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ON A NEW TYPE OF MULTIPLE INTEGRALS OF MULTI-VALUED FUNCTIONS OF AN ABSTRACT SET

D. GOGUADZE AND P. KARCHAVA

Abstract. In the present paper we introduce a new type of multiple integrals of multi-valued functions of abstract sets. Fundamental relations are established between double upper and lower and between repeated upper and lower integrals from which we obtain generalizations of Fubini's theorem both for the functions of a set and for the functions of a point.

რეზიუმე. წინამდებარე ნაშრომში შემოღებულია აბსტრაქტული სიმრავლის მრავალსახა ფუნქციის ახალი ტიპის ჯერადი ინტეგრალები. დადგენილია ფუნდამენტური დამოკიდებულებები ორჯერად ზედა და ქვედა და განმეორებით ზედა და ქვედა ინტეგრალებს შორის, რომლებიდანაც მიიღება ფუბინის თეორემის განზოგადოებები როგორც სიმრავლის, ასევე წერტილის ფუნქციებისათვის.

A class of sets which along with its any two sets contains their intersection is called multiplicative, and we denote it by $\mathfrak{M}$.

Let $\mathfrak{M}$ be the multiplicative class of sets, and $E \in \mathfrak{M}$. A finite or a countable class of pairwise nonintersecting sets $\{E_1, E_2, \dots\}$, belonging to the class $\mathfrak{M}$ and the union of which is equal to the set E, is called a partition of the set E and denoted by DE. Moreover, the sets E_k $(k = 1, 2, \dots)$ are called the components of a partition DE. If the partition contains only a finite number of components, then it is called finite and denoted by D^*E.

The partition D_1E of the set $E \in \mathfrak{M}$ is called a continuation of the partition DE of the set E, in the notation $DE \prec D_1E$, if every component of the partition D_1E is a subset of some component of the partition DE, or what is the same thing, if the partition D_1E contains the partition of every component of the partition DE.

Let $D_1E = \{E_1', E_2', \dots\}$ and $D_2E = \{E_1'', E_2'', \dots\}$ be two partitions of the set $E \in \mathfrak{M}$. The product of the partitions D_1E and D_2E is called a partition whose components are all possible intersections $E_i' \cap E_k''$ $(i, k =$

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$1, 2, \dots$), and we denote it by $(D_1 \cdot D_2)E$. Obviously, the product of two partitions is the continuation of both partitions.

By $\mathfrak{M}E$ we denote a set of all partitions of the set $E \in \mathfrak{M}$, and by $\sum\limits_{\mathfrak{M}}(E)$ a set of all components of all possible partitions of the set E.

A class of sets which along with its any two sets contains also the partition of their intersections, is called generalized-multiplicative, and denoted by $\mathfrak{M}$.

It is evident that the multiplicative class is also a generalized-multiplicative. Here we present an example of a generalized-multiplicative class which is not multiplicative. Let E be an arbitrary set with capacity more than or equal to a countable capacity, and let $P(E)$ be a class of all its subsets. We take a finite subset $e \subset E$ and remove from the class $P(E)$ the set e and all its subsets, with the exclusion of the subsets consisting of a single element. The remaining class we denote by $\mathfrak{M}$ and show that this class is generalized-multiplicative, and not multiplicative. Indeed, we take the set $E - e$ and represent it as $E - e = F_1 \cup F_2$, where $F_1 \cap F_2 = \varnothing$ ($\varnothing$ is an empty set). We construct the sets $E_1 = F_1 \cup e$ and $E_2 = F_2 \cup e$. Then $E_1 \cap E_2 = (F_1 \cup e) \cap (F_2 \cup e) = e$. Consequently, $E_1 \cap E_2 = e \notin \mathfrak{M}$. However, $\mathfrak{M}'$ contains the partition of the set $E_1 \cap E_2 = e$.

