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LIPSCHITZ ESTIMATES FOR MULTILINEAR COMMUTATOR OF LITTLEWOOD-PALEY OPERATOR

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Abstract. In this paper, we study the continuity of multilinear commutator generated by Littlewood-Paley operator and Lipschitz function on Triebel-Lizorkin, Hardy and Herz-Hardy space S.

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1. Introduction

We know, the commutator $[b, T](f)(x) = b(x)T(f)(x) - T(bf)(x)$ is bounded on $L^p(R^n)$ for $1 < p < \infty$ when T is the Calderón-Zygmund operator and $b \in BMO(R^n)$. Janson and Paluszynski study the commutator for the Triebel-Lizorkin space and the case $b \in \text{Lip}_\beta(R^n)$, where $\text{Lip}_\beta(R^n)$ is the homogenerous Lipschitz space. Chanillo (see [2]) proves a similiar result when T is replaced by the fractional operators. The main purpose of this paper is to discuss the boundedness of multilinear commutator generated by Littlewood-Paley operator and Lipschitz functions on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

2. Preliminaries and Definitions

Throughout this paper, $M(f)$ will denote the Hardy-Littlewood maximal function of f, and write $M_p(f) = (M(f^p))^{1/p}$ for $0 < p < \infty$. Q will denote a cube of R^n with side parallel to the axes. Let $f_Q = |Q|^{-1} \int_Q f(x)dx$ and $f^\#(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y) - f_Q|dy$. Denote the Hardy spaces by $H^p(R^n)$. It is well known that $H^p(R^n)(0 < p \leq 1)$ has the atomic decomposition characterization(see [11], [16], [17]). For $\beta > 0$ and $p > 1$, let

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$\dot{F}_p^{\beta,\infty}(R^n)$ be the homogeneous Tribel-Lizorkin space. The Lipschitz space $\mathrm{Lip}_\beta(R^n)$ is the space of functions f such that

$$||f||_{\mathrm{Lip}_\beta} = \sup_{\substack{x,y\in R^n\\ x\neq y}} \frac{|f(x)-f(y)|}{|x-y|^\beta} < \infty.$$

Lemma 1 (see [15]). *For $0<\beta<1$, $1<p<\infty$, we have*

$$\begin{aligned} ||f||_{\dot{F}_p^{\beta,\infty}} &\approx \left\| \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-f_Q|dx \right\|_{L^p} \approx \\ &\approx \left\| \sup_{\cdot\in Q} \inf_c \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-c|dx \right\|_{L^p}. \end{aligned}$$

Lemma 2 (see [15]). *For $0<\beta<1$, $1\le p\le\infty$, we have*

$$\begin{aligned} ||f||_{\mathrm{Lip}_\beta} &\approx \sup_Q \frac{1}{|Q|^{1+\frac{\beta}{n}}} \int_Q |f(x)-f_Q|dx \approx \\ &\approx \sup_Q \frac{1}{|Q|^{\frac{\beta}{n}}} \left(\frac{1}{|Q|} \int_Q |f(x)-f_Q|^p dx \right)^{1/p}. \end{aligned}$$

Lemma 3 (see [2]). *For $1\le r<\infty$ and $\beta>0$, let*

$$M_{\beta,r}(f)(x) = \sup_{x\in Q} \left(\frac{1}{|Q|^{1-\frac{\beta r}{n}}} \int_Q |f(y)|^r dy \right)^{1/r},$$

suppose that $r<p<n/\beta$, and $1/q = 1/p-\beta/n$, then

$$||M_{\beta,r}(f)||_{L^q} \le C||f||_{L^p}.$$

Lemma 4 (see [5]). *Let $Q_1\subset Q_2$, then*

$$|f_{Q_1} - f_{Q_2}| \le C||f||_{\dot{\Lambda}_\beta} |Q_2|^{\beta/n}.$$

Definition 1. Let $0<p,\ q<\infty$, $\alpha\in R$, $B_k=\{x\in R^n,\ |x|\le 2^k\}$, $A_k = B_k\backslash B_{k-1}$ and $\chi_k = \chi_{Ak}$ for $k\in\mathbf{Z}$.

1) The homogeneous Herz space is defined

$$\dot{K}_q^{\alpha,p}(R^n) = \{f\in L^q_{\mathrm{Loc}}(R^n\backslash\{0\}) : ||f||_{\dot{K}_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{\dot{K}_q^{\alpha,p}} = \left[\sum_{k=-\infty}^{\infty} 2^{k\alpha p} ||f\chi_k||^p_{L^q} \right]^{1/p};$$

2) The nonhomogeneous Herz space is defined by

$$K_q^{\alpha,p}(R^n) = \{f\in L^q_{\mathrm{Loc}}(R^n) : ||f||_{K_q^{\alpha,p}} < \infty\},$$

where

$$||f||_{K_q^{\alpha,p}} = \left[\sum_{k=1}^{\infty} 2^{k\alpha p}||f\chi_k||_{L^q}^p + ||f\chi_{B_0}||_{L^q}^p\right]^{1/p}.$$

Definition 2. Let $\alpha \in R$, $0 < p, q < \infty$.

(1) The homogeneous Herz type Hardy space is defined by

$$H\dot{K}_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in \dot{K}_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{H\dot{K}_q^{\alpha,p}} = ||G(f)||_{\dot{K}_q^{\alpha,p}};$$

(2) The nonhomogeneous Herz type Hardy space is defined by

$$HK_q^{\alpha,p}(R^n) = \{f \in S'(R^n) : G(f) \in K_q^{\alpha,p}(R^n)\},$$

and

$$||f||_{HK_q^{\alpha,p}} = ||G(f)||_{K_q^{\alpha,p}};$$

where $G(f)$ is the grand maximal function of f.

The Herz type Hardy spaces have the atomic decomposition characterization.

Definition 3. Let $\alpha \in R$, $1 < q < \infty$. A function $a(x)$ on R^n is called a central (α, q)-atom (or a central (a, q)-atom of restrict type), if

1) supp$a \subset B(0, r)$ for some $r > 0$ (or for some $r \geq 1$),
2) $||a||_{L^q} \leq |B(0, r)|^{-\alpha/n}$,
3) $\int_{R^n} a(x)x^\eta dx = 0$ for $|\eta| \leq [\alpha - n(1 - 1/q)]$.

