Proceedings of A. Razmadze
Mathematical Institute
Vol. **146** (2008), 45–55

INTEGRAL CHARACTERISTICS OF MAXIMAL FUNCTIONS ON THE LAGUERRE HYPERGROUP

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Abstract. In this paper we proof the boundedness of the maximal operator on the Laguerre hypergroup from the spaces $L_1(1+\ln^+ L_1)$ to the spaces L_1.

რეზიუმე. ნაშრომში დამტკიცებულია ლაგერის ჰიპერჯგუფზე განსაზღვრული მაქსიმალური ფუნქციის შემოსაზღვრულობა $L_1(1+\ln^+ L)$-დან L^1 სივრცეში.

1. Introduction

In the theory of functions the shift operator $f(x) \to f(x+y)$ and the connected with it techniques of Fourier analysis plays the important role. Natural generalization of shift operator on $\mathbb{R}$ is the Delsart-Levitan generalized [18] shift operator (GSO), particularly Bessel's GSO which can be constructed by arbitrary Shturm-Liouvill differential operator on $\mathbb{R}$. Generalized shift operators form one parametrical family, but nevertheless many problems of harmonic analysis can be generalized if we use generalized shift instead of ordinary ones.

The Hardy–Littlewood maximal function, fractional maximal function and fractional integrals are important technical tools in harmonic analysis, theory of functions and partial differential equations. The maximal function was firstly introduced by Hardy and Littlewood in 1930 (see [16]) for functions defined on the circle. It was extended to the Euclidean spaces, various Lie groups, symmetric spaces, and some weighted measure spaces (see [6], [7], [21], [24], [25]). In the setting of hypergroups versions of Hardy–Littlewood maximal functions were given in [4] for the Jacobi hypergroups of compact type, in [5] for the Jacobi-type hypergroups, in [2] for the one-dimensional Chebli-Trimeche hypergroups, in [23] for the one-dimensional Bessel-Kingman hypergroups, in [8] (see also [9, 10, 11]) for

2000 *Mathematics Subject Classification.* Primary 42B20, 42B25, 42B35; Secondary 47G10, 47B37.

Key words and phrases. Laguerre hypergroup, Generalized translation operator, Maximal operator.

the n-dimensional Bessel-Kingman hypergroups ($n \geq 1$), and in [12] for the Laguerre hypergroups.

In the papers [13, 14] we obtain necessary and sufficient conditions on the parameters for the boundedness of the fractional maximal operator and the fractional integral operator on the Laguerre hypergroup from the spaces $L_p(\mathbb{K})$ to the spaces $L_q(\mathbb{K})$ and from the spaces $L_1(\mathbb{K})$ to the weak spaces $WL_q(\mathbb{K})$. In this paper we consider the Laguerre generalized shift operator, by means of which are determined and investigated Mf maximal functions. The boundedness of Mf maximal functions from space $L_1(1+\ln^+ L_1)$ to space L_1 is proved. For Hardy-Littlewood maximal functions the analogous result was obtained in [15].

2. Main Result

Let $\alpha \geq 0$ be a fixed number, $\mathbb{K} = [0,\infty) \times \mathbb{R}$ and m_α be the weighted Lebesgue measure on $\mathbb{K}$, given by

$$dm_\alpha(x,t) = \frac{x^{2\alpha+1} dx dt}{\pi \Gamma(\alpha+1)}, \quad \alpha \geq 0.$$

For every $1 \leq p \leq \infty$, we denote by $L_p(\mathbb{K}) = L_p(\mathbb{K}; dm_\alpha)$ the spaces of complex-valued functions f, measurable on $\mathbb{K}$ such that

$$\|f\|_{L_p(\mathbb{K})} = \left(\int_{\mathbb{K}} |f(x,t)|^p \, dm_\alpha(x,t) \right)^{1/p} < \infty \quad \text{if} \quad p \in [1,\infty),$$

and

$$\|f\|_{L_\infty(\mathbb{K})} = \operatorname*{ess\,sup}_{(x,t)\in\mathbb{K}} |f(x,t)| \quad \text{if} \quad p = \infty.$$

For $1 \leq p < \infty$ we denote by $WL_p(\mathbb{K})$, the weak $L_p(\mathbb{K})$ spaces defined as the set of locally integrable functions $f(x,t)$, $(x,t) \in \mathbb{K}$ with the finite norm

$$\|f\|_{WL_p(\mathbb{K})} = \sup_{r>0} r\, f_*(r)^{1/p},$$

where $f_*(r) = m_\alpha \{(x,t) \in \mathbb{K} \ : \ |f(x,t)| > r\}$.

