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THE NOETHER PROPERTY OF A REGULAR OPERATOR WITH CONSTANT COEFFICIENTS IN A REGION

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Abstract. In this paper we proved the criterion for the regular operator with constant coefficients to be Noetherian. We show that the index of that operator in the region is equal to zero.

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The present work is devoted to the investigation of the Noether property of a linear differential regular operator with constant coefficients in a region. It is proved that for the operator to be Noetherian, it is necessary and sufficient for it to be regular. In particular, it is proved that the index of the regular operator with constant coefficients in the region is equal to zero.

The index theory of an elliptic operator has been studied by many authors. In particular, the two-dimensional theory (R^2) was, to a considerable extent, completed in 1961 by A.I. Volpert [1], who studied general boundary value problems for an arbitrary elliptic system in a simply-connected bounded region on a plane, and proved the equivalence of ellipticity and Noetherity of these problems in spaces of sufficiently smooth functions. M.S. Agronovich [2] proved that for a singular integro-differentail operator on a smooth manifold to be elliptic, it is necessary and sufficient for it to be Noetherian. L. A. Bagirov [3] has proved that if coefficients of the elliptic operator are sufficiently smooth in R^n, then the operator in certain weight spaces is Noetherian. However, the index theory of the regular operator is little studied. In [4] and [5] we proved that if some of the supplementary conditions on the symbol of the operator in R^n (in the statement of which there occur lowest terms) are fulfilled, the index of the semi-elliptical operator in certain weighted spaces is finite.

In what follows, the use will be made of the following standard notation: R^n is the n-dimensional Euclidean space, Z_+^n is a set of multiindices, i.e., of vectors $\alpha = (\alpha_1, \dots, \alpha_n)$ with integral, nonnegative components. For $x, \xi \in$

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R^n and $\alpha \in Z_+^n$ we put $|\alpha| = \alpha_1 + \cdot + \alpha_n$, $\alpha! = \alpha_1! \dots \alpha_n!$, $\xi^\alpha = \xi_1^{\alpha^1} \dots \xi_n^{\alpha_n}$ where $D_k = \dfrac{1}{i}\dfrac{\partial}{\partial x_k}$ $i^2 = -1$.

Let $\mathcal{N} = \{e^1, \dots, e^N\}$, $e^j \in Z_+^n$ $(j = 1, 2, \dots, N)$.

Definition 1. A characteristic polytope of a set of multiindices $\mathcal{N}$ is caled a smallest convex polytope $\mathcal{R} = \mathcal{R}(\mathcal{N})$ in R^n which contains all points of the set $\mathcal{N}$.

Definition 2. A nonempty polytope $\mathcal{R}$ is called complete if the origin of coordinates Z_+^n is the vertex of $\mathcal{R}$, and $\mathcal{R}$ has vertices on each of the coordinate axes Z_+^n, different from the origin.

The complete polytope $\mathcal{R}$ is called entirely perfect, if outer normals of all $(n-1)$-dimensional non-coordinate faces of $\mathcal{R}$ have only positive coordinates.

The multiindex $\alpha \in \mathcal{R}$ is called principal if it belongs to any $(n-1)$-dimensional non-coordinate face of the polytope $\mathcal{R}$. The set of all principal points from $\mathcal{R}$ we denote by $\partial\mathcal{R}$.

Let $\mathcal{R}$ be an arbitrary entirely perfect polytope, and k be a positive number. Suppose $\mathcal{R}^0 = \mathcal{R} \backslash \partial'\mathcal{R}$, $\mathcal{R}^k = \{k\alpha = (k\alpha_1, k\alpha_2, \dots, k\alpha_n), \alpha \in \mathcal{R}\}$. By $\alpha \in \mathcal{R}$ we mean $\alpha \in \mathcal{R} \cap Z_+^n$.

By $\mathcal{R}_k^{n-1}$ we denote $(k = 1, 2, \dots, I_{n-1})(n-1)$-dimensional faces of the polytope $\mathcal{R}$.

