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THE DIRICHLET PROBLEM FOR HARMONIC FUNCTIONS OF SMIRNOV CLASSES IN DOUBLY-CONNECTED DOMAINS WITH ARBITRARY PIECEWISE SMOOTH BOUNDARIES

G. KHUSKIVADZE AND V. PAATASHVILI

Abstract. In doubly connected domains with arbitrary piecewise smooth boundaries we investigate the Dirichlet problem for harmonic functions from Smirnov classes. The conditions of solvability which depend essentially on the geometry of the boundary are established. Depending on the angle sized of the boundary the homogeneous problem may have nontrivial solutions. A number of linearly independent solutions are calculated.

რეზიუმე. ნებისმიერი უბან-უბან გლუვი წირებით შემოსაზღვრულ ორადბმულ არეში შესწავლილია დირიხლეს ამოცანა სმირნოვის კლასის ჰარმონიული ფუნქციებისთვის. დადგენილია ამოხსნადობის პირობები, რომლებიც არსებითად განისაზღვრება საზღვრის გეომეტრიით. იმისდა მიხედვით თუ როგორია საზღვრის კუთხეთა სიდიდეები ერთგვაროვან ამოცანას შეიძლება გააჩნდეს არანულოვანი ამოხსნები. დათვლილია წრფივად დამოუკიდებელ ამონახსნთა რაოდენობა.

Analytic functions from Smirnov classes E^p for $p \geq 1$ are representable by the Cauchy integrals (see, for e.g., [1], Ch. X). According to that fact, the real parts of that functions are harmonic ones, representable by a combination of simple and double layer potentials ([2], §12). Consequently, they can be useful for applications. On the other hand, such functions provide us with solutions of boundary value problems in some cases in which they are unsolvable in the traditional classes of smooth functions. The Smirnov classes turn also out useful when we study the problems in domains with non-smooth boundaries. In [3], [4] and [5] we have considered various boundary value problems in simply-connected domains with arbitrary piecewise smooth boundaries by means of the methods of the Cauchy type integrals and the theory of functions. In [6], analogous results for piecewise-Ljapunov

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curves have been obtained by using singular integral equations and functional analysis.

In [7] and [8], we investigated the Dirichlet problem for harmonic functions from Smirnov classes in doubly-connected domains bounded by simple piecewise smooth curves. In particular, the cases of the unique solvability have been elucidated. However, our method of investigation did not allow us to cover any kind of boundaries of the class under consideration: the cases for which the boundary had angular points of large angle sizes ($\geq \pi p$), i.e, those for which the Dirichlet problem in simply-connected domains was solvable non-uniquely, remained out of investigation. Using the results obtained in [5] and [8], we have now managed to consider the Dirichlet problem in a manner allowing one to cover doubly-connected domains with arbitrary piecewise smooth boundaries. In this paper we present the obtained results.

1^0. Notation, Definitions and Auxiliary Statements

Let Γ_1 and Γ_2 be the Jordan curves bounding the doubly-connected domain D, where Γ_2 lies in a inside domain bounded by the curve Γ_1, and let $\omega(z)$ be the function, analytic in D.

Definition 1. We say that the analytic in D function Φ belongs to the class $E^p(D;\omega)$, $p>0$, if there is an increasing sequence of doubly-connected domains $\{D^i\}$ with rectifiable boundaries $\mathcal{L}^i$, exhausting the domain D and such that

$$\sup_i \int\limits_{\mathcal{L}^i} \big|\Phi(z)\,\omega(z)\big|^p\,|dz| < \infty$$

(see, for e.g., [9]).

For simply-connected domains G bounded by a rectifiable curve, we assume that

$$E^p(G;w) = \bigg\{\Phi : \Phi \ \text{ is analytic in } \ G \ \text{ and } \ \sup_{0<r<1} \int\limits_{\Gamma_r} |\Phi(\zeta)\,\omega(\zeta)|^p\,|d\zeta| < \infty\bigg\},$$

where Γ_r are the images of the circumferences $\{w : |w| = r\}$ under the conformal mapping $\zeta = \zeta(w)$ of the unit circle $U = \{w : |w| < 1\}$ onto the domain G, and also, if G is an infinite domain, then we assume that $\zeta(0) = \infty$.