Let $|(x,t)|_{\mathbb{K}} = (x^4 + 4t^2)^{1/4}$ be the homogeneous norm of $(x,t) \in \mathbb{K}$. For $r > 0$ we will denote by $\delta_r(x,t) = (rx, r^2 t)$ the dilation of $(x,t) \in \mathbb{K}$, and by $B_r(x,t) = \{(y,s) \in \mathbb{K} : |(x-y, t-s)|_{\mathbb{K}} < r\}$ the ball centered at (x,t) with radius r, and by B_r the ball $B_r(0,0)$.

For $(x,t),(y,s) \in \mathbb{K}$ and $\theta \in [0,2\pi[$, $r \in [0,1]$ let

$$((x,t),(y,s))_{\theta,r} = \left(\left(x^2 + y^2 + 2xyr\cos\theta\right)^{1/2}, t + s + xyr\sin\theta \right).$$

The generalized translation operators $T^{(\alpha)}_{(x,t)}$ on the Laguerre hypergroup are given for a suitable function f by

$$T^{(\alpha)}_{(x,t)}f(y,s)=\begin{cases}\frac{1}{2\pi}\int\limits_0^{2\pi} f\left(((x,t),(y,s))_{\theta,1}\right)d\theta, & \text{if } \alpha=0,\\ \frac{\alpha}{\pi}\int\limits_0^{1}\Big(\int_0^{2\pi} f\left(((x,t),(y,s))_{\theta,r}\right)d\theta\Big)r(1-r^2)^{\alpha-1}dr, & \text{if } \alpha>0.\end{cases}$$

Now on the Laguerre hypergroup we define the maximal function (see [12]) by

$$Mf(x,t)=\sup_{r>0}\big(m_\alpha B_r\big)^{-1}\int\limits_{B_r} T^{(\alpha)}_{(x,t)}|f(y,s)|\,dm_\alpha(y,s).$$

In [12] the following theorem was proved.

Theorem 1. [12] 1. *If* $f\in L_1(\mathbb{K})$*, then* $Mf\in WL_1(\mathbb{K})$ *and*

$$m_\alpha\left\{(x,t)\in\mathbb{K}: Mf(x,t)>\tau\right\}\le\frac{C_1}{\tau}\int\limits_{\mathbb{K}}|f(x,t)|dm_\alpha(x,t), \tag{1}$$

where $C_1>0$ *is independent of* f.

2. *If* $f\in L_p(\mathbb{K})$, $1<p\le\infty$, *then* $Mf\in L_p(\mathbb{K})$ *and*

$$\|Mf\|_{L_p(\mathbb{K})}\le C_2\|f\|_{L_p(\mathbb{K})},$$

where $C_2>0$ *is independent of* f.

Corollary 1. *If* $f\in L_{\mathrm{loc}}(\mathbb{K})$, *then*

$$\lim_{r\to0}\frac{1}{m_\alpha B_r}\int\limits_{B_r}\left|T^{(\alpha)}_{(x,t)}f(y,s)-f(x,t)\right|dm_\alpha(y,s)=0$$

for a. e. $(x,t)\in\mathbb{K}$.

The following theorem is our main result in which we proof the boundedness of the maximal operator on the Laguerre hypergroup from the spaces $L_1(1+\ln^+L_1)$ to the spaces L_1.

Theorem 2. *For any* $r>0$, *and any measurable on* K *function* f *for which* $\operatorname{supp} f\subset B_r$ *the following inequality holds:*

$$\|Mf\|_{L_1(B_r)}\le C\int\limits_{B_r}|f(x,t)|(1+\ln^+|f(x,t)|)dm_\alpha(x,t)+m_\alpha(B_r),$$

where $C>0$ *is independent of* f.

3. Preliminaries

Consider the following partial differential operators system:

$$\begin{cases} D_1 = \frac{\partial}{\partial t}, \\ D_2 = \frac{\partial^2}{\partial x^2} + \frac{2\alpha+1}{x}\frac{\partial}{\partial x} + x^2\frac{\partial^2}{\partial t^2}, \\ (x,t) \in]0,\infty[\times\mathbb{R} \ \text{ and } \ \alpha \in [0,\infty[. \end{cases}$$

For $\alpha = n-1$, $n \in \mathbb{N}\setminus\{0\}$, the operator D_2 is the radial part of the sub-Laplacian on the Heisenberg group $\mathbb{H}_n$.

