xi}{|X_2(\xi)|(\xi-\zeta)} - C\Bigg], \quad (2.15)$$
$$\operatorname{Im}\zeta > 0,$$

where

$$X_1(\zeta) = \sqrt{\frac{(\zeta+a_4)(\zeta-1)}{(\zeta+a_1)(\zeta-a_1)}}, \quad \operatorname{Im}\zeta > 0,$$
$$X_2(\zeta) = \sqrt{\frac{(\zeta+1)(\zeta-a_4)}{(\zeta+a_1)(\zeta-a_1)}}. \quad \operatorname{Im}\zeta > 0 \quad (2.16)$$

Since the functions $X_1(\zeta)$ and $X_2(\zeta)$ have singularities of 0.5 order at the points $\xi = \pm a_1$, therefore for the functions $\Phi(\zeta)$, $\Psi(\zeta)$ to be bounded in the vicinity of the points $a_1, -a_1$, it is necessary and sufficient that the following conditions be fulfilled:

$$\begin{cases} \displaystyle\int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi+a_1)} - 2ak\int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi+a_1)} + C = 0, \\ \displaystyle\int\limits_{-a_1}^{a_1} \frac{P\,d\xi}{|X_1(\xi)|(\xi-a_1)} - 2ak\int\limits_{1}^{a_1} \frac{d\xi}{|X_1(\xi)|(\xi-a_1)} + C = 0. \end{cases} \quad (2.17)$$

The above conditions is a system of two equations involving unknown parameters a_1, C, k.

Solving the system with respect to k and C, we obtain

$$k = \frac{P}{2a}\,\frac{\displaystyle\int\limits_{-a_1}^{a_1}\frac{d\xi}{|X_1(\xi)|(\xi^2-a_1^2)}}{\displaystyle\int\limits_{1}^{a_1}\frac{d\xi}{|X_1(\xi)|(\xi^2-a_1^2)}}\,, \tag{2.18}$$

$$C = 2ak\int\limits_{1}^{a_1}\frac{d\xi}{|X_1(\xi)|(a_1+\xi)} - P\int\limits_{-a_1}^{a_1}\frac{d\xi}{|X_1(\xi)|(a_1+\xi)}\,, \tag{2.19}$$

Analogously, the equation of the arc γ_2 is given by the formula

$$t = \omega(\xi_0) = \frac{\Phi_0(-\xi_0)-\Phi_0(\xi_0)}{2k}\sqrt{2} + i\,\frac{\Phi_0(\xi_0)+\Phi_0(-\xi_0)}{2k}\sqrt{2},$$
$$\xi_0 \in (-\infty,-a_4)\cup(a_4,\infty), \tag{2.20}$$
$$\Phi_0(\xi_0) = \frac{X_1(\xi_0)}{2\pi}\left[\int\limits_{-a_1}^{a_1}\frac{P\,d\xi}{|X_1(\xi)|(\xi-\xi_0)} - 2ak\int\limits_{1}^{a_1}\frac{d\xi}{|X_1(\xi)|(\xi-\xi_0)} + C\right],$$
$$\xi_0 \in (-\infty,-a_4)\cup(a_4,\infty),$$

and the equation of the arc γ_4 is given by the formula

$$t = \omega(\xi_0) = \frac{\Phi(-\xi_0)-\Phi_0(\xi_0)}{2k}\sqrt{2} + i\,\frac{\Phi(\xi_0)+\Phi_0(-\xi_0)-P}{2k}\sqrt{2}, \tag{2.21}$$
$$\xi_0 \in (-1,1),$$

where

$$\Phi_0(\xi_0) = \frac{X_1(\xi_0)}{2\pi}\left[\int\limits_{-a_1}^{a_1}\frac{P\,d\xi}{|X_1(\xi)|(\xi-\xi_0)} - 2ak\int\limits_{1}^{a_1}\frac{d\xi}{|X_1(\xi)|(\xi-\xi_0)} + C\right],$$
$$\xi_0 \in (-1,1).$$

Calculations and construction of graphs are performed by means of the Mathcad system. The graphs are constructed for the arcs γ_2 and γ_4. Since the problem is cyclically symmetric, the rest parts of the contour are constructed by turning the graph $\omega(\xi_0)$ by the angle $\frac{\pi}{2}$.