A class of sets is called normal and denoted by $\mathfrak{N}$, if for every $E \in \mathfrak{N}$ the set $\mathfrak{N}E$ of all partitions of the set E is a directed relation of the continuation $\succ$.

Theorem 1. *A class of the sets $\mathfrak{N}$ is normal if and only if for every set $E \in \mathfrak{N}$ the set $\sum\limits_{\mathfrak{N}}(E)$ of all components of all possible partitions of the set E is generalized-multiplicative one.*

Proof. The necessity. Let $E', E'' \in \sum\limits_{\mathfrak{N}}(E)$. Then there exist the partitions D_1E and D_2E of the set E, such that the sets E' and E'' are the components of the partitions D_1E and D_2E, respectively.

Let $DE = \{e_1, e_2, \dots\}$ be the continuation of the partitions D_1E and D_2E. Then according to the definition of a continuation of partitions,

$$E' = \bigcup_i e'_i \quad \text{and} \quad E'' = \bigcup_i e''_i,$$

where $\{e'_i\} \subset \{e_1, e_2, \dots\}$ and $\{e''_i\} \subset \{e_1, e_2, \dots\}$.

Let us consider the intersection

$$E' \cap E'' = \bigcup_i e'_i \cap \bigcup_k e''_k = \bigcup_i \bigcup_k \, e'_i \cap e''_k.$$

Since $\{e_1, e_2, \dots\}$ is the partition of the set E,

$$e'_i \cap e''_k = \varnothing \quad \text{or} \quad e'_i = e''_k \quad (i, k = 1, 2, \dots).$$

Thus we have found that the class $\sum\limits_{\mathfrak{N}}(E)$ contains the partition of the intersection $E' \cap E''$.

The sufficiency. Let the class $\sum_{\mathfrak{N}}(E)$ be generalized-multiplicative, and $D_1E = \{E_1', E_2', \dots\}$ and $D_2E = \{E_1'', E_2'', \dots\}$ be two partitions of the set $E \in \mathfrak{N}$. Then by the definition of products, we have

$$D_1E \cdot D_2E = \underset{i}{\cup}\,\underset{k}{\cup}\,(E_i' \cap E_k'') = \underset{i}{\cup}\,\underset{k}{\cup}\,\underset{j}{\cup}\,E_{ik}^j,$$

where E_{ik}^j $(j = 1, 2, \dots)$ is the partition of the intersection $E_i' \cap E_k''$.

Thus the partition E_{ik}^j $(i, k, j = 1, 2, \dots)$ of the set E is the continuation of the partitions D_1E and D_2E.

The generalized-multiplicative class $\mathfrak{M}$ is said to be a semi-ring denoted by P, if it contains an empty set $\varnothing$, and it follows from $E_1, E \in P$ and $E_1 \subset E$ that P contains the partition of the set E whose component is the set E_1.

Let X and Y be arbitrary sets. The set of all ordered pairs (x, y), where $x \in X$ and $y \in Y$, is called the Cartesian product of the sets X and Y and denoted by $X \times Y$. If $A \subset X$ and $B \subset Y$, then the set $E = A \times B$, contained in $X \times Y$, is called a rectangle, and the sets A and B are called the sides of that rectangle.

Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be the normal classes. The class of all rectangles $A \times B$, where $A \in \mathfrak{N}_1$ and $B \in \mathfrak{N}_2$, is called a product of the normal classes $\mathfrak{N}_1$ and $\mathfrak{N}_2$ and denoted by $\mathfrak{N}_1 \otimes \mathfrak{N}_2$.

It can immediately be verified that the product of normal classes is likewise a normal class.