Let $z = z(w)$ be the conformal mapping onto the domain D of the circular ring $K = \{w :\ \rho < |w| < 1\}$ with the boundary $\gamma = \gamma_1 \cup \gamma_2$, $\gamma_1 = \{\tau :\ |\tau| = 1\}$, $\gamma_2 = \{\tau :\ |\tau| = \rho < 1\}$. Suppose $z(\gamma_i) = \Gamma_i$, $i = 1,2$.

Statement 2. ([9]–[10]). (i) For every function Φ from $E^p(D;\omega)$, we can take as $\mathcal{L}^i$ the images of circumferences with center $w = 0$ of radius r, $\rho < r < 1$ under the conformal mapping of K onto D; (ii) if $\omega(z)$ has angular boundary values almost everywhere on $\Gamma = \Gamma_1 \cup \Gamma_2$, then $\Phi(z)$

has likewise angular boundary values $\Phi^+(t)$ almost for all $t \in \Gamma$, where $\Phi^+ \in L^p(\Gamma;\omega)$; (iii) $\Phi(z)$ belongs to $E^p(D;\omega)$, if and only if the function $\Phi(z(w))\omega(z(w))\sqrt[p]{z'(w)}$ belongs to $E^p(K)$; [8] (iv) the class $E^p(D;\omega)$ coincides with the class of functions Φ representable in the form $\Phi = \Phi_1 + \Phi_2$, $\Phi_i \in E^p(D_i;\omega_i)$, where D_i is that domain bounded by Γ_i which contains D, and ω_i is the narrowing on Γ_i of the function ω.

Moreover, $z' \in E^1(K)$, ([8]).

Definition 3. We say that the given on the unit circumference function $\omega(\tau)$ belongs to the class W^p, if the operator

$$T : \varphi \to T\varphi,\ (T\varphi)(\tau) = \frac{\omega(\tau)}{\lambda_i} \int\limits_{\gamma_1} \frac{\varphi(\zeta)}{\omega(\zeta)} \frac{d\zeta}{\zeta - \tau}\,,\ \ \tau \in \gamma_1,$$

is continuous in $L^p(\gamma_1)$.

Let D be the doubly-connected domain bounded by the simple piecewise smooth curves Γ_1 and Γ_2. By $t_1, t_2, \dots, t_n$ we denote angular points of the boundary $\Gamma = \Gamma_1 \cup \Gamma_2$. Assume that at the given points the angle sizes, interior with respect to the domain D, are equal to $\pi\nu_k$, $0 \le \nu_k \le 2$, $k = \overline{1, n}$. A set of such boundaries we denote by $C^1(t_1, t_2, \dots, t_n; \nu_1, \nu_2, \dots, \nu_n)$.

Statement 4. ([8]). If the doubly-connected domain D is bounded by the boundary from $C^1(t_1, \dots, t_n; \nu_1, \dots, \nu_n)$, $0 \le \nu_k \le 2$, and if $z = z(w)$ is the conformal mapping of the ring K onto D, such that $z(\gamma_i) = \Gamma_i$, $i = 1, 2$, $z(a_k) = t_k$, then

$$z'(w) = \prod_{k=1}^{n} (w - a_k)^{\nu_k - 1} z_0(w), \tag{1}$$

where

$$z_0^{\pm 1} \in \bigcap_{\delta > 1} E^\delta(K), \tag{2}$$

but if $0 < \nu_k < \min(2;p)$, then

$$\sqrt[p]{z'(e^{i\mu})},\ \sqrt[p]{z'(\rho e^{i\mu})} \in W^p. \tag{3}$$

2^0. On the Dirichlet Problem in Smirnov Classes in Simply-Connected Domains

Let G be the simply-connected domain. Aassume

$$e^p(G;\omega) = \big\{u :\ u = \operatorname{Re}\Phi,\ \Phi \in E^p(G;\omega)\big\},\ \ e^p(G) \equiv e^p(G;1).$$

In [3] and [5] (Ch. IV), we investigated the Dirichlet problem in the class $e^p(G)$, when G was the domain bounded by the piecewise smooth Jordan curve. The obtained results can be understood as a solution of the Dirichlet problem in a certain weighted Smirnov class for a circle. Indeed, the following theorem is valid.