Graphs of equi-strong contours for particular cases

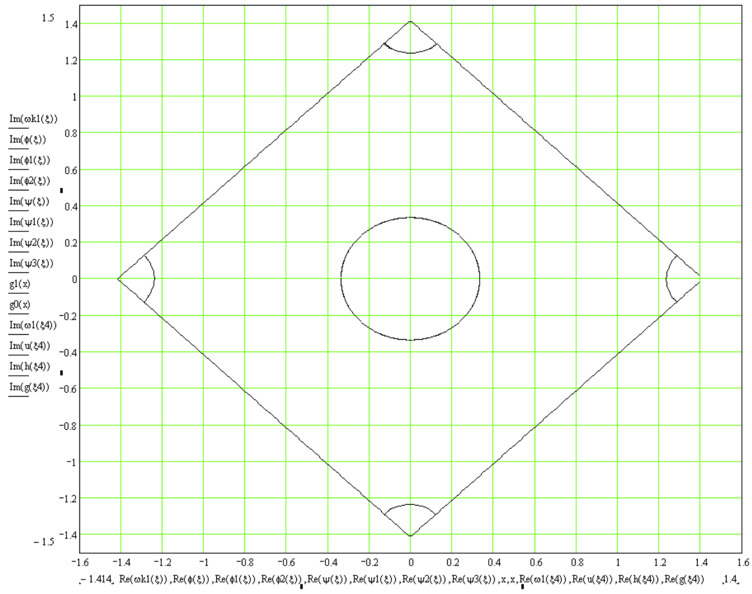

Fig. 2
$P=-10$, $a=1$, $a_1=9$, $a_4=24$, $K=-11.289$, $C=16.802$.

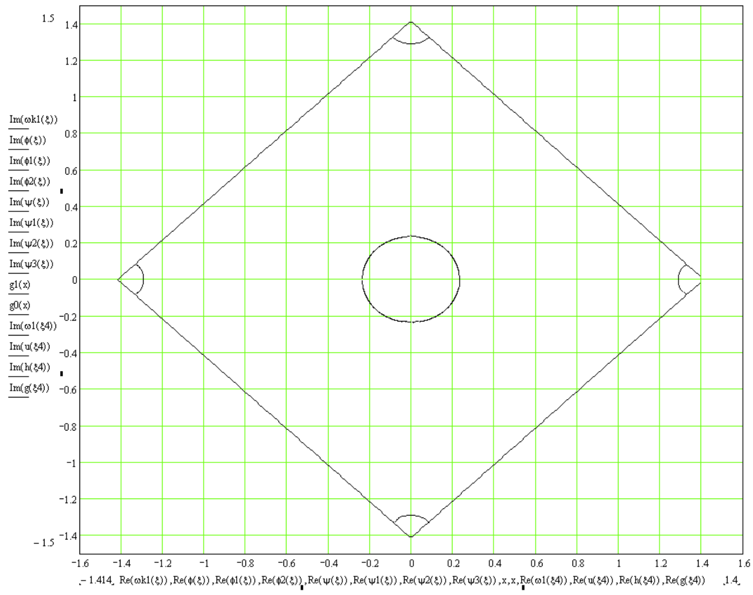

Fig. 3
$P=-10$, $a=1$, $a_1=19$, $a_4=104$, $K=-10.499$, $C=10.949$.

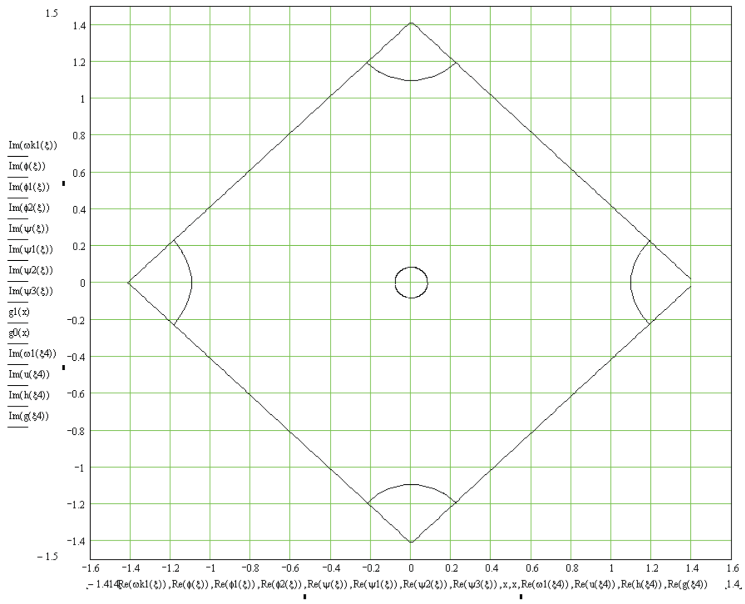

Fig. 4
$P = -10$, $a = 1$, $a_1 = 3$, $a_4 = 124$, $K = -10.914$, $C = 4.007$.