Towards this end, according to Theorem 1, it suffices to show that for any $A \times B \in \mathfrak{N}_1 \otimes \mathfrak{N}_2 = \mathfrak{N}$ the class $\sum_{\mathfrak{N}}(A \times B)$ is generalized-multiplicative. Indeed, let $A_1 \times B_1, A_2 \times B_2 \in \mathfrak{N}_1 \otimes \mathfrak{N}_2$. Then we have

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2) =$$

$$= \underset{i=1}{\cup} A_i \times \underset{j=1}{\cup} B_j = \underset{i=1}{\cup}\,\underset{j=1}{\cup}\,(A_i \times B_j). \qquad \square$$

Proposition 1. *If P_1 and P_2 are the semi-rings, then their product $P = P_1 \otimes P_2$ is likewise a semi-ring* (*see* [1]).

The partition $D(A \times B)$ of the rectangle $A \times B \in \mathfrak{N}$ is said to be netting, if it has the form $\{A_i \times B_k\}$ $(i, k = 1, 2, \dots)$, where $\{A_i\}$ $(i = 1, 2, \dots)$ is the partitions of the side A, and $\{B_k\}$ $(k = 1, 2, \dots)$ is that of the side B.

The partition $D(A \times B)$ of the rectangle $A \times B \in \mathfrak{N}$ is said to be the right generalized-netting if it has the form $\{A_i \times B_k^i\}$ $(i, k = 1, 2, \dots)$, where $\{A_i\}$ $(i = 1, 2, \dots)$ is the partition of the side A, and $\{B_k^i\}$ $(k = 1, 2, \dots)$ is that of the side B, corresponding to the component $\{A_i\}$ $(i = 1, 2, \dots)$.

The partition $D(A \times B)$ of the rectangle $A \times B \in \mathfrak{N}$ is said to be the left generalized-netting if it has the form $\{A_i^k \times B_k\}$ $(k, i = 1, 2, \dots)$, where

$\{B_k\}$ $(k=1,2,\dots)$ is the partitions of the side B, and $\{A_i^k\}$ $(i=1,2,\dots)$ is that of the side A, corresponding to the component $\{B_k\}$ $(k=1,2,\dots)$.

The partition $D(A\times B)$ of the rectangle $A\times B\in\mathfrak{N}$ is said to be twice netting if has the form $\{A_i^k\times B_j^l\}$ $(i,j,k,l=1,2,\dots)$, where $\{A_i\times B_j\}$ $(i,j=1,2,\dots)$ is the netting partition of the rectangle $A\times B$, and $\{A_i^k\times B_j^l\}$ $(k,l=1,2,\dots)$ is that of the rectangle $A_i\times B_j$.

Let $\{A_i\times B_j\}$ $(i,j=1,2,\dots)$ and $\{C_k\times D_l\}$ $(k,l=1,2,\dots)$ be two netting partitions of the rectangle $A\times B$. Then their product

$$\big\{(A_i\times B_j)\cap(C_k\times D_l)\big\}\quad (i,j,k,l=1,2,\dots)$$

is likewise the netting partition of the rectangle $A\times B$.

Indeed, we have

$$\begin{aligned}&\big\{(A_i\times B_j)\cap(C_k\times D_l)\big\}\ (i,j,k,l=1,2,\dots)=\\&=\big\{(A_i\times C_k)\cap(B_j\times D_l)\big\}\ (i,j,k,l=1,2,\dots).\end{aligned}$$

Since $\mathfrak{N}_1$ is the normal class of sets, it contains, by Theorem 1, the partition $\{S_p^{ik}\}$ $(p=1,2,\dots)$ of the intersection $A_i\cap C_k$ $(i,k=1,2,\dots)$. Analogously, since $\mathfrak{N}_2$ is the normal class of sets, it contains, by Theorem 1, the partition $\{T_q^{jl}\}$ $(q=1,2,\dots)$ of the intersection $B_j\cap D_l$ $(j,l=1,2,\dots)$. Taking the above-said into consideration, we obtain

$$\begin{aligned}&\big\{(A_i\times B_j)\cap(C_k\times D_l)\big\}\ (i,j,k,l=1,2,\dots)=\\&=\Big\{\bigcup_{p=1}^{\infty}S_p^{ik}\times\bigcup_{q=1}^{\infty}T_q^{jl}\Big\}\ (i,j,k,l=1,2,\dots)=\\&=\big\{S_p^{ik}\times T_q^{jl}\big\}\ (i,j,k,l,p,q=1,2,\dots),\end{aligned}$$

where $\{S_p^{ik}\}$ $(i,k,p=1,2,\dots)$ is the partition of the side A, and $\{T_q^{jl}\}$ $(j,l,q=1,2,\dots)$ is that of the side B.