Theorem 5. *Let $\zeta=\zeta(w)$ be the conformal mapping U onto the simply-connected domain with the boundary $C^1(b_1,\dots,b_n;\nu_1,\dots,\nu_n)$, $0\le\nu_k\le 2$ and $z(a_k)=b_k$, $k=\overline{1,n}$. Then for the problem*

$$\begin{cases}\Delta u=0, & u\in e^p(U;\sqrt[p]{\zeta'}),\\ u\big|_{\gamma_1}=\varphi, & \varphi\in L^p(\gamma_1;\sqrt[p]{\zeta'}),\end{cases}\tag{4}$$

the following statements are valid.

All the solutions of the homogeneous problem (i.e., of the problem (4) for $\varphi=0$) are given by the equality

$$u_0(w)=\sum_{k=1}^{n}A_k(p)\ \mathrm{Re}\,\frac{a_k+w}{a_k-w}\,,\tag{5}$$

where

$$A_k(p)=\begin{cases}0, \ \text{if } 0\le\nu_k<p, \ \text{or } \nu_k=p \ \text{and}\\ \qquad X_k\overline{\in}E^p(U), \ X_k=(w-a_k)^{-1/p}z_0^{1/p},\\ A_k \ \text{is an arbitrary real constant for } p<\nu_k\le 2,\\ \qquad \text{or } \nu_k=p \ \text{and } X_k\in E^p(U).\end{cases}\tag{6}$$

The inhomogeneous problem is, generally speaking, unsolvable if there are angular points ν_k from the set $\{0;p\}$. If, however, instead of the condition $\varphi\in L^p(\gamma_1;\sqrt[p]{\zeta'})$ is fulfilled the condition

$$\varphi(\tau)\ln\Big|\prod_{\nu_k\in\{0;p\}}(w(\tau)-a_k)\Big|\in L^p(\gamma_1;\sqrt[p]{\zeta'}),\tag{7}$$

where $w=w(\zeta)$ is the function, inverse to $\zeta=\zeta(w)$, then the problem (4) is solvable.

In all cases for which the problem (4) is solvable, its solution is given by the equality

$$u(w)=u_0(w)+u_\varphi(w),$$

where u_0 is the function, defined by the equality (5), and

$$u_\varphi(w)=\mathrm{Re}\bigg[\bigg(\frac{1}{2\pi i}\int\limits_{\gamma_1}\frac{\varphi(\tau)\,\rho(\tau)}{\tau-w}\,d\tau+ \\ +\frac{(-1)^{n_i}}{2\pi i}\,w^{n_1+1}\int\limits_{\gamma_1}\frac{\varphi(\tau)\,\overline{\rho}(\tau)}{\tau(\tau-w)}\,d\tau\bigg)\frac{1}{\rho(w)}\bigg].\tag{8}$$

Here n_1 is a quantity of numbers ν_k from the interval $(p,2]$ (for $p=2$ we put $(2;2]=\varnothing$), $\rho(w)=\prod\limits_{\nu_k\in(p;2]}(w-a_k)$ and $\rho(\tau)=1$, if $\{\nu_k:\nu_k\in(p;2]\}=\varnothing$.

Indeed, if $u=\mathrm{Re}\,\Phi$, then we write the problem (4) as follows:

$$\Phi^+(\tau)+\overline{\Phi^+(\tau)}=2\varphi(t),,$$

whence, assuming $\Phi(z)=\overline{\Phi\big(\frac{1}{\bar z}\big)}$ for $|z|>1$, we obtain

$$\sqrt[p]{\zeta'(\tau)}\,\Phi^+(\tau)+\frac{\sqrt[p]{\zeta'(\tau)}}{\overline{\sqrt[p]{\zeta'(\tau)}}}\,\overline{\sqrt[p]{\zeta'(\tau)}}\,\Phi^-(\tau)=2\sqrt[p]{\zeta'(\tau)}\,\varphi(\tau).$$