Lemma 5 (see [6], [14])**.** *Let $0 < p < \infty$, $1 < q < \infty$ and $\alpha \geq n(1-1/q)$. A temperate distribution f belongs to $H\dot{K}_q^{\alpha,p}(R^n)$ (or $HK_q^{\alpha,p}(R^n)$) if and only if there exist central (α, q)-atoms (or central (α, q)-atoms of restrict type) a_j supported on $B_j = B(0, 2^j)$ and constants λ_j, $\sum_j |\lambda_j|^p < \infty$ such that $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$ (or $f = \sum_{j=0}^{\infty} \lambda_j a_j$) in the $S'(R^n)$ sense, and*

$$||f||_{H\dot{K}_q^{\alpha,p}}(or\ ||f||_{HK_q^{\alpha,p}}) \sim \left(\sum_j |\lambda_j|^p\right)^{1/p}.$$

Definition 4. Let $0 < \delta < n$, $\varepsilon > 0$ and ψ be a fixed function which satisfies the following properties:

1) $\int_{R^n} \psi(x)dx = 0$,
2) $|\psi(x)| \leq C(1 + |x|)^{-(n+1-\delta)}$,
3) $|\psi(x + y) - \psi(x)| \leq C|y|^\varepsilon(1 + |x|)^{-(n+1+\varepsilon-\delta)}$ when $2|y| < |x|$.

Let m be a positive integer and $b_j (1 \le j \le m)$ be the locally integrable function, set $\vec{b} = (b_1, \ldots, b_m)$. The multilinear commutator of Littlewood-Paley operator is defined by

$$g_{\psi,\delta}^{\vec{b}}(f)(x) = \left(\int_0^\infty |F_t^{\vec{b}}(x)|^2 \frac{dt}{t} \right)^{1/2},$$

where

$$F_t^{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^m (b_j(x) - b_j(y)) \psi_t(x-y) f(y) dy,$$

and $\psi_t(x) = t^{-n+\delta}\psi(x/t)$ for $t > 0$. Set $F_t(f) = \psi_t * f$. We also define that

$$g_{\psi,\delta}(f)(x) = \left(\int_0^\infty |F_t(f)(x)|^2 \frac{dt}{t} \right)^{1/2},$$

which is the Littlewood-Paley g function (see [17]).

Let H be the space $H = \{h : ||h|| = (\int_0^\infty |h(t)|^2 dt/t)^{1/2} < \infty\}$, then, for each fixed $x \in R^n$ $F_t(f)(x)$ may be viewed as a mapping from $[0, +\infty)$ to H, and it is clear that

$$g_{\psi,\delta}(f)(x) = ||F_t(f)(x)|| \text{ and } g_{\psi,\delta}^{\vec{b}}(f)(x) = ||F_t^{\vec{b}}(f)(x)||.$$

Note that when $b_1 = \cdots = b_m$, $g_{\psi,\delta}^{\vec{b}}$ is just the m order commutator. It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-4][7-10][12][15]). Our main purpose is to establish the boundedness of the multilinear commutator on Triebel-Lizorkin space, Hardy space and Herz-Hardy space.

Given a positive integer m and $1 \le j \le m$, we set $||\vec{b}||_{\mathrm{Lip}_\beta} = \prod_{j=1}^m ||b_j||_{\mathrm{Lip}_\beta}$ and denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{\mathrm{Lip}_\beta} = ||b_{\sigma(1)}||_{\mathrm{Lip}_\beta} \cdots ||b_{\sigma(j)}||_{\mathrm{Lip}_\beta}$.

Lemma 6 (see [10]). *Let $0 < \beta \le 1, 0 < \delta < n$, $1 < p < n/\beta$, $1/q = 1/p - \beta/n$ and $b \in \mathrm{Lip}_\beta(R^n)$. Then $g_{\psi,\delta}^b$ is bounded from $L^p(R^n)$ to $L^q(R^n)$.*

3. Theorems and Proofs.

Theorem 1. *Let $0 < \delta < n$, $0 < \beta < \min(1, \varepsilon/m)$, $1 < p < \infty$, $\vec{b} = (b_1, \ldots, b_m)$ with $b_j \in \mathrm{Lip}_\beta(R^n)$ for $1 \le j \le m$ and $g_{\psi,\delta}^{\vec{b}}$ be the multilinear commutator of Littlewood-Paley operator as in Definition 4. Then*

(a) *$g_{\psi,\delta}^{\vec{b}}$ is bounded from $L^p(R^n)$ to $\dot{F}_p^{m\beta,\infty}(R^n)$ for $1 < p < n/\delta$ and $1/p - 1/q = \delta/n$.*

(b) $g^{\vec{b}}_{\psi,\delta}$ *is bounded from* $L^p(R^n)$ *to* $L^q(R^n)$ *for* $1/p - 1/q = m\beta + \delta/n$ *and* $1/p > m\beta + \delta/n$.

Proof. (a). Fixed a cube $Q = (x_0, l)$ and $\tilde{x} \in Q$, see [10] when $m = 1$. Consider now the case $m \geq 2$. Set $\vec{b}_Q = ((b_1)_Q, \ldots, (b_m)_Q)$, where $(b_j)_Q = |Q|^{-1} \int_Q b_j(y)dy,\ 1 \leq j \leq m$. Write $f = f_1 + f_2$, where $f_1 = f\chi_{2Q}, f_2 = f\chi_{R^n \setminus 2Q}$, we have

$$\begin{aligned}
F^{\vec{b}}_t(f)(x) &= \int_{R^n} (b_1(x) - b_1(y)) \cdots (b_m(x) - b_m(y))\psi_t(x-y)f(y)dy = \\
&= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)F_t(f)(x) + \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f)(x) + \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C^m_j} (-1)^{m-j} (b(x) - \vec{b}_Q)_\sigma \times \\
&\quad \times \int_{R^n} (b(y) - \vec{b}_Q)_{\sigma^c} \psi_t(x-y)f(y)dy = \\
&= (b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)F_t(f)(x) + \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_1)(x) + \\
&\quad +(-1)^m F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_2)(x) + \\
&\quad + \sum_{j=1}^{m-1} \sum_{\sigma \in C^m_j} (-1)^{m-j} (b(x) - \vec{b}_Q)_\sigma F_t((b - \vec{b}_Q)_{\sigma^c} f)(x),
\end{aligned}$$