Consequently, the set of all netting partitions of the rectangle $A\times B\in\mathfrak{N}$ is the directed relation of the continuation.

Therefore the mapping of the sets of all netting partitions of the rectangle $A\times B\in\mathfrak{N}$ into the set of real numbers is the directedness, and to them can be applied the generalized theory of limits (see [2], Ch.II).

Assume that $\{A_i\times B_k^i\}$ $(i,k=1,2,\dots)$ and $\{C_j\times D_l^j\}$ $(j,l=1,2,\dots)$ are two the right generalized-netting partitions of the rectangle $A\times B$. Then their product

$$\big\{(A_i\times B_k^i)\cap(C_j\times D_l^j)\big\}\ (i,j,k,l=1,2,\dots)$$

is likewise the right generalized-netting partition of the rectangle $A\times B$.

Since $\{A_i\}$ $(i=1,2,\dots)$ and $\{C_j\}$ $(j=1,2,\dots)$ are the partitions of the side A of the rectangle $A\times B$, therefore the class $\mathfrak{N}$, by Theorem 1, contains the partition $\{S_p^{ij}\}$ $(p=1,2,\dots)$ of the intersection $A_i\cap C_j$.

Analogously, since $\{B_k^i\}$ $(k = 1,2,\dots)$ and $\{D_l^j\}$ $(j = 1,2,\dots)$ are the partitions of the side B, corresponding to the components A_i and C_k, because of the fact that the class $\mathfrak{N}_2$ is normal, it contains the partition $D^{ij}B$ following the partitions $\{B_k^i\}$ $(k = 1,2,\dots)$ and $\{D_l^j\}$ $(l = 1,2,\dots)$.

Thus we have found that for any $i = 1,2,\dots$ and $j = 1,2,\dots$, to every component $\{S_p^{ij}\}$ $(p = 1,2,\dots)$ of the product $\{S_p^{ij}\}$ $(i,j,p = 1,2,\dots)$ of the partitions $\{A_i\}$ $(i = 1,2,\dots)$ and $\{C_j\}$ $(j = 1,2,\dots)$ of the side A there corresponds the partition $D^{ij}B$ of the side B following the partitions $\{B_k^i\}$ $(k = 1,2,\dots)$ and $\{D_l^j\}$ $(l = 1,2,\dots)$.

Therefore we have

$$\big\{(A_i\cap C_j)\times(B_k^i\cap D_l^j)\big\}\,(i,j,k,l=1,2,\dots)=\big\{S_p^{ij}\times D^{ij}B\big\}\,(i,j,p=1,2,\dots).$$

Thus the set of all the right generalized-netting partitions of the rectangle $A\times B\in\mathfrak{N}$ is the directed relation of the continuation.

Similarly, the set of all left generalized-netting partitions of the rectangle $A\times B\in\mathfrak{N}$ is the directed relation of the continuation.

That is why the mappings of the sets of all the right generalized-netting partitions of the rectangle $A\times B\in\mathfrak{N}$ and the sets of all left generalized-netting partitions of the rectangle $A\times B\in\mathfrak{N}$ into the set of real numbers is the directedness, and to them can be applied the generalized theory of limits.