If we put $g(t)=2\sqrt[p]{\zeta'(\tau)}\varphi(\tau)$, $\Psi(w)=2\sqrt[p]{\zeta'(w)}\Phi(w)$, $|w|<1$ and

$$\Omega(w)=\begin{cases}\Psi(w), & |w|<1,\\ \overline{\Psi\Big(\dfrac{1}{\overline{w}}\Big)}, & |w|>1,\end{cases}\qquad \Omega_*(w)=\overline{\Omega\Big(\frac{1}{\overline{w}}\Big)},\ \ |w|\neq 1,$$

then we find that

$$\begin{cases}\Omega^+(\tau)=-\dfrac{\sqrt[p]{\zeta'(\tau)}}{\overline{\sqrt[p]{\zeta'(\tau)}}}\,\Omega^-(\tau)+g(\tau), & g\in L^p(\gamma_1),\\ \Omega(w)=\Omega_*(w),\quad |w|\neq 1, & \Omega\in E^p(U),\end{cases}\tag{9}$$

i.e. we obtain the problem (1.6) from [5], (p. 156) for which Theorem 1.2 from Chapter IV of [5], (p. 168) is valid. Thus the above Theorem 5 is the reformulated version of that theorem with respect to the problem (4).

Remark 6. If we consider the problem (4) on the complement $C\overline{U}$ of the unit circle U, then because of the fact that the functions from $e^p(C\overline{U})$ are representable by the Poisson integrals and equal at infinity to zero, the statement of the above Theorem 5 somewhat varies. Namely, for the problem to be solvable, it is necessary (and in the presence of angular points with ν_k from $\{0;p\}$ for φ with the condition (7)) and sufficient that the condition

$$\int\limits_0^{2\pi}\varphi(e^{i\mu})\,d\mu+\sum_{k=1}^n A_k(p)=0\tag{10}$$

be fulfilled.

3^0. The Dirichlet Problem in the Domain with the Boundary from the Class $C^1(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$

Let D be the doubly-connected domain with the boundary $\Gamma=\Gamma_1\cup\Gamma_2$ from the class $C^1(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$, $0\le\nu_k\le 2$. We consider the Dirichlet problem which is formulated as follows: find the function $u(z)$ satisfying the conditions

$$\begin{cases}\Delta u=0, & u\in e^p(D),\quad p>1,\\ u\big|_{\Gamma_i}=f_i, & f_i\in L^p(\Gamma_i),\quad i=1,2.\end{cases}\tag{11}$$

According to Statement 2, the function $V(w)=u(z(w))$ belongs to $e^p(U;\sqrt[p]{z'})$ and therefore the problem (11) can equivalently be reduced to the problem

$$\begin{cases} \Delta V = 0, & V \in e^p(U; \sqrt[p]{z'}), \\ V\big|_{\gamma_i} = g_i, & g_i(\tau) = f_i(z(\tau)) \in L^p(\gamma_i, \sqrt[p]{z'}). \end{cases} \tag{12}$$

(for simply-connected domains, for details see [5], pp. 156–157).

Any function V from $e^p(U;\omega)$, $\omega = \sqrt[p]{z'}$ is representable in the form $V(w) = V_1(w) + V_2(w)$, $V_i \in e^p(K_i)$, where K_i is that domain, bounded by γ_i, which contains K. Hence

$$V(w) = V(re^{i\mu}) = \frac{1}{2\pi} \int_0^{2\pi} \delta(\alpha)\, \frac{1-r^2}{1+r^2-2r\cos(\alpha-\mu)}\, d\alpha + $$

$$+ \frac{1}{2\pi} \int_0^{2\pi} \lambda(\alpha)\, \frac{\rho^2-r^2}{\rho^2+r^2-2r\rho\cos(\alpha-\mu)}\, d\alpha,$$

where the functions δ and λ belong to $L^p(I,\omega)$, $I = [0, 2\pi]$ and also

$$\int_0^{2\pi} \lambda(\mu)\, d\mu = 0. \tag{13}$$

(see [8]).