then

$$\begin{aligned}
&|g^{\vec{b}}_{\psi,\delta}(f)(x) - g_{\psi,\delta}(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m)f_2)(x_0)| \leq \\
\leq\ & ||F^{\vec{b}}_t(f)(x) - F_t(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m)f_2)(x_0)|| \leq \\
\leq\ & ||(b_1(x) - (b_1)_Q) \cdots (b_m(x) - (b_m)_Q)F_t(f)(x)|| + \\
&+ \sum_{j=1}^{m-1} \sum_{\sigma \in C^m_j} ||(b(x) - \vec{b}_Q)_\sigma F_t((b - \vec{b}_Q)_{\sigma^c} f)(x)|| + \\
&+||F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_1)(x)|| + \\
&+||F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_2)(x) - \\
&-F_t((b_1 - (b_1)_Q) \cdots (b_m - (b_m)_Q)f_2)(x_0)|| = \\
=\ & I_1(x) + I_2(x) + I_3(x) + I_4(x),
\end{aligned}$$

so

$$\frac{1}{|Q|^{1+\frac{m\beta}{n}}} \int_Q |g^{\vec{b}}_{\psi,\delta}(f)(x) - g_{\psi,\delta}((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m)f_2)(x_0)|dx \leq$$

$$\begin{aligned}
&\leq \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_1(x)dx+\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_2(x)dx+\\
&+ \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_3(x)dx+\frac{1}{|Q|^{1+\frac{m\beta}{n}}}\int_Q I_4(x)dx=\\
&= I+II+III+IV.
\end{aligned}$$

For I, by using **Lemma 2**, we have

$$\begin{aligned}
I &\leq \frac{1}{|Q|^{1+\frac{m\beta}{n}}}\sup_{x\in Q}|b_1(x)-(b_1)_Q|\cdots|b_m(x)-(b_m)_Q|\int_Q |g_{\psi,\delta}(f)(x)|dx\leq\\
&\leq C||\vec{b}||_{\mathrm{Lip}_\beta}\frac{1}{|Q|^{1+\frac{m\beta}{n}}}|Q|^{\frac{m\beta}{n}}\int_Q |g_{\psi,\delta}(f)(x)|dx\leq\\
&\leq C||\vec{b}||_{\mathrm{Lip}_\beta}M(g_{\psi,\delta}(f))(\tilde{x}).
\end{aligned}$$

For II, taking $1<r<p<q<n/\delta$, $1/q'+1/q=1, 1/s'+1/s=1$, $1/q=1/p-\delta/n$, $ps=r$ by using the Hölder's inequality and the boundedness of $g_{\psi,\delta}$ from L^p to L^q and **Lemma 2**, we get

$$\begin{aligned}
II &\leq \sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{1+m\beta/n}}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma||g_{\psi,\delta}((\vec{b}-\vec{b}_Q)_{\sigma^c}f)(x)|dx\leq\\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_\sigma|^{q'}dx\right)^{1/q'}\times\\
&\quad\times\left(\frac{1}{|Q|}\int_{R^n}|g_\psi((\vec{b}-\vec{b}_Q)_{\sigma^c}f\chi_Q)(x)|^q dx\right)^{1/q}\leq\\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{\mathrm{Lip}_\beta}|Q|^{|\sigma|\beta/m}\frac{1}{|Q|^{1/q}}\times\\
&\quad\times\left(\int_{R^n}|(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}f\chi_Q|^p dx\right)^{1/p}\leq\\
&\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{\mathrm{Lip}_\beta}|Q|^{(-1/q)+(1/ps')+(1-\delta ps/n)/ps}\times\\
&\quad\times\left(\frac{1}{|Q|}\int_Q |(\vec{b}(x)-\vec{b}_Q)_{\sigma^c}|^{ps'}\right)^{1/ps'}\times
\end{aligned}$$

$$\times\left(\frac{1}{|Q|^{1-\frac{\delta ps}{n}}}\int\limits_Q |f(x)|^{ps}dx\right)^{1/ps} \le$$

$$\le C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|^{m\beta/n}}||\vec{b}_\sigma||_{\mathrm{Lip}_\beta}|Q|^{|\sigma|\beta/n}||\vec{b}_{\sigma^c}||_{\mathrm{Lip}_\beta}|Q|^{|\sigma^c|\beta/n}M_{r,\delta}(f)(x)\le$$

$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}M_{r,\delta}(f)(\tilde{x}).$$

For III, we choose $1<r<p<q<n/\delta$, $0<\delta<n$, $1/q=1/p-\delta/n$, $r=ps$, by the boundness of $g_{\psi,\delta}$ from $L^p(R^n)$ to $L^q(R^n)$ and Hölder's inequality with $1/s+1/s'=1$, we get

$$\begin{aligned}
III &= \frac{1}{|Q|^{1+m\beta/n}}\int\limits_Q |g_{\psi,\delta}((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_1)(x)|dx \le\\
&\le C\frac{1}{|Q|^{m\beta/n}}\left(\frac{1}{|Q|}\int\limits_{R^n}\left|g_{\psi,\delta}(\prod_{j=1}^m(b_j(y)-(b_j)_Q)f\chi_Q)(x)\right|^q dx\right)^{1/q} \le\\
&\le C\frac{1}{|Q|^{m\beta/n}}\frac{1}{|Q|^{1/q}}\left(\int\limits_{R^n}\left|\prod_{j=1}^m(b_j(y)-(b_j)_Q)\right|^p|f\chi_Q|^p dx\right)^{1/p} \le\\
&\le C\frac{1}{|Q|^{m\beta/n}}|Q|^{(-1/q)+1/ps'-(1-(\delta ps/n)/ps)}\times\\
&\quad\times\left(\frac{1}{|Q|}\int\limits_Q\left|\prod_{j=1}^m(b_j(y)-(b_j)_Q)\right|^{ps'}dx\right)^{1/ps'}\times\\
&\quad\times\left(\frac{1}{|Q|^{1-\delta ps/n}}\int\limits_Q |f(x)|^{ps}dx\right)^{1/ps} \le C||\vec{b}||_{\mathrm{Lip}_\beta}M_{r,\delta}(f)(\tilde{x}).
\end{aligned}$$