Let $\{A_i^k\times B_j^l\}$ $(i,j,k,l = 1,2,\dots)$ and $\{C_m^r\times D_n^s\}$ $(m,n,r,s = 1,2,\dots)$ be two twice netting partitions of the rectangle $A\times B$, where $\{A_i\times B_j\}$ $(i,j = 1,2,\dots)$ and $\{C_m\times D_n\}$ $(m,n = 1,2,\dots)$ are the netting partitions of the rectangle $A\times B$, and $\{A_i^k\times B_j^l\}$ $(k,l = 1,2,\dots)$ is the netting partition of the rectangle $A_i\times B_j$, and $\{C_m^r\times D_n^s\}$ $(r,s = 1,2,\dots)$ is the netting partition of the rectangle $C_m\times D_n$.

Indeed, we have

$$\begin{aligned}&\big\{(A_i^k\times B_j^l)\cap(C_m^r\times D_n^s)\big\}\ (i,j,k,l,m,n,r,s = 1,2,\dots) =\\ &\quad=\big\{(A_i^k\cap C_m^r)\times(B_j^l\cap D_n^s)\big\}\ (i,j,k,l,m,n,r,s = 1,2,\dots)\end{aligned}$$

and

$$\big\{(A_i\cap C_m)\times(B_j\cap D_n)\big\}\ \ (i,m,j,n = 1,2,\dots)$$

is the netting partition of the rectangle $A\times B$, since $\{A_i\times B_j\}$ $(i,j = 1,2,\dots)$ and $\{C_m\times D_n\}$ $(m,n = 1,2,\dots)$ are the netting partitions of the rectangle $A\times B$. From its side,

$$\big\{(A_i^k\cap C_m^r)\times(B_j^l\cap D_n^s)\big\}\ \ (k,l,r,s = 1,2,\dots)$$

is the netting partition of the rectangle $(A_i\cap C_m)\times(B_j\cap D_n)$, since $\{A_i^k\times B_j^l\}$ $(k,l = 1,2,\dots)$ is the netting partition of the rectangle $A_i\times B_j$, and $\{C_m^r\times D_n^s\}$ $(r,s = 1,2,\dots)$ is the netting partition of the rectangle $C_m\times D_n$.

Thus the set of all twice netting partitions of the rectangle $A \times B \in \mathfrak{N}$ is the directed relation of the continuation, and hence its mappings into the set of real numbers is the directedness, and to them can be applied the generalized theory of limits.

Proposition 2. *The partition $D(A \times B)$ of the rectangle $A \times B$ is twice netting if and only if it is a product of the left and of the right generalized-netting partitions of the rectangle $A \times B$.*

Proof. The necessity. Let $\{A_i^k \times B_j^l\}$ $(i,j,k,l = 1,2,\dots)$ be the twice netting partition of the rectangle $A \times B$. Then $\{A_i \times B_j\}$ $(i,j = 1,2,\dots)$ is the netting partition of the rectangle $A \times B$, and $\{A_i^k \times B_j^l\}$ $(k,l = 1,2,\dots)$ is the netting partition of the rectangle $A_i \times B_j$.

But $\{A_i \times B_j^l\}$ $(i,j,l = 1,2,\dots)$ is the right generalized-netting partition of the rectangle $A \times B$.

Analogously, $\{A_i^k \times B_j\}$ $(i,k,j = 1,2,\dots)$ is the left generalized-netting partition of the rectyangle $A \times B$.

Constructing the product of the partitions $\{A_i \times B_j^l\}$ $(i,j,l = 1,2,\dots)$ and $\{A_i^k \times B_j\}$ $(i,k,j = 1,2,\dots)$, we obtain an unknown partition

$$\begin{aligned}&\big\{(A_i \times B_j^l) \cap (A_i^k \times B_j)\big\} \ (i,k,j,l = 1,2,\dots) = \\ &= \big\{(A_i \cap A_i^k) \times (B_j \cap B_j^l)\big\} \ (i,k,j,l = 1,2,\dots) = \\ &= \big\{A_i^k \times B_j^l\big\} \ (i,k,j,l = 1,2,\dots).\end{aligned}$$