Let μ^k $(k = 1, \dots, I_{n-1})$ be such outer normal of the face of $\mathcal{R}_k^{n-1}$ for which for all $\alpha \in \mathcal{R}_k^{n-1}$, $(\mu^k, \alpha) = 1$. By $a^k = (0, \dots, 0, a_k, 0, \dots, 0) \in \partial'\mathcal{R}$ $a_k \neq 0$ we denote the vertex of the polytope which lies on the k-th coordinate axis. Assume $\gamma_k = \frac{1}{a_k}$ $(k = 1, \dots, n)$, $\lambda_{\max} = \max\limits_{1 \le j \le n} \lambda_j$, $\lambda_{\min} = \min\limits_{1 \le j \le n} \lambda_j$.

For $x \in R^n$, $\mu = (\mu_1, \mu_2, \dots, \mu_n)$ we denote $|x|_\mu = \left(\sum_{i=1}^n |x_i|^{\frac{2}{\mu_i}} \right)^{\frac{1}{2}}$.

Definition 3. For the polytope $\mathcal{R}$ and for the bounded region $\Omega \subset R^n$ we denote by $H^{\mathcal{R}}(\Omega)$ a set of measurable functions $\{u\}$ with a finite norm

$$\|u\|_{\mathcal{R}}(\Omega) \equiv \left(\sum_{\alpha \in \mathcal{R}} \int\limits_\Omega |D^\alpha u(x)|^2 dx \right)^{\frac{1}{2}}, \tag{1}$$

and by $H^{\mathcal{R}}(\Omega)$ we denote the closure of the set $C_0^\infty(\Omega)$ with respect to the norm (1).

For an open bounded cube Δ $(\overline{\Omega} \subset \Delta)$ and for the function $\Phi \in \dot{H}^{\mathcal{R}}(\Delta)$ we denote $H^{\mathcal{R}}(\Omega, \Phi) = \{u \in H^{\mathcal{R}}(\Omega); (u - \Phi) \in \dot{H}^{\mathcal{R}}(\Omega)\}$.

In the sequel, it will be assumed that k is a natural number, $\Phi \in \dot{H}^{\mathcal{R}+1}(\Delta)$ is a fixed function, the region $\Omega \subset R^n$ satisfies the condition of the rectangle (see, for e.g., [6]), $\mathcal{R}$ is the entirely perfect polytope, and all multiindices from $\partial'\mathcal{R}$ have even coordinates.

We will consider the operators having only real coefficients.

Definition 4. We say that the linear differential operator $P(D) = \sum\limits_{\alpha\in\mathcal{R}} p_\alpha D^\alpha$ is regular, if for some constant $\chi > 0$

$$|P^0(\xi)| \geq \chi \sum_{\alpha\in\partial'\mathcal{R}} |\xi^\alpha|, \quad \forall\xi \in R^n,$$

where $P^0(\xi) \equiv \sum\limits_{\alpha\in\partial'\mathcal{R}} p_\alpha \xi^\alpha$.

Definition 5. We say that the linear bounded operator A acting from the Banach space B_1 to the Banach space B_2 is Noetheria, if:

(1) a subspace of solutions of the equation $Au = 0$ in B_1 is of finite dimension, i.e., $\dim \operatorname{Ker} A < \infty$;

(2) the range of values $\{A : B_1\}$ of the operator A in B_2 is closed;

(3) the factor-space $B_2/\{A : B_1\}$ is of finite dimension, i.e., $\dim \operatorname{Coker} A < \infty$.

The difference $\operatorname{Ind} A = \dim \ker A - \dim \operatorname{Coker} A$ is called the index of the operator A.

It is well-known from the theory of Noetherian operators that

$$\dim \operatorname{Ker} A^* = \dim \operatorname{Coker} A, \quad \text{and} \quad \dim \operatorname{Coker} A^* = \dim \operatorname{Ker} A,$$

where the operator A^* is formally conjugate to A.

Let $P_0(D) \equiv \sum_{\alpha\in\partial'\mathcal{R}} p_a l D^\alpha + p_0$ be the regular operator with constant coefficients for which $P_0(\xi) \equiv \sum_{\alpha\in\partial'\mathcal{R}} p_\alpha \xi^\alpha + p_0 \neq 0$ for all $\xi \in R^n$. Then it is evident that for some constant $\chi > 0$,

$$|P_0(\xi)| \geq \chi\biggl(\sum_{\alpha\in\partial'\mathcal{R}} |\xi^\alpha| + 1\biggr) \quad \forall\xi \in R^n. \tag{2}$$

The following theorem is known (see [7]).