For IV, since $|x_0-y|\approx|x-y|$ for $y\in(2Q)^c$, by **Lemma 4** and the condition on ψ, we have

$$\begin{aligned}
&||F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x)-\\
&-F_t((b_1-(b_1)_Q)\cdots(b_m-(b_m)_Q)f_2)(x_0)|| \le\\
&\le\left[\int\limits_0^\infty\left(\int\limits_{(2Q)^c}|\psi_t(x-y)-\psi_t(x_0-y)||f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy\right)^2\frac{dt}{t}\right]^{1/2}\le\\
&\le C\left[\int\limits_0^\infty\left(\int\limits_{(2Q)^c}\frac{t|x-x_0|^\varepsilon}{(t+|x_0-y|)^{n+1+\varepsilon-\delta}}|f(y)|\prod_{j=1}^m|b_j(y)-(b_j)_Q|dy\right)^2\frac{dt}{t}\right]^{1/2}\le
\end{aligned}$$

$$\leq C \int\limits_{(2Q)^c} |x_0 - x|^{\varepsilon} |x_0 - y|^{-(n+\varepsilon-\delta)} |f(y)| \prod_{j=1}^{m} |b_j(y) - (b_j)_Q| dy \leq$$

$$\leq C \sum_{k=1}^{\infty} \int\limits_{2^{k+1}Q \setminus 2^k Q} |x_0 - x|^{\varepsilon} |x_0 - y|^{-(n+\varepsilon-\delta)} |f(y)| \prod_{j=1}^{m} |b_j(y) - (b_j)_Q| dy \leq$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} |2^{k+1}Q|^{-1} \times$$

$$\times \int\limits_{2^{k+1}Q} |f(y)| \prod_{j=1}^{m} (|b_j(y) - (b_j)_{2^{k+1}Q}| + |(b_j)_{2^{k+1}} - (b_j)_Q|) dy \leq$$

$$\leq C \sum_{k=1}^{\infty} 2^{-k\varepsilon} |2^{k+1}Q|^{\delta/n} \frac{1}{|2^{k+1}Q|^{1-\delta/n}} ||\vec{b}||_{\mathrm{Lip}_\beta} M_{r,\delta}(f) \leq$$

$$\leq C ||\vec{b}||_{\mathrm{Lip}_\beta} |Q|^{\frac{m\beta}{n}} M_{r,\delta}(f)(\tilde{x}),$$

thus

$$IV \leq C ||\vec{b}||_{\mathrm{Lip}_\beta} M_{r,\delta}(f)(\tilde{x}).$$

We put these estimates together, by using **Lemma 1** and taking the supremum over all Q such that $x \in Q$, we obtain

$$||g^{\vec{b}}_{\psi,\delta}(f)||_{\dot{F}_p^{m\beta,\infty}} \leq C ||\vec{b}||_{\mathrm{Lip}_\beta} ||f||_{L^p}.$$

This complete the proof of (a).

(b). By some argument as in proof of (a), we have

$$\frac{1}{|Q|} \int\limits_Q |g^{\vec{b}}_{\psi,\delta}(f)(x) - g_{\psi,\delta}(((b_1)_Q - b_1) \cdots ((b_m)_Q - b_m) f_2)(x_0)| dx \leq$$

$$\leq \frac{1}{|Q|} \int\limits_Q I_1(x) dx + \frac{1}{|Q|} \int\limits_Q I_2(x) dx + \frac{1}{|Q|} \int\limits_Q I_3(x) dx + \frac{1}{|Q|} \int\limits_Q I_4(x) dx \leq$$

$$\leq C ||\vec{b}||_{\mathrm{Lip}_\beta} (M_{m\beta,1}(g_{\psi,\delta}(f)) + M_{m\beta+\delta,r}(f)),$$

thus

$$(g^{\vec{b}}_{\psi,\delta}(f))^{\#}(\tilde{x}) \leq C ||\vec{b}||_{\mathrm{Lip}_\beta} (M_{m\beta,1}(g_{\psi,\delta}(f)) + M_{m\beta+\delta,r}(f)).$$

By using **Lemma 3** and the boundedness of $g_{\psi,\delta}$, we have

$$||g^{\vec{b}}_{\psi,\delta}(f)||_{L^q} \leq C ||(g^{\vec{b}}_{\psi,\delta}(f))^{\#}||_{L^q} \leq$$

$$\leq C ||\vec{b}||_{\mathrm{Lip}_\beta} (||M_{m\beta,1}(g_{\psi,\delta}(f)) + M_{m\beta+\delta,r}(f))||_{L^q} \leq C ||\vec{b}||_{\mathrm{Lip}_\beta} ||f||_{L^p}.$$

This complete the proof of (b) and the theorem. □

Theorem 2. *Let* $0 < \delta < n$, $0 < \beta + \delta/m < \min(\gamma/m, 1/2m)$, $n/(n+\beta+\delta/m) < p \le 1$, $1/q = 1/p - (m\beta+\delta)/n$, $\vec{b} = (b_1, \ldots, b_m)$ *with* $b_j \in \mathrm{Lip}_\beta(R^n)$ *for* $1 \le j \le m$. *Then* $g^{\vec{b}}_{\psi,\delta}$ *is bounded from* $H^p(R^n)$ *to* $L^q(R^n)$.

Proof. It suffices to show that there exists a constant $C > 0$ such that for every H^p-atom a,

$$||g^{\vec{b}}_{\psi,\delta}(a)||_{L^q} \le C.$$

Let a be a H^p-atom, that is that a supported on a cube $Q = Q(x_0, r)$, $||a||_{L^\infty} \le |Q|^{-1/p}$ and $\int_{R^n} a(x)x^\gamma dx = 0$ for $|\gamma| \le [n(1/p-1)]$.