The sufficiency. Let $D_1(A \times B) = \{A_i \times B_k^i\}$ $(i,k = 1,2,\dots)$ be the right generalized-netting partition of the rectangle $A \times B$, and $D_2(A \times B) = \{A_l^j \times B_j\}$ $(j,l = 1,2,\dots)$ be the left generalized-netting partition of the rectangle $A \times B$. Then by the definition of the partition product, we obtain

$$\begin{aligned}(D_1 \cdot D_2)(A \times B) &= \big\{(A_i \times B_k^i) \cap (A_l^j \times B_j)\big\} \ (i,k,j,l = 1,2,\dots) = \\ &= \big\{(A_i \cap A_l^j) \times (B_j \cap B_k^i)\big\} \ (i,k,j = 1,2,\dots).\end{aligned}$$

But

$$\big\{(A_i \cap A_l^j) \times (B_j \cap B_k^i)\big\} \ (k,l = 1,2,\dots)$$

is the netting partition of the rectangle $A_i \times B_j$, and $\{A_i \times B_j\}$ $i,j = 1,2,\dots)$ is the netting partition of the rectangle $A \times B$. Consequently,

$$\big\{(A_i \cap A_l^j) \times (B_j \cap B_k^i)\big\} \ (k,l = 1,2,\dots)$$

is the twice netting partition of the rectangle $A \times B$. □

Proposition 3. *If $A_1, \dots, A_m$ is an arbitrary finite class from the semi-ring P, then their union can be represented as*

$$A_1 \cup \ldots \cup A_m = A_1^1 \cup \ldots \cup A_1^{k_1} \cup \ldots \cup A_m^1 \cup \ldots \cup A_m^{k_m},$$

where $A_i^1, \dots, A_i^{k_i}$ are contained in the set A_i $(i = 1, \dots, m)$, and all sets the right belong to the semi-ring P and do not intersect pairwise.

For the proof, see [1].

This proposition is, as is erroneously mentioned in [3], p.132, invalid for the countable classes. As a counterexample we can mention the countable class

$$\left\{\left[\frac{m-1}{2^n}, \frac{m}{2^n}\right]\right\} \ (n = 1, 2, \dots;\ m = 1, 2, \dots, 2^n)$$

in a semi-ring P of all semi-segments from the semi-segment $[0, 1)$.

Proposition 4. *Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be the normal classes of the sets, $\mathfrak{N}_1 \otimes \mathfrak{N}_2$ be their product, the rectangle $A \times B \in \mathfrak{N}_1 \otimes \mathfrak{N}_2$, and $D(A\times B) = \{A_k \times B_k\}$ $(k = 1, 2, \dots)$ be the partition of the rectangle $A \times B$. For the countable continuation of the partition $D(A \times B)$ to exist, it is necessary and sufficient that there exist the partition $\{\widetilde{A}_1, \widetilde{A}_2, \dots\} \subset \mathfrak{N}_1$ of the set A and the partition $\{\widetilde{B}_1, \widetilde{B}_2, \dots\} \subset \mathfrak{N}_2$ of the set B, such that every A_k $(k = 1, 2, \dots)$ (respectively, B_k $(k = 1, 2, \dots)$) is the union of a finite or a countable number of sets from $\{\widetilde{A}_1, \widetilde{A}_2, \dots\}$ (respectively, from $\{\widetilde{B}_1, \widetilde{B}_2, \dots\}$).*

Proof. The necessity. Let there exist a countable continuation

$$\{\widetilde{A}_i \times \widetilde{B}_j\} \ (i, j = 1, 2, \dots)$$

of the partition $D(A \times B)$. Then $\{\widetilde{A}_1, \widetilde{A}_2, \dots\}$ is the partition of the set A, and $\{\widetilde{B}_1, \widetilde{B}_2, \dots\}$ is that of the set B. According to the definition of the continuation of partition, every $A_k \times B_k$ is a finite or countable union of the rectangles from $\{\widetilde{A}_i \times \widetilde{B}_j\}$ $(i, j = 1, 2, \dots)$,

$$A \times B = \bigcup_i \bigcup_j \big(\widetilde{A}_i \times \widetilde{B}_j\big).$$