Theorem 1. *Let $P_0(D)$ satisfy the condition* (2). *Then there exists the only one function $u \in H^{\mathcal{R}^{a/2}}(\Omega, \Phi)$ which is a solution of the equation $P_0(D)u = 0$.*

Corollary 1. *The kernel of the operator $P_0(D)$, acting from $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$ to $H^{\mathcal{R}^k}(\Omega, \Phi)$, is of zero dimension, i.e.,*

$$\dim \operatorname{Ker} P_0 = 0.$$

Proof. Since $H^{\mathcal{R}^{k+1}}(\Omega, \Phi) \subset u \in H^{\mathcal{R}^{a/2}}(\Omega, \Phi)$, from Theorem 1 it immediately follows that the equation $P_0(D)u = 0$ can have not more than one solution in $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$.

It is not difficult to see that the operator $P_0(D)$, considered as that from $H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$ to $H^{\mathcal{R}^k}(\Omega,\Phi)$, is self-conjugate. Consequently,

$$\dim\ker P_0 = 0.$$

By virtue of the fact that the region of values of the operator $P_0(D)$, considered as that from $H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$ to $H^{\mathcal{R}^k}(\Omega,\Phi)$, is closed, the operator $P_0(D)$, acting from $H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$ to $H^{\mathcal{R}^k}(\Omega,\Phi)$, is Noetherian, and

$$\operatorname{Ind} P_0 = \dim\operatorname{Ker} P_0 - \dim\operatorname{Co}\ker P_0 = 0. \quad \square$$

Thus we have proved the following

Theorem 2. *The operator $P_0(D)$, acting from $H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$ to $H^{\mathcal{R}^k}(\Omega,\Phi)$, is Noetherian, and its index is equal to zero.*

The following theorem is known (see [2]).

Theorem 3. *Let the operator $A(D)$ be bounded from $\dot H^{\mathcal{R}^{k+1}}(\Omega)$ to $\dot H^{\mathcal{R}^k}(\Omega)$ for which for any $\varepsilon>0$ and a constant $M_\varepsilon>0$*

$$\|A(D)u\|_{\mathcal{R}^k}(\Omega) \le \varepsilon\|u\|_{\mathcal{R}^{k+1}}(\Omega) + M_\varepsilon\|u\|_{L_2}(\Omega) \quad \forall u\in \dot H^{cR^{k+1}}(\Omega).$$

Then $A(D)$ is the compact operator from $\dot H^{cR^{k+1}}(\Omega)$ to $\dot H^{cR^k}(\Omega)$.

Let $P_1(D)$ be the linear differential operator with constant coefficients of the type

$$P_1(D) = \sum_{\alpha\in\mathcal{R}^0} p_\alpha D^\alpha.$$

Theorem 4. *$P_1(D)$ is the compact operator from $\dot H^{cR^{k+1}}(\Omega)$ to $\dot H^{cR^k}(\Omega)$.*

Since for any $\varepsilon>0$ and $\alpha\in Z_+^n$ $\alpha\in\mathcal{R}^0$ (see [6], Ch. VI) there exists the constant $C_{\varepsilon,\alpha}>0$, such that

$$\|D^\alpha u\|_{\mathcal{R}^k}(\Omega) \le \varepsilon\|u\|_{\mathcal{R}^{k+1}}(\Omega) + C_{\varepsilon,\alpha}\|u\|_{L_2}(\Omega) \quad \forall u\in \dot H^{\mathcal{R}+1}(\Omega),$$

we have

$$\|P_1(D)u\|_{\mathcal{R}^k}(\Omega) = \left\|\sum_{\alpha\in\mathcal{R}^0} p_\alpha D^\alpha u\right\|_{\mathcal{R}^k}(\Omega) + M_\varepsilon\|u\|_{L_2}(\Omega) \quad \forall u\in \dot H^{\mathcal{R}+1}(\Omega),$$

where $C=\operatorname{card}\{\alpha\in Z_+^n;\ \alpha\in\mathcal{R}^0\}\cdot\max_{\alpha\in\mathcal{R}^0}|p_\alpha|$, $M_\varepsilon=\max_{\alpha\in\mathcal{R}^0} C_{\varepsilon,\alpha}$.