When $m = 1$ see [10]. Now consider the case $m \ge 2$. Write

$$\begin{aligned} ||g^{\vec{b}}_{\psi,\delta}(a)(x)||_{L^q} &\le \left(\int\limits_{|x-x_0|\le 2r} |g^{\vec{b}}_{\psi,\delta}(a)(x)|^q dx\right)^{1/q} + \\ &+ \left(\int\limits_{|x-x_0|>2r} |g^{\vec{b}}_{\psi,\delta}(a)(x)|^q dx\right)^{1/q} = I + II. \end{aligned}$$

For I, choose $1 < p_1 < n/(m\beta+\delta)$ and q_1 such that $1/q_1 = 1/p_1 - m\beta + \delta/n$. By the boundednss of $g^{\vec{b}}_{\psi,\delta}$ from $L^{p_1}(R^n)$ to $L^{q_1}(R^n)$(see **Theorem 1**), we get

$$\begin{aligned} I \le C||g^{\vec{b}}_{\psi,\delta}(a)||^q_{L^{q_1}}|Q(x_0,2r)|^{1-q/q_1} &\le C||a||^q_{L^{p_1}}||\vec{b}||_{\mathrm{Lip}_\beta}|Q|^{1-q/q_1} \le \\ &\le C||\vec{b}||_{\mathrm{Lip}_\beta}|Q|^{-q/p+q/p_1+1-q/q_1} \le C||\vec{b}||_{\mathrm{Lip}_\beta}. \end{aligned}$$

For II, let $\tau, \tau' \in N$ such that $\tau + \tau' = m$, and $\tau' \ne 0$. We get

$$\begin{aligned} |F^{\vec{b}}_t(a)(x)| \le\ & |(b_1(x) - b_1(x_0)) \cdots (b_m(x) - b_m(x_0)) \times \\ & \times \int\limits_B (\psi_t(x-y) - \psi_t(x-x_0))a(y)dy| + \\ & + \sum_{j=1}^m \sum_{\sigma \in C^m_j} |(b(x) - b(x_0))_{\sigma^c} \int\limits_B (b(y) - b(x_0))_\sigma \psi_t(x-y)a(y)dy| \le \\ & \le C||\vec{b}||_{\mathrm{Lip}_\beta}|x-x_0|^{m\beta} \cdot \int\limits_B |\psi_t(x-y) - \psi_t(x-x_0)||a(y)|dy + \\ & + C||\vec{b}||_{\mathrm{Lip}_\beta} \sum_{\tau+\tau'=m} |x-x_0|^{\tau\beta} \int\limits_B |y-x_0|^{\tau'\beta}|\psi_t(x-y)||a(y)|dy \le \\ & \le C||\vec{b}||_{\mathrm{Lip}_\beta} \frac{|x-x_0|^{m\beta}t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}} \int\limits_B |x_0-y|^\varepsilon |a(y)|dy + \end{aligned}$$

$$+ C||\vec{b}||_{\mathrm{Lip}_\beta} \sum_{\tau+\tau'=m} |x-x_0|^{\tau\beta} \frac{t}{(t+|x-x_0|)^{n+1-\delta}} \times$$
$$\times \int_B |y-x_0|^{\tau'\beta}|a(y)|dy \le C||\vec{b}||_{\mathrm{Lip}_\beta} \frac{t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}} \times$$
$$\times\, r^{m\beta+\varepsilon+n(1-\frac{1}{p})} + C||\vec{b}||_{\mathrm{Lip}_\beta} \frac{t}{(t+|x-x_0|)^{n+1-\delta}} \cdot r^{m\beta+n(1-\frac{1}{p})},$$

thus

$$|g^{\vec{b}}_{\psi,\delta}(a)(x)| \le C||\vec{b}||_{\mathrm{Lip}_\beta} \left(\int_0^\infty \left(\frac{t}{(t+|x-x_0|)^{n+1+\varepsilon-\delta}} \right)^2 \frac{dt}{t} \right)^{1/2} \times$$
$$\times\, r^{m\beta+\varepsilon+n(1-\frac{1}{p})} +$$
$$+ C||\vec{b}||_{\mathrm{Lip}_\beta} \left(\int_0^\infty \left(\frac{t}{(t+|x-x_0|)^{n+1-\delta}} \right)^2 \frac{dt}{t} \right)^{1/2} \times$$
$$\times\, r^{m\beta+n(1-\frac{1}{p})} \le$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta} |x-x_0|^{-n+\delta} \cdot r^{m\beta+n(1-\frac{1}{p})},$$

so

$$II \le C||\vec{b}||_{\mathrm{Lip}_\beta} \cdot r^{m\beta+n(1-\frac{1}{p})} \left(\int_{|x-x_0|>2r} |x-x_0|^{-nq+q\delta} dx \right)^{1/q} \le$$
$$\le C||\vec{b}||_{\mathrm{Lip}_\beta}.$$

This complete the proof of Theorem 2. □

Theorem 3. *Let* $0 < \beta \le 1$, $0 < \delta < n$, $0 < p < \infty$, $1 < q_1, q_2 < \infty$, $1/q_1 - 1/q_2 = m\beta + \delta/n$, $n(1-1/q_1) \le \alpha < n(1-1/q_1) + \beta + \delta/m$, $\vec{b} = (b_1, \dots, b_m)$ *with* $b_j \in \mathrm{Lip}_\beta(R^n)$ *for* $1 \le j \le m$. *Then* $g^{\vec{b}}_{\psi,\delta}$ *is bounded from* $H\dot{K}^{\alpha,p}_{q_1}(R^n)$ *to* $\dot{K}^{\alpha,p}_{q_2}(R^n)$.