But then

$$A = \bigcup_i \widetilde{A}_i, \quad B = \bigcup_j \widetilde{B}_j.$$

The sufficiency. Let there exist the partitions $\{\widetilde{A}_1, \widetilde{A}_2, \dots\} \subset \mathfrak{N}_1$ of the set A and $\{\widetilde{B}_1, \widetilde{B}_2, \dots\} \subset \mathfrak{N}_2$ those of the set B, such that every A_k $(k = 1, 2, \dots)$ (respectively, every B_k $(k = 1, 2, \dots)$) is the union of a finite or a countable number of sets from $\{\widetilde{A}_1, \widetilde{A}_2, \dots\}$ (respectively, from $\{\widetilde{B}_1, \widetilde{B}_2, \dots\}$).

Let us show that the netting partition $\{\widetilde{A}_i \times \widetilde{B}_j\}$ is unknown. Indeed, let $A_{k_0} \times B_{k_0}$ be an arbitrary rectangle from $\{A_k \times B_k\}$ $(k = 1, 2, \dots)$. Then by the assumption,

$$A_{k_0} = \bigcup_i \widetilde{A}_i, \quad B_{k_0} = \bigcup_j \widetilde{B}_j,$$

and hence

$$A_{k_0} \times B_{k_0} = \big(\bigcup_i \widetilde{A}_i\big) \times \big(\bigcup_j \widetilde{B}_j\big) = \bigcup_i \bigcup_j \big(\widetilde{A}_i \times \widetilde{B}_j\big). \qquad \square$$

Corollary 1. *Let P_1 and P_2 be the semi-rings, and $P_1 \otimes P_2$ be their product. Then for every finite partition $\{A_k \times B_k\}$ $(k = 1, \dots, n)$ of the rectangle $A \times B \in P_1 \otimes P_2$ there exists its netting continuation.*

Proof. From the equality

$$A \times B = \bigcup_{k=1}^{n} (A_k \times B_k),$$

it follows that

$$A = \bigcup_{k=1}^{n} A_k, \quad B = \bigcup_{k=1}^{n} B_k.$$

Then by Proposition 2, there exist the finite partitions $\{\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_r\} \subset P_1$ and $\{\widetilde{B}_1, \widetilde{B}_2, \dots\} \subset \mathfrak{N}_2$ of the sets A and B, such that every A_k $(k = 1, 2, \dots)$ (respectively, every B_k $(k = 1, 2, \dots)$) is the union of a finite number of some sets from $\{\widetilde{A}_1, \widetilde{A}_2, \dots, \widetilde{A}_r\}$ (respectively, from $\{\widetilde{B}_1, \widetilde{B}_2, \dots, \widetilde{B}_s\}$). $\square$

By Proposition 4, the netting partition $\{A_i \times B_j\}$ $(i = 1, \dots, r; j = 1, \dots, s)$ is unknown.

Corollary 1 does not extend to the countable partitions. Indeed, let P_1 be the semi-ring of all semi-segments $[a, b)$ contained in $[0, 1)$, and let P_2 be the semi-ring of all semi-segments contained in $[a, b)$ contained in $[0, +\infty)$ and $P = P_1 \otimes P_2$. Consider the countable partition

$$\left\{ \left[\frac{m-1}{2^n}, \frac{m}{2^n} \right] \times [n-1, n) \right\} \ (n = 1, 2, \dots; \ m = 1, 2, \dots, 2^n)$$

of the rectangle $[0, 1) \times [0, +\infty)$. This partition is the right generalized-netting, however it fails to have a countable continuation.