Consequently, by Theorem 2, $P_1(D)$ is the compact operator from $\dot H^{\mathcal{R}^{k+1}}(\Omega)$ to $\dot H^{\mathcal{R}^k}(\Omega)$.

Corollary 2. *$P_1(D)$ is the compact operator from $\dot H^{\mathcal{R}^{k+1}}(\Omega)$ to $\dot H^{\mathcal{R}^k}(\Omega)$.*

Proof. Let $u_n\in H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$ $(n=1,2,\dots)$. Then $v\equiv u_n-\Phi\in \dot H^{\mathcal{R}^{k+1}}(\Omega)$. By Theorem 4, we can select from $\{P_1(D)v_n\}$ a convergent subsequence

$\{P_1(D)v_{n_k}\}$ $\Big(P_1(D)v_{n_k} \overset{k\to\infty}{\longrightarrow} g_v \in \dot{H}^{\mathcal{R}^k}(\Omega)\Big)$. But, on the other hand, we have

$$P_(D)u_{n_k} = P_1(D)(v_{n_k} + \Phi) =$$
$$= P_1(D)v_{n_k} + P_1(D)\Phi \overset{k\to\infty}{\longrightarrow} g_v + P_1(D)\Phi \in H^{\mathcal{R}^k}(\Omega, \Phi). \qquad \square$$

From the theory of Noether operators the following theorem is well-known (see [8], [9], [10]).

Theorem 5. *Let $P(D)$ be the Noetherian operator, and $A(D)$ be the compact operator from $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$ to $H^{\mathcal{R}^k}(\Omega, \Phi)$. Then the operator $P(D) + A(D)$ from $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$ to $H^{\mathcal{R}^k}(\Omega, \Phi)$ is likewise Noetherian, and* $\operatorname{Ind}(P + A) = \operatorname{Ind} P$.

Theorem 6. *The index of the regular operator with constant coefficients from $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$ to $H^{\mathcal{R}^k}(\Omega, \Phi)$ is finite and equal to zero.*

Proof. Let $P(D)$ be the linear differential regular operator with constant coefficients. Then for any number p_0' the operator $P(D)$ can be represented in the form

$$P(D) \equiv \sum_{\alpha\in\mathcal{R}} p_\alpha D^\alpha = P_0(D) + P_1(D),$$

where $P_0(D) \equiv \sum_{\alpha\in\partial'\mathcal{R}} p_\alpha D^\alpha + p_0'$ and $P_1(D) \equiv \sum_{\alpha\in\partial'\mathcal{R}^0} p_\alpha D^\alpha + p_0'$.

Let the number p_0' be chosen in such a way that $P_0(\xi) \neq 0$ for all $\xi \in R^n$ (this is possible on the strength of regularity and owing to the fact that the coefficients of the operator $P(D)$ are real).

From Theorem 2 and Corollary 2 it follows that $P_0(D)$ in the Noether operator ($\operatorname{Ind} P = 0$), and $P_1(D)$ is the compact one from $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$ to $H^{\mathcal{R}^k}(\Omega, \Phi)$.

Consequently, by Theorem 5, the operator $P(D)$, acting from $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$ to $H^{\mathcal{R}^k}(\Omega, \Phi)$, is Noethrian, and its index is equal to zero.

The theorem below is well-known (see [2]). $\square$

Theorem 7. *Let $P(D) = \sum_{\alpha\in\mathcal{R}} p_\alpha D^\alpha$ be the linear differential operator with constant coefficients, acting from $\dot{H}^{\mathcal{R}^{k+1}}(\Omega)$ to $\dot{H}^{\mathcal{R}^k}(\Omega)$. Then for the operator $P(D)$ to have a finite-dimensional kernel in $\dot{H}^{\mathcal{R}^{k+1}}(\Omega)$ and a closed range of values in $\dot{H}^{\mathcal{R}^k}(\Omega)$, it is necessary and sufficient that the inequality*

$$\|u\|_{\mathcal{R}^{k+1}}(\Omega) \leq C\Big(\|P(D)u\|_{\mathcal{R}^k}(\Omega) + \|u\|_{L_2}(\Omega)\Big) \quad \forall u \in \dot{H}^{\mathcal{R}^{k+1}}(\Omega),$$

holds; here C is some constant, not depending on the function u.