Let V be a solution of the problem (12), then $V = V_1 + V_2$, $V_i \in e^p(K_i; w)$. The summand V_2 satisfies the conditions

$$\begin{cases} \Delta V_2 = 0, \quad V_2 \in e^p(K_2;\omega), \quad K_2 = \{w : |w| > \rho\}, \\ V_2\big|_{\gamma_2} = g_2(\rho e^{i\mu}) - V_1(\rho e^{i\mu}). \end{cases} \tag{14}$$

If on γ_2 there are the points a_k for which $\nu_k \in \{0, p\}$, then there is the function $g_2 \in L^p(\gamma_2;\omega)$, such that the problem (14) is unsolvable (see [5], pp. 165–168), and thus the problem (11) is likewise unsolvable. If, however, for g_2 is fulfilled the assumption of the form (7) in Theorem 5, i.e., if

$$g_2(\tau)\ \ln \prod_{a_k\in\gamma_2,\, \nu_k\in\{0,p\}} |\tau - a_k| \in L^p(\gamma_2;\omega) \tag{15}$$

then the problem (14) is solvable. Its solution is given by the formula of type (8). Consequently, the narrowing of the function $V_2(re^{i\mu})$ on γ_1 is contained in the set of functions $V_2(e^{i\mu}) + \sum\limits_{a_k\in\gamma_2} A_k(p)\ \mathrm{Re}\, \frac{e^{i\mu}+a_k}{e^{i\mu}-a_k} = V_2(e^{i\mu}) + V_{2,0}(e^{i\mu})$, where $V_{2,0}(e^{i\mu}) = \sum\limits_{a_k\in\gamma_2} A_k(p)\ \mathrm{Re}\, \frac{e^{i\mu}+a_k}{e^{i\mu}-a_k}$, and the real constants $A_k(p)$ are defined according to equality (6). By Remark 6, we have

$$\int_0^{2\pi} V_2(\rho e^{i\mu}) d\mu + \sum_{a_k\in\gamma_2} A_k(p) = 0. \tag{16}$$

V_1 is now contained in the set of functions satisfying the conditions

$$\begin{cases} V_1 \in e^p(K_1;\omega), \quad K_1 = \{w : |w| < 1\}, \\ V_1\big|_{\gamma_1} = g_1(e^{i\mu}) - V_2(e^{i\mu}) - \sum\limits_{a_k \in \gamma_2} A_k(p) \operatorname{Re} \dfrac{e^{i\mu} + a_k}{e^{i\mu} - a_k} = \\ \qquad = g_1 - V_2 - V_{2,0}. \end{cases} \tag{17}$$

By Theorem 5, we have

$$V_1(w) = \sum_{a_k \in \gamma_1} A_k(p) \operatorname{Re} \frac{w + a_k}{w - a_k} + V_{\widetilde{g}_1}(e^{i\mu}), \quad w \in K_1. \tag{18}$$

where $A_k(p)$ are the real constants defined by equality (6), and $V_{\widetilde{g}_1}$ is a particular solution of the problem (17) which is representable by the Poisson integral, and therefore

$$\begin{aligned} V_{\widetilde{g}_1}(re^{i\mu}) = & \frac{1}{2\pi} \int_0^{2\pi} g_1(\alpha) \frac{1 - r^2}{1 + r^2 - 2r\cos(\alpha - \mu)}\, d\alpha - \\ & - \frac{1}{2\pi} \int_0^{2\pi} V_2(e^{i\alpha}) \frac{1 - r^2}{1 + r^2 - 2r\cos(\alpha - \mu)}\, d\alpha - \\ & - \sum_{a_k \in \gamma_2} \frac{1}{2\pi} \int_0^{2\pi} A_k(p) \operatorname{Re} \frac{e^{i\alpha} + a_k}{e^{i\alpha} - a_k} \, \frac{1 - r^2}{1 + r^2 - 2r\cos(\alpha - \mu)}\, d\alpha = \\ = & (Pg_1)(r,\mu) - P(V_2)(r,\mu) - P(V_{2,0})(r,\mu); \;\; re^{i\mu} \in K_1, \end{aligned} \tag{19}$$

whence it follows that in the ring K

$$\begin{aligned} V(w) = \sum_{a_k \in \gamma_1} & A_k(p) \operatorname{Re} \frac{w + a_k}{w - a_k} - P(V_2)(w) + P(g_1)(w) - \\ & - P(V_{2,0})(w) + V_2(w). \end{aligned}$$