For $(\lambda, m) \in \mathbb{R}\times\mathbb{N}$, the initial problem

$$\begin{cases} D_1 u = i\lambda u, \\ D_2 u = -4|\lambda|\left(m + \frac{\alpha+1}{2}\right)u; \\ u(0,0) = 1, \ \frac{\partial u}{\partial x}(0,t) = 0 \ \text{ for all } \ t \in \mathbb{R}, \end{cases}$$

has a unique solution $\varphi_{\lambda,m}$ given by

$$\varphi_{\lambda,m}(x,t) = e^{i\lambda t}\mathcal{L}_m^{(\alpha)}\left(|\lambda|x^2\right), \quad (x,t)\in\mathbb{K},$$

where $\mathcal{L}_m^{(\alpha)}$ is the Laguerre functions defined on $\mathbb{R}_+$ by

$$\mathcal{L}_m^{(\alpha)}(x) = e^{-x/2}L_m^{(\alpha)}(x)/L_m^{(\alpha)}(0)$$

and $L_m^{(\alpha)}$ is the Laguerre polynomial of degree m and order α (see [1]).

For $f \in L_1(\mathbb{K})$ the Fourier-Laguerre transform $\mathcal{F}$ is defined by

$$\mathcal{F}(f)(\lambda,m) = \int_{\mathbb{K}} \varphi_{-\lambda,m}(x,t)f(x,t)dm_\alpha(x,t)$$

such that

$$\|\mathcal{F}(f)\|_{L_\infty(\mathbb{K})} \le \|f\|_{L_1(\mathbb{K})}$$

(see [1, 20]).

The generalized translation operators $T_{(x,t)}^{(\alpha)}$ on the Laguerre hypergroup satisfies the following properties (see [1, 20])

$$T_{(x,t)}^{(\alpha)}f(y,s) = T_{(y,s)}^{(\alpha)}f(x,t), \quad T_{(0,0)}^{(\alpha)}f(y,s) = f(y,s),$$
$$\|T_{(x,t)}^{(\alpha)}f\|_{L_p(\mathbb{K})} \le \|f\|_{L_p(\mathbb{K})} \ \text{ for all } \ f\in L_p(\mathbb{K}), \ 1\le p\le\infty,$$
$$\mathcal{F}\big(T_{(x,t)}^{(\alpha)}f\big)(\lambda,m) = \mathcal{F}(f)(\lambda,m)\ \varphi_{\lambda,m}(x,t).$$

The translation operator $T_{(x,t)}^{(\alpha)}$ is defined by

$$T_{(x,t)}^{(\alpha)}f(y,s) = \int_{\mathbb{K}} f(z,v)W_\alpha((x,t),(y,s),(z,v))z^{2\alpha+1}dzdv,$$

where $dzdv$ is the Lebesgue measure on $\mathbb{K}$, and W_α is an appropriate kernel satisfying

$$\int_{\mathbb{K}} W_\alpha((x,t),(y,s),(z,v)) z^{2\alpha+1} dzdv = 1$$

(see [19]). For all $(\lambda, m) \in \mathbb{R} \times \mathbb{N}$, the function $\varphi_{\lambda,m}(x,t)$ satisfies the following product formula

$$\varphi_{\lambda,m}(x,t)\, \varphi_{\lambda,m}(y,s) = T^{(\alpha)}_{(x,t)} \varphi_{\lambda,m}(y,s).$$

By using the generalized translation operators $T^{(\alpha)}_{(x,t)}$, $(x,t) \in \mathbb{K}$, we define a generalized convolution product $*$ on $\mathbb{K}$ by

$$\left(\delta_{(x,t)} * \delta_{(y,s)}\right)(f) = T^{(\alpha)}_{(x,t)} f(y,s),$$

where $\delta_{(x,t)}$ is the Dirac measure at (x,t).