Proof. By **Lemma 5**, let $f \in H\dot{K}^{\alpha,p}_{q_1}(R^n)$ and $f = \sum_{j=-\infty}^{\infty} \lambda_j a_j$, $suppa_j \subset B_j = B(0, 2^j)$, a_j be a central (α, q)−atom, and $\sum_{j=-\infty}^{\infty} |\lambda_j|^p < \infty$. We have

$$||g^{\vec{b}}_{\psi,\delta}(f)||^p_{\dot{K}^{\alpha,p}_{q_2}} \le C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=-\infty}^{k-2} |\lambda_j| ||g^{\vec{b}}_{\psi,\delta}(a_j)\chi_k||_{L^{q^2}} \right)^p +$$
$$+ C \sum_{k=-\infty}^{\infty} 2^{k\alpha p} \left(\sum_{j=k-1}^{\infty} |\lambda_j| ||g^{\vec{b}}_{\psi,\delta}(a_j)\chi_k||_{L^{q_2}} \right)^p =$$
$$= I + II.$$

For II, by the boundedness of $g_{\psi,\delta}^{\vec{b}}$ on (L^{q_1}, L^{q_2}), we have

$$II \le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty} |\lambda_j| ||a_j||_{L^{q_1}}\Big)^p \le$$

$$\le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum_{j=k-1}^{\infty} |\lambda_j| \cdot 2^{-j\alpha}\Big)^p \le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \times$$

$$\times \begin{cases} \sum_{k=-\infty}^{\infty} \sum\limits_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{(k-j)\alpha p}, & 0 < p \le 1 \\ \sum\limits_{k=-\infty}^{\infty} 2^{k\alpha p}\Big(\sum\limits_{j=k-1}^{\infty} |\lambda_j|^p \cdot 2^{-j\alpha p/2}\Big)\Big(\sum\limits_{j=k-1}^{\infty} 2^{-j\alpha p'/2}\Big)^{p/p'}, & 1<p<\infty \end{cases} \le$$

$$\le C||\vec{b}||^p_{\mathrm{Lip}_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$$

For I, when $m = 1$, we have

$$\begin{aligned}
|F_t^{b_1}(a_j)(x)| &\le \Big|(b_1(x) - b_1(0)) \int_{B_j} (\psi_t(x-y) - \psi_t(x)) a_j(y) dy\Big| + \\
&\quad + \Big|\int_{B_j} \psi_t (b_1(y) - b_1(0)) a_j(y) dy\Big| \le \\
&\le C||b_1||_{\mathrm{Lip}_\beta} \Big[\int_{B_j} \frac{|x|^\beta |y|^\varepsilon t}{(t+|x|)^{n+1+\varepsilon-\delta}} \cdot |a_j(y)| dy + \\
&\quad + \int_{B_j} \frac{t|y|^\beta}{(t+|x-y|)^{n+1-\delta}} \cdot |a_j(y)| dy\Big] \le \\
&\le C||b_1||_{\mathrm{Lip}_\beta} \Big[\frac{|x|^\beta t}{(t+|x|)^{n+1+\varepsilon-\delta}} \int_{B_j} |y|^\varepsilon |a_j(y)| dy + \\
&\quad + \frac{t}{(t+|x|)^{n+1-\delta}} \int_{B_j} |y|^\varepsilon |a_j(y)| dy\Big] \le \\
&\le C||b_1||_{\mathrm{Lip}_\beta} \Big[\frac{|x|^\beta t}{(t+|x|)^{n+1+\varepsilon-\delta}} \cdot 2^{j(\varepsilon + n(1-\frac{1}{q_1}) - \alpha)} + \\
&\quad + \frac{t}{(t+|x|)^{n+1-\delta}} \cdot 2^{j(\beta + n(1-\frac{1}{q_1}) - \alpha)}\Big],
\end{aligned}$$

thus

$$\begin{aligned} g^{b_1}_{\psi,\delta}(a_j)(x) &\le C||b_1||_{\mathrm{Lip}_\beta}\Big[\Big(\int_0^\infty\Big(\frac{t}{(t+|x|)^{n+1+\varepsilon-\delta}}\Big)^2\Big)^{1/2}\times\\ &\times |x|^\beta\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}+\\ &+\Big(\int_0^\infty\Big(\frac{t}{(t+|x|)^{n+1-\delta}}\Big)^2\frac{dt}{t}\Big)^{1/2}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\Big]\le\\ &\le C||b_1||_{\mathrm{Lip}_\beta}\Big[|x|^{-(n+\varepsilon-\delta)}\cdot|x|^\beta\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)}\le\\ &\le |x|^{-n+\delta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\Big]\le\\ &\le C||b_1||_{\mathrm{Lip}_\beta}|x|^{-n+\delta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}, \end{aligned}$$

from that we have

$$\begin{aligned} ||g^{b_1}_{\psi,\delta}(a_j)\chi_k||_{L^{q_2}} &\le C||b_1||_{\mathrm{Lip}_\beta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\times\\ &\times\Big(\int B_k|x|^{-nq_2+q_2\delta}dx\Big)^{1/q_2}\le\\ &\le C||b_1||_{\mathrm{Lip}_\beta}\cdot 2^{j(\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot 2^{-kn(1-\frac{1}{q_2})+k\delta}\le\\ &\le C||b_1||_{\mathrm{Lip}_\beta}\cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}, \end{aligned}$$

so

$$\begin{aligned} &I\le C||b_1||^p_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\times\\ &\times\Big(\sum_{j=-\infty}^{\infty}|\lambda_j|\cdot 2^{[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\Big)^p\le C||b_1||^p_{\mathrm{Lip}_\beta}\times\\ &\times\begin{cases}\sum\limits_{k=-\infty}^{\infty}\sum\limits_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\le 1\\ \sum\limits_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum\limits_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{\frac{p}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\Big)\times & \\ \quad\times\Big(\sum\limits_{j=-\infty}^{k-2}2^{\frac{p'}{2}[j(\beta+n(1-\frac{1}{q_1})-\alpha)-k(\beta+n(1-\frac{1}{q_1}))]}\Big)^{p/p'}, & 1<p<\infty\end{cases}\le\\ &\le C||b_1||^p_{\mathrm{Lip}_\beta}\times\\ &\times\begin{cases}\sum\limits_{j=-\infty}^{\infty}|\lambda_j|^p\sum\limits_{k=j+2}^{\infty}2^{(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\le 1\\ \sum\limits_{j=-\infty}^{\infty}|\lambda_j|^p\sum\limits_{k=j+2}^{\infty}2^{\frac{p}{2}[(j-k)(\beta+n(1-\frac{1}{q_1})-\alpha)]}, & 1<p<\infty\end{cases}\le \end{aligned}$$

$$\leq C||b_1||^p_{\mathrm{Lip}_\beta} \sum_{j=-\infty}^{\infty} |\lambda_j|^p.$$