Corollary 3. *Let $P(D)$ be the linear differential self-conjugate operator with constant coefficients of the type*

$$P(D) = \sum_{\alpha \in \mathcal{R}} p_\alpha D^\alpha,$$

acting from $\dot{H}^{\mathcal{R}^{k+1}}(\Omega)$ to $\dot{H}^{\mathcal{R}^k}(\Omega)$. Then for the operator $P(D)$ to be Noetherian, it is necessary and sufficient that the inequality

$$\|u\|_{\mathcal{R}^{k+1}}(\Omega) \leq C\Big(\|P(D)u\|_{\mathcal{R}^k}(\Omega) + \|u\|_{L_2}(\Omega)\Big) \quad \forall u \in \dot{H}^{\mathcal{R}^{k+1}}(\Omega),$$

holds; here C is aome constant, not depending on the function u.

Corollary 4. *Let $P(D)$ be the linear differential regular operator with constant coefficients, acting from $\dot{H}^{\mathcal{R}^{k+1}}(\Omega)$ to $\dot{H}^{\mathcal{R}^k}(\Omega)$. Then for some constant $C > 0$, the following inequality holds:*

$$\|u\|_{\mathcal{R}^{k+1}}(\Omega) \leq C\Big(\|P(D)u\|_{\mathcal{R}^k}(\Omega) + \|u\|_{L_2}(\Omega)\Big) \quad \forall u \in \dot{H}^{\mathcal{R}^{k+1}}(\Omega).$$

The proof follows from Theorems 6 and 7.

Theorem 8. *Let $P(D) = \sum_{\alpha \in \mathcal{R}} p_\alpha D^\alpha$ be the linear differential operator with constant coefficients, acting from $\dot{H}^{\mathcal{R}^{k+1}}(\Omega)$ to $\dot{H}^{\mathcal{R}^{k+1}}(\Omega)$, for which for some constant $C > 0$,*

$$\|u\|_{\mathcal{R}^{k+1}}(\Omega) \leq C\Big(\|P(D)u\|_{\mathcal{R}^k}(\Omega) + \|u\|_{L_2}(\Omega)\Big) \quad \forall u \in \dot{H}^{\mathcal{R}^{k+1,v}}(\Omega). \qquad (3)$$

Then the operator $P(D)$ is regular.

Proof. Let $\mathcal{R}_j^{n-1}$ $j = 1, \dots, I_{n-1}$ be one of the $(n-1)$-dimensional faces of the polytope $\mathcal{R}$, and μ^j be its outer normal. Let M be an arbitrary positive number, and $\xi \in R^n$. Assume $M^{\frac{\mu^j}{k+1}}\xi = \Big(M^{\frac{\mu_1^j}{k+1}}\xi_1, M^{\frac{\mu_2^j}{k+1}}\xi_2, \dots, M^{\frac{\mu_n^j}{k+1}}\xi_n\Big)$.

Let $\varphi \in C_0^\infty(\Omega)$ and $\|\varphi\|_{L_2}(\Omega) = 1$. Denote $u_{\mu^j}(x) \equiv e^{i\left(M^{\frac{\mu^j}{k+1}}\xi, x\right)}\varphi(x)$. Let $(\mu^j, \alpha) \leq k+1$ $\alpha \in Z_+^n$. Then we have

$$D^\alpha u_{\mu^j}(x) = D^\alpha\Big(e^{i\left(M^{\frac{\mu^j}{k+1}}\xi, x\right)}\varphi(x)\Big) = \xi^\alpha M^{\frac{(\mu^j,\alpha)}{k+1}} e^{i\left(M^{\frac{\mu^j}{k+1}}\xi, x\right)}\varphi(x) +$$