We define the convolution product on the space $M_b(\mathbb{K})$ of bounded Radon measures on $\mathbb{K}$ by

$$(\mu * \nu)(f) = \int_{\mathbb{K}\times\mathbb{K}} T^{(\alpha)}_{(x,t)} f(y,s)\, d\mu(x,t)\, d\nu(y,s).$$

If $\mu = h \cdot m_\alpha$ and $\nu = g \cdot m_\alpha$, then we have

$$\mu * \nu = (h * \check{g}) \cdot m_\alpha, \quad \text{with} \quad \check{g}(y,s) = g(y,-s),$$

where h and g belong to the space $L_1(\mathbb{K})$ of the integrable functions on $\mathbb{K}$ with respect to the measure $dm_\alpha(x,t)$, and $h * g$ is the convolution product defined by

$$(h * g)(x,t) = \int_{\mathbb{K}} T^{(\alpha)}_{(x,t)} h(y,s)\, g(y,-s)\, dm_\alpha(y,s), \quad \text{for all} \quad (x,t) \in \mathbb{K}.$$

Note that, for the convolution operators the Young inequality is valid: If $1 \le p,\, r \le q \le \infty$, $1/p' + 1/q = 1/r$, $f \in L_p(\mathbb{K})$, and $g \in L_r(\mathbb{K})$, then $f * g \in L_q(\mathbb{K})$ and

$$\|f * g\|_{L_q(\mathbb{K})} \le \|f\|_{L_p(\mathbb{K})} \|g\|_{L_r(\mathbb{K})},$$

where $p' = p/(p-1)$.

$(M_b(\mathbb{K}), *, i)$ is an involutive Banach algebra, where i is the involution on $\mathbb{K}$ given by $i(x,t) = (x,-t)$ and the convolution product $*$ satisfies all the conditions of Jewett (see [3], [17]). Hence $(\mathbb{K}, *, i)$ is a hypergroup in the sense of Jewett and the functions $\varphi_{\lambda,m}$ are characters of $\mathbb{K}$. If $\alpha = n-1$ is a nonnegative integer, then the Laguerre hypergroup $\mathbb{K}$ can be identified with the hypergroup of radial functions on the Heisenberg group $\mathcal{H}_n$.

4. Some Lemmas

Let's note that the first part of the Theorem 1 is possible to be amplified. Namely, it is valid

Lemma 1. *Let* $\varepsilon \in (0,1), C_3 = C_1/(1-\varepsilon)$, *then for any* $\tau > 0$ *the following inequality is valid*

$$m_\alpha \{(x,t) \in \mathbb{K} : Mf(x,t) > \tau\} \leq \frac{C_3}{\tau} \int\limits_{\{(x,t)\in\mathbb{K}:|f(x,t)|>\varepsilon\tau\}} |f(x,t)| dm_\alpha(x,t).$$

Proof. For $\varepsilon = 0$ this is inequality (1). For $0 < \varepsilon < 1$ let's determine $f_1(x,t)$ so:

$$f_1(x,t) = \begin{cases} f(x,t), & |f(x,t)| > \varepsilon\tau, \\ 0, & |f(x,t)| \leq \varepsilon\tau. \end{cases}$$

Then $|f(x,t)| \leq |f_1(x,t)| + \varepsilon\tau$ and $Mf(x,t) \leq Mf_1(x,t) + \varepsilon\tau$. So, if $Mf(x,t) > \tau$, then $Mf_1(x,t) > (1-\varepsilon)\tau$, i.e.

$$\{(x,t) \in \mathbb{K} : Mf(x,t) > \tau\} \subset \{(x,t) \in \mathbb{K} : Mf_1(x,t) > (1-\varepsilon)\tau\}.$$

So, according to (1)

$$\begin{aligned} &m_\alpha \{(x,t) \in \mathbb{K} : \ Mf(x,t) > \tau\} \leq \\ &\leq m_\alpha \{(x,t) \in K : \ Mf_1(x,t) > (1-\varepsilon)\tau\} \leq \\ &\leq \frac{C_1}{(1-\varepsilon)\tau} \int\limits_{\mathbb{K}} |f_1(x,t)| dm_\alpha(x,t) = \\ &= \frac{(1-\varepsilon)^{-1} C_1}{\tau} \int\limits_{\{(x,t)\in\mathbb{K}:\ |f(x,t)|>\varepsilon\tau\}} |f(x,t)| dm_\alpha(x,t) = \\ &= \frac{C_3}{\tau} \int\limits_{\{(x,t)\in\mathbb{K}:|f(x,t)|>\varepsilon\tau\}} |f(x,t)| dm_\alpha(x,t). \end{aligned}$$

The Lemma 1 has been proved. □

From Theorem 1 and Lemma 1 follows

Corollary 2. 1) *Let* $f \in L_1(B_r)$, $\operatorname{supp} f \subset B_r$, $r > 0$, *then for any* $\tau > 0$

$$m_\alpha \{(x,t) \in B_r : \ Mf(x,t) > \tau\} \leq \frac{C_1}{\tau} \int\limits_{B_r} |f(x,t)| dm_\alpha(x,t),$$

when C_1 *is independent of* f.