Then

$$||g^{b_1}_{\psi,\delta}(f)||_{\dot K^{\alpha,p}_{q_2}} \leq C||b_1||_{\mathrm{Lip}_\beta}(\sum_{j=-\infty}^{\infty} |\lambda_j|^p)^{1/p} \leq C||f||_{H\dot K^{\alpha,p}_{q_1}}.$$

When $m > 1$, we have

$$\begin{aligned}
|F^{\vec b}_t(a_j))(x)| \leq & |(b_1(x)-b_1(0))\cdots(b_m(x)-b_m(0))\times \\
& \times \int_{B_j} (\psi_t(x-y)-\psi_t(x))a_j(y)dy| + \\
& + \sum_{j=1}^{\infty} \sum_{\sigma\in C^m_j} |(b(x)-b(0))_{\sigma^c} \times \\
& \times \int_{B_j} (b(y)-b(0))_\sigma \psi_t(x-y)a_j(y)dy| \leq \\
\leq & C||\vec b||_{\mathrm{Lip}_\beta} |x|^{m\beta} \int_{B_j} |\psi_t(x-y)-\psi_t(x)||a_j(y)|dy + \\
& + C||\vec b||_{\mathrm{Lip}_\beta} \sum_{\tau+\tau'=m} |x|^{\tau\beta} \int_{B_j} |y|^{\tau'\beta} |\psi_t(x-y)||a_j(y)|dy \leq \\
\leq & C||\vec b||_{\mathrm{Lip}_\beta} \frac{|x|^{m\beta} t}{(t+|x|)^{n+1+\varepsilon-\delta}} \int_{B_j} |y|^{\varepsilon}|a_j(y)|dy + \\
& + C||\vec b||_{\mathrm{Lip}_\beta} \sum_{\tau+\tau'=m} \frac{|x|^{\tau\beta} t}{(t+|x|)^{n+1-\delta}} \int_{B_j} |y|^{\tau'\beta}|a_j(y)|dy \leq \\
\leq & C||\vec b||_{\mathrm{Lip}_\beta} \frac{|x|^{m\beta} t}{(t+|x|)^{n+1+\varepsilon-\delta}} \cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)} + \\
& + C||\vec b||_{\mathrm{Lip}_\beta} \sum_{\tau+\tau'=m} \frac{|x|^{\tau\beta} t}{(t+|x|)^{n+1-\delta}} \cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)},
\end{aligned}$$

thus

$$\begin{aligned}
g^{\vec b}_{\psi,\delta}(a_j)(x) &= \left(\int_0^\infty |F^{\vec b}_t(a_j)(x)|^2 \frac{dt}{t} \right)^{1/2} \leq \\
&\leq C||\vec b||_{\mathrm{Lip}_\beta} |x|^{m\beta} \cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)} \times
\end{aligned}$$

$$\begin{aligned}
\times\quad & \Big(\int\limits_0^\infty \Big(\frac{t}{(t+|x|)^{n+1+\varepsilon-\delta}}\Big)^2\frac{dt}{t}\Big)^{1/2} + \\
& +C||\vec{b}||_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)}\times \\
\times\quad & \Big(\int\limits_0^\infty \Big(\frac{t}{(t+|x|)^{n+1-\delta}}\Big)^2\frac{dt}{t}\Big)^{1/2} \le \\
\le\quad & C||\vec{b}||_{\mathrm{Lip}_\beta}|x|^{m\beta}|x|^{-(n+\varepsilon-\delta)}\cdot 2^{j(\varepsilon+n(1-\frac{1}{q_1})-\alpha)} + \\
& +C||\vec{b}||_{\mathrm{Lip}_\beta}\sum_{\tau+\tau'=m}|x|^{\tau\beta}|x|^{-n+\delta}\cdot 2^{j(\tau'\beta+n(1-\frac{1}{q_1})-\alpha)} \le \\
\le\quad & C||\vec{b}||_{\mathrm{Lip}_\beta}|x|^{-n+\delta}\cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)},
\end{aligned}$$

then

$$\begin{aligned}
& ||g^{\vec{b}}_{\psi,\delta}(a_j)\chi_k||_{L^{q_2}} \le \\
\le\quad & C||\vec{b}||_{\mathrm{Lip}_\beta}\cdot 2^{j(m\beta+n(1-\frac{1}{q_1})-\alpha)}\cdot\Big(\int\limits_{B_j}|x|^{-nq_2+q_2\delta}dx\Big)^{1/q_2} \le \\
\le\quad & C||\vec{b}||_{\mathrm{Lip}_\beta}\cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]},
\end{aligned}$$

so

$$\begin{aligned}
I \le & C||\vec{b}||^p_{\mathrm{Lip}_\beta}\sum_{k=-\infty}^{\infty}2^{k\alpha p}\times \\
& \times\Big(\sum_{j=-\infty}^{k-2}|\lambda_j|\cdot 2^{[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\Big)^p \le \\
& \le C||\vec{b}||^p_{\mathrm{Lip}_\beta}\times \\
& \times\begin{cases}
\sum\limits_{k=-\infty}^{\infty}\sum\limits_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{(j-k)(m\beta+n(1-\frac{1}{q_1})-\alpha)p}, & 0<p\le 1\\
\sum\limits_{k=-\infty}^{\infty}2^{k\alpha p}\Big(\sum\limits_{j=-\infty}^{k-2}|\lambda_j|^p\cdot 2^{\frac{p}{2}[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\Big)\times & \\
\quad\times\Big(\sum\limits_{j=-\infty}^{k-2}2^{\frac{p'}{2}[j(m\beta+n(1-\frac{1}{q_1})-\alpha)-k(m\beta+n(1-\frac{1}{q_1}))]}\Big)^{p/p'}, & 1<p<\infty
\end{cases} \le \\
& \le C||\vec{b}||^p_{\mathrm{Lip}_\beta}\sum_{j=-\infty}^{\infty}|\lambda_j|^p.
\end{aligned}$$