$$+ \sum_{0 \leq \beta < \alpha} \xi^\beta M^{\frac{\mu^j,\beta}{k+1}} e^{i\left(M^{\frac{\mu^j}{k+1}}\xi, x\right)} D^{\alpha-\beta}\varphi(x) =$$

$$= \xi^\alpha M^{\frac{\mu^j,\beta}{k+1}} e^{i\left(M^{\frac{\mu^j}{k+1}}\xi, x\right)}\varphi(x)(1 + \overline{o}(1)), \quad \text{for} \quad M \to \infty$$

whence

$$D^\alpha P_j(D)u_{\mu^j}(x) = \xi^\alpha M^{\frac{\mu^j,\alpha}{k+1}} e^{i\left(M^{\frac{\mu^j}{k+1}}\xi, x\right)} \sum_{\beta \in \mathcal{R}_j^{n-1}} p_\beta \xi^\beta M^{\frac{\mu^j,\beta}{k+1}}\varphi(x)(1 + \overline{o}(1)),$$

where $P_j(D)$ is the suboperator of the operator $P(D)$, which corresponds to the face $\mathcal{R}_j^{n-1}$. Substitution into (3) yields

$$\sum_{\alpha\in(\mathcal{R}_j^{n-1})^{k+1}} |\xi^\alpha| M^{\frac{\mu^j,\alpha}{k+1}} \|\varphi\|_{L_2}(\Omega)(1+\overline{o}(1)) = \sum_{\alpha\in(\mathcal{R}_j^{n-1})^{k+1}} \|D^\alpha u_{\mu_j}\|_{L_2}(\Omega) \leq$$

$$\leq \sum_{\alpha\in(\mathcal{R})^{k+1}} \|D^\alpha u_{\mu_j}\|_{L_2}(\Omega) \leq C\Big(\sum_{\alpha\in(\mathcal{R})^k} \|D^\alpha P(D) u_{\mu^j}\|_{L_2}(\Omega) + \|u_{\mu^j}\|_{L_2}(\Omega)\Big) =$$

$$= C\Big(\sum_{\alpha\in(\mathcal{R})^k} \Big\| D^\alpha \sum_{\alpha\in\mathcal{R}_j^{n-1}} p_\beta D^\alpha u_{\mu^j}\Big\|_{L_2}(\Omega) +$$
$$+ \sum_{\alpha\in(\mathcal{R})^k} \Big\| D^\alpha \sum_{\beta\in\mathcal{R}\setminus\mathcal{R}_j^{n-1}} p_\beta D^\alpha u_{\mu^j}\Big\|_{L_2}(\Omega) + \|u_{\mu^j}\|_{L_2}(\Omega)\Big) = \tag{4}$$

$$= C\Big(\sum_{\alpha\in(\mathcal{R})^k} |\xi^\alpha| M^{\frac{\mu^j,\alpha}{k+1}} \Big|\sum_{\beta\in\mathcal{R}_j^{n-1}} p_\beta \xi^\beta M^{\frac{\mu^j,\beta}{k+1}}\Big|\Big) \|\varphi\|_{L_2}(\Omega)(1+\overline{o}(1)) +$$

$$+ \sum_{\alpha\in(\mathcal{R})^k} |\xi^\alpha| M^{\frac{\mu^j,\alpha}{k+1}} \Big|\sum_{\beta\in\mathcal{R}_j^{n-1}} p_\beta \xi^\beta M^{\frac{\mu^j,\beta}{k+1}}\Big|\Big) \|\varphi\|_{L_2}(\Omega)(1+\overline{o}(1)) + \|\varphi\|_{L_2}(\Omega).$$

Since the characteristic polytope $\mathcal{R}$ of the operator $P(D)$ is entirely perfect, it follows for any $\beta \in Z_+^n \cap (\mathcal{R}\setminus\mathcal{R}_j^{n-1})$ that $(\mu^j, \beta) < 1$. Consequently, tending in (4) $M \to \infty$ and dividing hitherto by $M(\|\varphi\|_{L_2}(\Omega) = 1)$, we obtain

$$\sum_{\alpha\in\partial'(\mathcal{R}_j^{n-1})^{k+1}} |\xi^\alpha| \leq C \sum_{\alpha\in\partial'(\mathcal{R}_j^{n-1})^k} |\xi^\alpha| \Big|\sum_{\alpha\in\partial'\mathcal{R}_j^{n-1}} p_\beta \xi^\beta\Big|$$

and hence

$$\sum_{\alpha\in\partial'\mathcal{R}_j^{n-1}} |\xi^\alpha| < C|P_j^0(\xi)|, \tag{5}$$

where

$$P_j^0(\xi) = \sum_{\alpha\in\partial'\mathcal{R}_j^{n-1}} p_\alpha \xi^\alpha.$$