Since V is the solution of the problem (12), we have $V(\rho e^{i\mu}) = g_2(\rho e^{i\mu})$, and the last equality yields

$$\begin{aligned} V_2(\rho e^{i\mu}) = & \frac{1}{2\pi} \int_0^{2\pi} V_2(e^{i\alpha}) \frac{1 - \rho^2}{1 + \rho^2 - 2\rho\cos(\alpha - \mu)}\, d\alpha + P(g_1)(\rho,\mu) - \\ & - P(V_{2,0})(\rho,\mu) + \sum_{a_k \in \gamma_1} A_k(p) \operatorname{Re} \frac{\rho e^{i\mu} + a_k}{\rho e^{i\mu} - a_k} = g_2(\rho e^{i\mu}). \end{aligned} \tag{20}$$

As far as

$$V_2(re^{i\alpha}) = \frac{1}{2\pi} \int_0^{2\pi} V_2(\rho e^{i\beta})) \frac{r^2 - \rho^2}{r^2 + \rho^2 - 2r\rho\cos(\beta - \alpha)}\, d\beta,$$

therefore

$$V_2(e^{i\alpha})=\frac{1}{2\pi}\int\limits_0^{2\pi}V_2(\rho e^{i\beta})\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\beta)}\,d\beta.$$

We substitute this value $V_2(e^{i\alpha})$ into (20) and obtain

$$\begin{aligned}V_2(\rho e^{i\mu})-\frac{1}{2\pi}\int\limits_0^{2\pi}\Big[\frac{1}{2\pi}\int\limits_0^{2\pi}V_2(\rho e^{i\beta})\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\beta)}\,d\beta\Big]\times\\ \times\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\mu)}\,d\alpha+P(g_1)(\rho,\mu)-P(V_{2,0})(\rho,\mu)+\\ +\sum_{a_k\in\gamma_1}A_k(p)\ \mathrm{Re}\,\frac{\rho e^{i\mu}+a_k}{\rho e^{i\mu}-a_k}=g_2(\rho e^{i\mu}),\end{aligned}$$

i.e.

$$V_2(\rho e^{i\mu})+(KV_2)(\rho,\mu)=\widetilde{g}_2(\rho,\mu), \tag{21}$$

where

$$\begin{aligned}(KV_2)(\rho,\mu)=-\frac{1}{2\pi}\int\limits_0^{2\pi}\Big[\frac{1}{2\pi}\int\limits_0^{2\pi}V_2(\rho e^{i\beta})\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\beta)}\,d\beta\Big]\times\\ \times\frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\mu)}\,d\alpha,\\ \widetilde{g}_2(\rho,\mu)=g_2(\rho e^{i\mu})-(Pg_1)(\rho,\mu)+P(V_{2,0})(\rho,\mu)-\\ -\sum_{a_k\in\gamma_1}A_k(p)\ \mathrm{Re}\,\frac{\rho e^{i\mu}+a_k}{\rho e^{i\mu}-a_k}\,.\end{aligned} \tag{22}$$

The equation of type (21) in the classes $L^p(I,\omega)$ has been investigated in [8] for $\omega\in W_E^\rho$ (see [8], equation (15)). The class W_E^ρ is, in fact, the set of functions ω-analytic in K, belonging to $\underset{\delta>0}{U}E^\delta(K)$, and their narrowing on $\gamma_i-\omega_1=\omega(e^{i\mu})$ and $\omega_2=\omega(\rho e^{i\mu})$ belong to W^p.

Equation (21) is the Fredholm one in the class $L^p(I,\omega)$. It is solvable, if and only if

$$\begin{aligned}\int\limits_0^{2\pi}(Pg_1)(\rho,\mu)\,d\mu+\sum_{a_k\in\gamma_1}A_k(p)\int\limits_0^{2\pi}\mathrm{Re}\,\frac{\rho e^{i\mu}+a_k}{\rho e^{i\mu}-a_k}\,d\mu-\\ -\int\limits_0^{2\pi}P(V_{2,0})(\rho,\mu)\,d\mu=\int\limits_0^{2\pi}g_2(\rho e^{i\mu})\,d\mu.\end{aligned} \tag{23}$$

(see [8]).