From I and II, we have

$$||g_{\psi,\delta}^{\vec{b}}(f)|| \leq C||\vec{b}||_{\text{Lip}_\beta}\left(\sum_{j=-\infty}^{\infty}|\lambda_j|^p\right)^{1/p} \leq C||f||_{H\dot{K}_{q_1}^{\alpha,p}}.$$

This completes the proof of Theorem 3. □

Theorem 4. *Let* $0 < \beta < \min(\gamma/m, 1/2m)$, $0 < p \leq 1$, $1 < q_1, q_2 < \infty$, $0 < \delta < n$, $1/q_2 = 1/q_1 - (m\beta + \delta)/n$, $\vec{b} = (b_1, \ldots, b_m)$ *with* $b_j \in \text{Lip}_\beta(R^n)$ *for* $1 \leq j \leq m$. *Then* $g_{\psi,\delta}^{\vec{b}}$ *maps* $H\dot{K}_{q_1}^{n(1-1/q_1)+\beta+\delta/m,p}(R^n)$ *continuously into* $W\dot{K}_{q_2}^{n(1-1/q_1)+\beta+\delta/m,p}(R^n)$.

Proof. We write $f = \sum_{k=-\infty}^{\infty}\lambda_k a_k$, where each a_k is a central $(n(1-1/q_1)+\beta+\delta/m, q_1)$ atom supported on B_k and $\sum_{k=-\infty}^{\infty}|\lambda_k|^p < \infty$. Write

$$\begin{aligned}
&||g_{\psi,\delta}^{\vec{b}}||_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta+\delta/m,p}} \leq \\
\leq\ & \sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta+\delta/m)p}|\ \times\right.\\
\times\ & \left.\left\{x \in E_l : |g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=l-3}^{\infty}\lambda_k a_k\right)(x)| > \lambda/2\right\}|^{p/q_2}\right\}^{1/p} + \\
+\ & \sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{\infty}2^{l(n(1-1/q_1)+\beta+\delta/m)p}|\ \times\right.\\
\times\ & \left.\left\{x \in E_l : |g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=-\infty}^{l-4}\lambda_k a_k\right)(x)| > \lambda/2\right\}|^{p/q_2}\right\}^{1/p} = \\
=\ & G_1 + G_2.
\end{aligned}$$

By the (L^{q_1}, L^{q_2}) boundedness of $g_{\psi,\delta}^{\vec{b}}$ and an estimate similar to that for I_1 in Theorem 3, we get

$$\begin{aligned}
G_1^p &\leq C\sum_{l=-\infty}^{\infty}2^{lp(n(1-1/q_1)+\beta+\delta/m)}||g_{\psi,\delta}^{\vec{b}}\left(\sum_{l-3}^{\infty}\lambda_k a_k\right)(x)\chi_l||_{q_2}^p \leq \\
&\leq C||\vec{b}||_{\text{Lip}_\beta}^p\sum_{k=-\infty}^{\infty}|\lambda_k|^p.
\end{aligned}$$

To estimate G_2, let us now use the estimate

$$|g_{\psi,\delta}^{\vec{b}}(a_k)| \quad \leq \quad C||\vec{b}||_{\text{Lip}_\beta}|x|^{\delta-n}(2^k)^{m\beta+n(1-1/q_1)-\alpha},$$

which we get in the proof of Theorem 3. Note that when $x \in E_l$, $\alpha = n(1-1/q_1)+\beta+\delta/m$,

$$\begin{aligned}\lambda < \sum_{k=-\infty}^{l-4} |\lambda_k||g_{\psi,\delta}^{\vec{b}}(a_k)| &\le C||\vec{b}||_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{l-4} |\lambda_k||x|^{\delta-n}(2^k)^{m\beta+n(1-1/q_1)-\alpha} \le \\ &\le C||\vec{b}||_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{l-4}|\lambda_k||2^l|^{\delta-n}\sum_{k=-\infty}^{l-4}(2^k)^{m\beta+n(1-1/q_1)-\alpha} \le \\ &\le C||\vec{b}||_{\mathrm{Lip}_\beta} \sum_{k=-\infty}^{l-4} |\lambda_k|(2^l)^{((m-1)\beta+\delta-n-\delta/m)} \le \\ &\le C||\vec{b}||_{\mathrm{Lip}_\beta} 2^{l((m-1)\beta+\delta-n-\delta/m)}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p},\end{aligned}$$

for $\lambda > 0$, let l_λ be the maximal positive integer satisfying

$$2^{l_\lambda(n+\delta/m-(m-1)\beta-\delta)} \le C||\vec{b}||_{\mathrm{Lip}_\beta}\lambda^{-1}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p},$$

then if $l > l_\lambda$, we have

$$\left|\left\{x \in E_l : \left|g_{\psi,\delta}^{\vec{b}}\left(\sum_{k=-\infty}^{l-4}\lambda_k a_k\right)\right| > \lambda/2\right\}\right| = 0.$$

So we obtain

$$\begin{aligned}G_2 &\le \sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{l_\lambda} 2^{l(n(1-1/q_1)+\beta+\delta/m)p}(2^l)^{np/q_2}\right\}^{1/p} \le \\ &\le \sup_{\lambda>0}\lambda\left\{\sum_{l=-\infty}^{l_\lambda}(2^l)^{(n-(m-1)\beta-\delta)}\right\}^{1/p} \le \\ &\le \sup_{\lambda>0}\lambda 2^{l_\lambda(n-(m-1)\beta-\delta)} \le \\ &\le C||\vec{b}||_{\mathrm{Lip}_\beta}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p}.\end{aligned}$$

Now, combining the above estimates for G_1 and G_2, we obtain

$$||g_{\psi,\delta}^{\vec{b}}(f)||_{W\dot{K}_{q_2}^{n(1-1/q_1)+\beta+\delta/m,p}} \le C||\vec{b}||_{ip_\beta}\left(\sum_{k=-\infty}^{\infty}|\lambda_k|^p\right)^{1/p}.$$

Theorem 4 follows by taking the infimum over all central atomic decompositions. □

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