It follows from (5) that the suboperator $P_j(D)$ of the operator $P(D)$, corresponding to the face $\mathcal{R}_j^{n-1}$, is regular. Since j is arbitrary, we find that all suboperators $P_j(D)$ $(j = 1, 2, \dots, I_{n-1})$ of the operator $P(D)$ are regular. This implies that the operator $P(D)$ is regular (see [11]). □

Lemma 1. *Let $P(D) = \sum_{\alpha\in cR} p_\alpha D^\alpha$ be the linear differential operator with constant coefficients, for which for some constant $C > 0$,*

$$\|v\|_{\mathcal{R}^{k+1}}(\Omega) \leq C\Big(\|P(D)v\|_{\mathcal{R}^k}(\Omega) + \|v\|_{L_2}(\Omega)\Big). \tag{6}$$

Then for some (maybe another one) constant $C_1 > 0$ the inequality

$$\|u\|_{\mathcal{R}^{k+1}}(\Omega) \le C_1\Big(\|P(D)u\|_{\mathcal{R}^k}(\Omega) + \|u\|_{L_2}(\Omega) + \|\Phi\|_{\mathcal{R}^{k+1}}\Big) \quad \forall u \in H^{\mathcal{R}^{k+1}}(\Omega, \Phi).$$

holds.

Proof. By virtue of the triangle inequality, we have

$$\begin{gathered} \|u\|_{\mathcal{R}^{k+1}}(\Omega) - \|\Phi\|_{\mathcal{R}^{k+1}}(\Omega) \le \|v\|_{\mathcal{R}^{k+1}}(\Omega), \\ \|P(D)v\|_{\mathcal{R}^{k}}(\Omega) \le \|P(D)u\|_{\mathcal{R}^{k}}(\Omega) + \|P(D)\Phi\|_{\mathcal{R}^{k}}(\Omega) \\ \|v\|_{L_2}(\Omega) \le \|u\|_{L_2}(\Omega) + \|\Phi\|_{L_2}(\Omega), \end{gathered} \tag{7}$$

where $v \equiv u - \Phi\Big(v \in \dot{H}^{\mathcal{R}^{k+1}}(\Omega)\Big)$.

From the estimates (6) and (7), taking into account that Φ is the fixed function, we have

$$\begin{gathered} \|u\|_{\mathcal{R}^{k+1}}(\Omega) \le \|v\|_{\mathcal{R}^{k+1}}(\Omega) + \|\Phi\|_{\mathcal{R}^{k+1}}(\Omega) \le \\ \le C\Big(\|P(D)v\|_{\mathcal{R}^k}(\Omega) + \|v\|_{L_2}(\Omega)\Big) + \|\Phi\|_{\mathcal{R}^{k+1}}(\Omega) \le \\ \le C\Big(\|P(D)u\|_{\mathcal{R}^k}(\Omega) + \|P(D)\Phi\|_{\mathcal{R}^k}(\Omega) + \\ + \|u\|_{L_2}(\Omega) + \|\Phi\|_{L_2(\Omega)}\Big) + \|\Phi\|_{\mathcal{R}^{k+1}}(\Omega) \le \\ \le C\Big(\|P(D)u\|_{\mathcal{R}^k}(\Omega) + \|u\|_{L_2}(\Omega)\Big) + \|\Phi\|_{\mathcal{R}^{k+1}}(\Omega). \end{gathered}$$ □

Corollary 5. *Let $P(D)$ be the linear differential regular operator with constant coefficients, acting from $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$ to $H^{\mathcal{R}^k}(\Omega, \Phi)$. Then for some constant $C > 0$ the inequality*

$$\|u\|_{\mathcal{R}^{k+1}}(\Omega) \le C(\|P(D)u\|_{\mathcal{R}^{k}}(\Omega) + \|u\|_{L_2}(\Omega) + \|\Phi\|_{\mathcal{R}^{k+1}}(\Omega)) \quad \forall u \in H^{\mathcal{R}^{k+1}}(\Omega, \Phi).$$

holds.