2) *Let* $f \in L_1(B_r)$, $\operatorname{supp} f \subset B_r$, $\varepsilon \in (0,1)$, $C_3 = \frac{C_1}{1-\varepsilon}$, *then for any* $\tau > 0$ *the following inequality is valid*

$$m_\alpha \{(x,t) \in B_r : \ Mf(x,t) > \tau\} \leq \frac{C_3}{\tau} \int\limits_{\{(x,t)\in B_r:|f(x,t)|>\varepsilon\tau\}} |f(x,t)| dm_\alpha(x,t).$$

Let $1 \le p < \infty$. Well known that for the measurable on $\mathbb{K}$ function f it is valid (see, for example [22])

$$\|f\|_{L_p(\mathbb{K})} = \left(p\int_0^\infty \tau^{p-1} f_*(\tau)\, d\tau\right)^{\frac{1}{p}} = \left(-\int_0^\infty \tau^p\, d\, f_*(\tau)\right)^{\frac{1}{p}},$$

In the particular case $p = 1$

$$\|f\|_{L_1(\mathbb{K})} = \|f_*\|_{L_1(0,\infty)}.$$

The following equality is valid.

Lemma 2. *Let f be measurable on $\mathbb{K}$ function, then for any $s > 0$*

$$\int\limits_{\{(x,t)\in\mathbb{K}:|f(x,t)|>s\}} |f(x,t)|dm_\alpha(x,t) = sf_*(s) + \int_s^\infty f_*(\tau)d\tau.$$

Proof. Let $g(x,t)$ be the function, defined as follows

$$g(x,t) = \begin{cases} f(x,t), & |f(x,t)| > s, \\ 0, & |f(x,t)| \le s. \end{cases} \tag{2}$$

For any $\tau \in [s, \infty)$

$$g_*(\tau) = m_\alpha\{(x,t) \in K : |g(x,t)| > \tau\} = \\ = m_\alpha \{(x,t) \in K : |f(x,t)| > \tau\} = f_*(\tau). \tag{3}$$

For any $\tau \in (0, s)$

$$g_*(\tau) = m_\alpha \{(x,t) \in \mathbb{K} : |g(x,t)| > \tau\} = \\ = m_\alpha \{(x,t) \in \mathbb{K} : |f(x,t)| > s\} = f_*(s). \tag{4}$$

Then from (2), (3), (4) we have

$$\int\limits_{\{(x,t)\in K:|f(x,t)|>s\}} |f(x,t)|dm_\alpha(x,t) =$$

$$= \int\limits_{\mathbb{K}} |g(x,t)|dm_\alpha(x,t) = \int_0^\infty g_*(\tau)d\tau =$$

$$= \int_0^s g_*(\tau)d\tau + \int_s^\infty g_*(\tau)d\tau = sf_*(s) + \int_s^\infty f_*(\tau)d\tau.$$

Lemma 2 has been proved. □

Corollary 3. *Let f be measurable on B_r function, then for any $s>0$*

$$\int\limits_{\{(x,t)\in B_r:|f(x,t)|>s\}} |f(x,t)|dm_\alpha(x,t) =$$

$$= s\ m_\alpha\{(x,t)\in B_r : |f(x,t)|>s\} + \int\limits_s^\infty m_\alpha\{(x,t)\in B_r : |f(x,t)|>\tau\}d\tau.$$

From Corollary 2 directly implies the following

Lemma 3. *Let $f\in L_1(B_r)$, $\operatorname{supp} f\subset B_r$, then for every $\tau>0$*

$$\int\limits_1^\infty (\chi_{B_r}Mf)_*(\tau)\,d\tau \le C_4\int\limits_1^\infty \Bigg(\int\limits_{\{(x,t)\in B_r:|f(x,t)|>\tau/2\}} |f(x,t)|dm_\alpha(x,t)\Bigg)\frac{d\tau}{\tau},$$

where $C_4>0$ is independent of f.