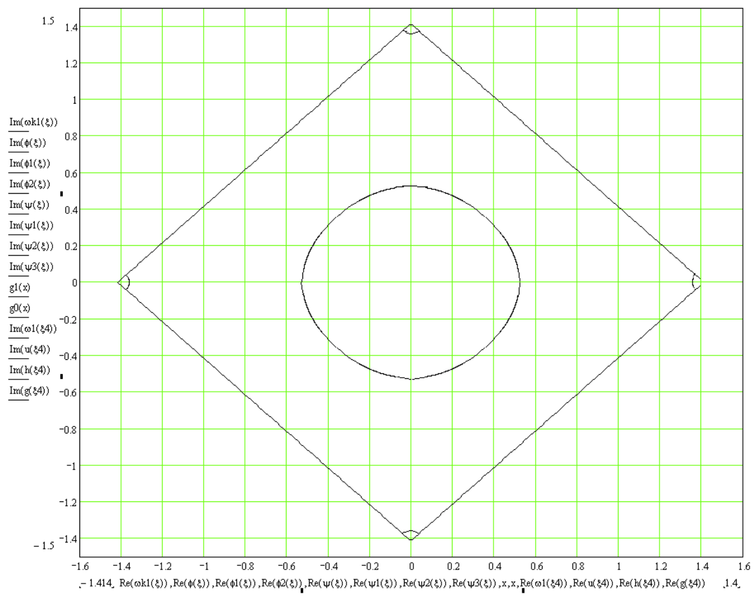

Fig. 5
$P = -10$, $a = 1$, $a_1 = 79$, $a_4 = 104$, $K = -12.754$, $C = 29.642$.

Acknowledgement

The present work is carried out under the financial support of the National Scientific Fund of Georgia (Grant No. GNSF/ST06/3-006).

References

1. N. V. Banichuk, Optimization of forms of elastic bodies. (Russian) *Nauka, Moscow*, 1980.
2. R. D. Bantsuri, Solution of the third basic problem for doubly connected domains bounded by broken lines. (Russian) *Dokl. AN SSSR*, **243**(1978), No. 4, 882–885.
3. R. D. Bantsuri, Some inverse problems of plane elasticity and of bending of thin plates. *Proc. of Intern. Symp. Dedicated to Centenary of Acad. N. Muskhelishvili. Tbilisi, Georgia*, 1993, 100–106.
4. R. D. Bantsuri and A. S. Isakhanov, Some inverse problems of the theory of elasticity. (Russian) *Trudy Tbiliss. Mat. Inst.* **87**(1987), 3–20.
5. G. M. Ivanov and A. S. Kosmodamianskii, To the solution of the problems with an unknown boundary in the presence of cyclic symmetry. (Russian) *Trudy Nikolaevskogo Karablestroitelnogo Institute*, 1973.
6. G. M. Ivanov and A. S. Kosmodamianskii, The inverse problems of bending of thin isotropic plates. (Russian) *Izv. AN SSSR. MTT*, 1974, No. 5, 53–56.
7. M. V. Keldysh and L. D. Sedov, The effective solution of some boundary value problems for harmonic functions. (Russian) *Dokl. Akad. Nauk. SSSR*, **XVI**(1937), No. 1, 7–10.
8. N. I. Muskhelishvili, Some Basic Problems of the Mathematical Theory of Elasticity. *Nauka, Moscow*, 1966.
9. N. I. Muskhelishvili, Singular Integral Equations. *Nauka, Moscow*, 1966.
10. G. M. Polozhii, Solution of some problems of the plane theory of elasticity for domains with angular points. (Russian) *Ukr. Mat. J.*, No. 4, 1949, 16–41.
11. G. M. Polozhii, Solution of the third basic problem of the plane theory of elasticity for an arbitrary finite convex polygon. (Russian) *Dokl. AN SSSR.* **73**(1950), No. 1, 49–52.
12. G. P. Cherepanov, Some problems of elasticity and plasticity with an unknown boundary. Applications of the function theory in solid mechanics. (Russian) *Sb./M.* No. 1 (1965), 135–150.
13. G. P. Cherepanov, Inverse problems of the plane theory of elasticity. (Russian) *Prikl. Math. Mekh.* **38**(1974), No. 6, 963–980.

(Received 05.02.2008)

Author's address:

Iv. Dzhavakhishvili State University
Faculty of Exact and Natural Sciences
Tbilisi, Georgia