Corollary 6. *Let $P(D) = \sum_{\alpha \in \mathcal{R}} p_\alpha D^\alpha$ be the linear differential operator with constant coefficients, acting from $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$ to $H^{\mathcal{R}^{k+1}}(\Omega, \Phi)$, for which for some constant $C > 0$,*

$$\|u\|_{\mathcal{R}^{k+1}}(\Omega) \le C(\|P(D)u\|_{\mathcal{R}^{k}}(\Omega) + \|u\|_{L_2}(\Omega) + \|\Phi\|_{\mathcal{R}^{k+1}}(\Omega)) \quad \forall u \in H^{\mathcal{R}^{k+1}}(\Omega, \Phi).$$

Then the operator $P(D)$ is regular.

The proof follows immediately from Theorem 8.

Let A be the bounded operator from the Banach space B_1 into the Banach space B_2. The bounded operator R_1 from B_2 to B_1 is called the left regularizer for A, if

$$R_1 = I_1 + T_1,$$

where T_1 and I_1 are, respectively, the unit and the compact operators in B_2.

The bounded operator R_2 from B_2 to B_1 is called the right regularizer for A, if

$$AR_2 = I_2 + T_2,$$

where T_2 and I_2 are, respectively, the unit and the compact operators in B_2.

If the operator A possesses the left regularizer R_1 and the right one R_2, then

$$R_1 A R_2 = R_1(I_2 + T_2) = (I_1 + T_1)R_2,$$

whence

$$R_1 - R_2 = T_1 R_2 - R_1 T_2.$$

Thus $R_1 - R_2$ is the compact operator from B_2 to B_1. This implies that in this case each of the operators R_1 and R_2 is simultaneously the left and the right regularizer. The operator which is simultaneously the left and the right regularizer, is called the regularizer for A.

The theorem below is well-known from the theory of Noether operators (see [8]).

Theorem 9. *Let A be the bounded operator from B_1 to B_2. Then:*

(1) *if a possesses the left regularizer R_1, then the kernel of the operator A in B_1 is finite-dimensional;*

(2) *if A possesses the right regularizer R_2, then the region of values of the operator A is closed in B_2, and there takes place the finite-dimensional co-kernel.*

Theorem 10. *The closed linear operator is Noetherian, if and only if it possesses the bounded left and right regularizer.*

The Basic Theorem. *Let $P(D) = \sum\limits_{\alpha\in\mathcal{R}} p_\alpha F^\alpha$ be the linear differential operator with constant coefficients, acting from $H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$ to $H^{\mathcal{R}^k}(\Omega,\Phi)$. Then the following conditions are equivalent:*

(1) *$P(D)$ is the regular operator;*

(2) *$P(D)$ is the Noether operator from $H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$ to $H^{\mathcal{R}^k}(\Omega,\Phi)$;*

(3) *for some constant $C > 0$ the estimate*

$$\|u\|_{\mathcal{R}^{k+1}}(\Omega) \leq$$
$$\leq C\Big(\|P(D)u\|_{\mathcal{R}^k}(\Omega) + \|u\|_{L_2}(\Omega) + \|\Phi\|_{\mathcal{R}^{k+1}}(\Omega)\Big) \;\; \forall u \in H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$$

holds;

(4) *the operator $P(D)$ is bounded from $H^{\mathcal{R}^{k+1}}(\Omega,\Phi)$ to $H^{\mathcal{R}^k}(\Omega,\Phi)$ and posseses the regularizer.*

Proof. $1 \Rightarrow 2$. Follows from Theorem 6.
$2 \Rightarrow 3$. Follows from Theorem 7.
$3 \Rightarrow 1$. Follows from Corollary 6.
$2 \Longleftrightarrow 4$. Follows from Theorem 10. □

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