Lemma 4. *Let $f\in L_1(B_r)$, $\operatorname{supp} f\subset B_r$, then for every $\tau>0$*

$$\int\limits_{B_r} |f(x,t)|dm_\alpha(x,t) \le (m_\alpha B_r)_{\mathbb{K}} + 2(\chi_{B_r}f)_*(1) + 4\int\limits_1^\infty (\chi_{B_r}f)_*(\tau)d\tau.$$

Proof. We have

$$\int\limits_{B_r} |f(x,t)|dm_\alpha(x,t) =$$

$$= \int\limits_{\{(x,t)\in B_r:|f(x,t)|\le 1\}} |f(x,t)|dm_\alpha(x,t) +$$

$$+ \int\limits_{\{(x,t)\in B_r:|f(x,t)|>1\}} |f(x,t)|dm_\alpha(x,t) \le m_\alpha(B_r) +$$

$$+ \sum_{k=0}^\infty \int\limits_{\{(x,t)\in B_r\,:\,2^k<|f(x,t)|\le 2^{k+1}\}} |f(x,t)|dm_\alpha(x,t) \le$$

$$\le m_\alpha(B_r) + \sum_{k=0}^\infty 2^{k+1} \int\limits_{\{(x,t)\in B_r\,:\,|f(x,t)|>2^k\}} dm_\alpha(x,t) \le$$

$$\le m_\alpha(B_r) + 2\sum_{k=0}^\infty 2^k\,(\chi_{B_r}f)_*\left(2^k\right) \le$$

$$\le m_\alpha(B_r) + 2\,(\chi_{B_r}f)_*\,(1) + 2\sum_{k=1}^\infty 2^k\,(\chi_{B_r}f)_*\left(2^k\right) \le$$

$$\leq m_\alpha(B_r) + 2\left(\chi_{B_r} f\right)_*(1) + 4\sum_{k=0}^{\infty} 2^k \left(\chi_{B_r} f\right)_*\left(2^{k+1}\right) \leq$$

$$\leq m_\alpha(B_r) + 2\left(\chi_{B_r} f\right)_*(1) + 4\int\limits_1^{\infty} \left(\chi_{B_r} f\right)_*(\tau)d\tau.$$

Lemma 4 has been proved. □

Proof of the Theorem 2. By virtue of Lemmas 1 - 4 and Corollary 1 we have

$$\int\limits_{B_r} Mf(x,t)dm_\alpha(x,t) \leq$$

$$\leq m_\alpha(B_r) + 2m_\alpha\left\{(x,t)\in B_r:\ Mf(x,t)>1\right\} +$$

$$+4\int\limits_1^{\infty}\left(\chi_{B_r} Mf\right)_*(\tau)d\tau \leq m_\alpha(B_r) + 2C_1\int\limits_{B_r}|f(x,t)|dm_\alpha(x,t)+$$

$$+8C_4\int\limits_1^{\infty}\left(\int\limits_{\{(x,t)\in B_r:|f(x,t)|>\frac{\tau}{2}\}}|f(x,t)|dm_\alpha(x,t)\right)\frac{d\tau}{\tau} \leq m_\alpha(B_r)+$$

$$+8C_4\int\limits_{\{(x,t)\in B_r:|f(x,t)|>\frac{1}{2}\}}|f(x,t)|\left(\int\limits_1^{2|f(x,t)|}\frac{d\tau}{\tau}\right)dm_\alpha(x,t)+$$

$$+2C_1\int\limits_{B_r}|f(x,t)|dm_\alpha(x,t) = m_\alpha(B_r) + 2C_1\int\limits_{B_r}|f(x,t)|dm_\alpha(x,t)+$$

$$+8C_4\int\limits_{\{(x,t)\in B_r:|f(x,t)|>\frac{1}{2}\}}|f(x,t)|\ln\left(2|f(x,t)|\right)dm_\alpha(x,t) \leq$$

$$\leq m_\alpha(B_r) + C\int\limits_{B_r}|f(x,t)|(1+\ln^+|f(x,t)|)dm_\alpha(x,t).$$

Theorem 2 has been proved.

Acknowledgement

This work was supported by the grant of the Azerbaijan-U. S. Bilateral Grants Program II (project ANSF Award / 16071) and the grant of INTAS (project 05-1000008-8157).

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(Received 10.10.2007)

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