It can be easily verified that

$$\int_0^{2\pi} (Pg_1)(\rho,\mu)d\mu = \int_0^{2\pi} g_1(e^{i\mu})\, d\mu.$$

Taking into account the equality

$$\frac{1}{2\pi i}\int_0^{2\pi} \frac{\rho e^{i\mu}+a_k}{\rho e^{i\mu}-a_k}\, \frac{d\rho e^{i\mu}}{\rho e^{i\mu}} = -1, \quad |a_k|=1,$$

we find that

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{\rho e^{i\mu}+a_k}{\rho e^{i\mu}-a_k}\, d\mu = -1.$$

Analogously

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Re}\, \frac{e^{i\mu}+a_k}{e^{i\mu}-a_k}\, d\mu = 1, \quad |a_k|=\rho.$$

Moreover,

$$\int_0^{2\pi} P(V_{2,0})d\mu = \int_0^{2\pi} \frac{1}{2\pi}\int_0^{2\pi} \sum_{a_k\in\gamma_2} A_k(p)\ \mathrm{Re}\, \frac{e^{i\mu}+a_k}{e^{i\mu}-a_k}\, \frac{1-\rho^2}{1+\rho^2-2\rho\cos(\alpha-\beta)}\, d\alpha\, d\mu =$$

$$= \sum_{a_k\in\gamma_2} A_k(p) \int_0^{2\pi} \mathrm{Re}\, \frac{e^{i\mu}+a_k}{e^{i\mu}-a_k}\, d\alpha = 2\pi \sum_{a_k\in\gamma_2} A_k(p).$$

Consequently, the condition (23) takes the form

$$\int_0^{2\pi} g_1(e^{i\mu})\, d\mu + 2\pi\Big(\sum_{a_k\in\gamma_1} A_k(p) - \sum_{a_k\in\gamma_2} A_k(p)\Big) = \int_0^{2\pi} g_2(\rho e^{i\mu})d\mu. \tag{24}$$

Thus if in equation (21): $\omega \in W_E^p$ and

$$\Big[g_2 - P(g_1) - P(V_{2,0}) - \sum_{a_k\in\gamma_1} A_k(p)\ \mathrm{Re}\, \frac{\rho e^{i\mu}+a_k}{\rho e^{i\mu}-a_k}\Big] \in L^p(I,\omega), \tag{25}$$

and the condition (24) is fulfilled, then it is solvable. These conditions are fulfilled if $\nu_k \overline{\in} \{0,p\}$. If, however, $\nu_k \in \{0,p\}$, then the inclusion (25) is valid if instead of the condition $g_1 \in L^p(\gamma_1;\omega)$ we require

$$g_1(\tau)\ \ln \prod_{a_k\in\gamma_1,\, \nu_k\in\{0,p\}} (\tau-a_k) \in L^p(\gamma_1;\omega). \tag{26}$$

If this condition and equalities (24) are fulfilled, then equation (21) is uniquely solvable.

Let

$$V_2(\rho e^{i\mu}) = M(\widetilde{g}_2)(\rho,\mu) \tag{27}$$

(see (21), (22)).

Using equalities (18) and (19), we are able to define the function $V_1(re^{i\mu})$:

$$V_1(re^{i\mu}) = \sum_{a_k\in\gamma_1} A_k(p)\ \mathrm{Re}\ \frac{re^{i\mu}+a_k}{re^{i\mu}-a_k} + (Pg_1)(r,\mu)- \\ - P(V_{2,0})(r,\mu) + P(V_2)(r,\mu). \tag{28}$$

In order for the function $V = V_1 + V_2$ to provide us with a solution of the problem (12), the condition (16) should, according to Remark 6, be fulfilled, i.e.,

$$\int_0^{2\pi} M(\widetilde{g}_2)(\rho,\mu)d\mu + \sum_{a_k\in\gamma_2} A_k(p) = 0. \tag{29}$$

Thus we have proved the following

Theorem 7. *Let the doubly-connected domain D be bounded by the Jordan curves Γ_1 and Γ_2; Γ_2 lies inside of Γ_1, while $\Gamma = \Gamma_1 + \Gamma_2$ belongs to $C^1(t_1,t_2,\dots,t_n;\nu_1,\nu_2,\dots,\nu_n)$, $0 \le \nu_k \le 2$. Further, let $z = z(w)$ be the conformal mapping of the ring $K = \{w : \rho < |w| < 1\}$ onto D, $z(a_k) = t_k$ and $z(\gamma_i) = \Gamma_i$ ($\gamma_1 = \{\tau : |\tau| = 1\}$, $\gamma_2 = \{\tau : |\tau| = \rho < 1\}$).*

If among the point t_k there are such for which $\nu_k \in \{0,p\}$, then the problem (11) is, generally speaking, unsolvable. If, however, instead of the conditions $f_i \in L^p(\Gamma_i)$ are fulfilled the conditions (15) and (26), and the real constants are defined by equality (6), then for the problem (11) to be solvable, it is necessary and sufficient that the conditions (24) and (29) be fulfilled. If they are fulfilled, then the solution is given by the equality $u(z) = V(w(z))$, where $w = w(z)$ is the function, inverse to $z = z(w)$, and $V(w) = V_1(w) + V_2(w)$, where

$$V_2(re^{i\mu}) = \int_0^{2\pi} V_2(\rho e^{i\alpha}) \frac{r^2-\rho^2}{r^2+\rho^2-2r\rho\cos(\alpha-\mu)}\,d\alpha. \tag{30}$$

Here $V_2(\rho e^{i\alpha})$ is the solution of equation (21) defined uniquely for any number of constants $A_k(p)$ with the condition (6) and satisfying the conditions (24) and (29); V_1 is the function given by equality (28).

Remark 8. If Γ is the piecewise-Lyapunov boundary, then for $\nu_k = p$ we have $X_k \overline{\in} E^p(u)$ (see [8]), and if all ν_k belong to $(0,p)$, then for any k we have $A_k(p) = 0$. Therefore the condition (24) takes the form

$$\int_0^{2\pi} g_1(e^{i\mu})d\mu = \int_0^{2\pi} g_2(\rho e^{i\mu})d\mu, \tag{31}$$

and under that condition $\int\limits_0^{2\pi} M(\widetilde{g}_2)d\mu = 0$, and hence (29) is fulfilled.

Thus for the domains with boundaries of the above-indicated type, the problem (11) is uniquely solvable.

This fact has been stated in [8].

Remark 9. If Γ is the piecewise-Lyapunov curve from the set $C^1(t_1,\dots,t_n;\nu_1,\dots,\nu_n)$, and also $\nu_k\overline{\in}\{0,\rho\}$, $k=\overline{1,n}$, then the solution contains $n_1+(n_2-1)$ arbitrary constants, where n_i is a number of points lying on Γ_i for which $\nu_k>p$. The difference of contribution of such points lying on different curves Γ_i is caused by the condition (16) which in its turn results from the fact that the summands V_i, in the representation $V=V_1+V_2$, belong to $e^p(K_i,\overline{)}$, and hence the function V_2 vanishes at infinity.

Remark 10. The problem (11) can also be considered in the weight class $e^p(D;r)$, where

$$r(t)=\prod_{k=1}^m (t-c_k)^{\alpha_k}, \quad c_k\in\Gamma, \quad -\frac{1}{p}<\alpha_k<\frac{1}{p'}\,, \quad p'=\frac{p}{p-1}\,,$$

just as it takes place in [8]. Depending on whether the points from the set $\{c_1,\dots,c_n\}$ coincide with some of the points t_k, $k=\overline{1,n}$, under the conditions of solvability of the Dirichlet problem, besides the condition $-\frac{1}{p}<\alpha_k<\frac{1}{p'}$ there appears the condition $-\frac{1}{p}<\frac{\nu_k-1}{p}+\alpha_k<\frac{1}{p'}$ (for $c_j=t_k$). We omit the details and refer the reader to [8].

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Authors' address:

A. Razmadze Mathematical Institute
1, Aleksidze St., Tbilisi 0193
Georgia