They say that to the class $\mathfrak{N}$ is assigned the multi-valued function of the set μ if to every set $E \in \mathfrak{N}$ there corresponds the uniquely defined set $\mu(E)$ of real numbers, and $\mu(\varnothing) = 0$.

Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be the normal classes of the sets, $\mathfrak{N} = \mathfrak{N}_1 \otimes \mathfrak{N}_2$ be their product, and on the rectangle $A \times B \in \mathfrak{N}$ be assigned an arbitrary multi-valued function of the rectangle μ.

The real number I is called a netting double integral of the multi-valued real function of the rectangle μ, denote by

$$(\mathfrak{N}) \iint_{A \times B} \mu(dA, dB)$$

if for every number $\varepsilon > 0$ there exists the netting partition $D_\varepsilon(A \times B)$ of the rectangle $A \times B$, such that for any its netting continuation $\{A_i \times B_j\}$ $(i, j = 1, 2, \dots)$ for every choice of the value $\mu(A_i, B_j)$ $(i, j = 1, 2, \dots)$ the inequality

$$\left|1-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu(A_i,B_j)\right|<\varepsilon$$

holds.

The real number I is called the right (the left) generalized-netting double integral of the multi-valued real function of the rectangle μ denoted by

$$(\mathfrak{N}^r)\iint\limits_{A\times B}\mu(dA,dB)\quad\Bigl((\mathfrak{N}^l)\iint\limits_{A\times B}\mu(dA,dB)\Bigr),$$

if for every number $\varepsilon>0$ there exists the right (the left) generalized netting partition $D_\varepsilon(A\times B)$ of the rectangle $A\times B$, such that for any its right (left) generalized-netting continuation $\{A_i\times B_j^i\}$ $(i,j=1,2,\dots)$ $(\{A_j^i\times B_i\}$ $(i,j=1,2,\dots))$ for every choice of the value $\mu(A_i,B_j^i)$ $(i,j=1,2,\dots)$ $(\mu(A_j^i,B_i)$ $(i,j=1,2,\dots))$ the inequality

$$\left|1-\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu(A_i,B_j^i)\right|<\varepsilon\quad\left(\left|1-\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\mu(A_j^i,B_i)\right|<\varepsilon\right)$$

holds.

Definition of the double upper and lower and of the repeated double upper and lower integrals for multiplicative classes can be found in [4].

Theorem 2. *Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be the normal classes of the sets, $\mathfrak{N}=\mathfrak{N}_1\otimes\mathfrak{N}_2$ be their product, and let on the rectangle $A\times B\in\mathfrak{N}$ be assigned an arbitrary real multi-valued function of the rectangle μ. Then the inequalities*

$$\begin{aligned}(\mathfrak{N}^r)\underline{\iint\limits_{A\times B}}\mu(dA,dB)&\le(\mathfrak{N}_1)\underline{\int\limits_{A}}\Bigl((\mathfrak{N}_2)\underline{\int\limits_{B}}\mu(dA,dB)\Bigr)\le\\ &\le(\mathfrak{N}_1)\overline{\int\limits_{A}}\Bigl((\mathfrak{N}_2)\overline{\int\limits_{B}}\mu(dA,dB)\Bigr)\le(\mathfrak{N}^r)\overline{\iint\limits_{A\times B}}\mu(dA,dB),\\ (\mathfrak{N}^l)\underline{\iint\limits_{A\times B}}\mu(dA,dB)&\le(\mathfrak{N}_2)\underline{\int\limits_{B}}\Bigl((\mathfrak{N}_1)\underline{\int\limits_{A}}\mu(dA,dB)\Bigr)\le\\ &(\mathfrak{N}_2)\overline{\int\limits_{B}}\Bigl((\mathfrak{N}_1)\overline{\int\limits_{A}}\mu(dA,dB)\Bigr)\le(\mathfrak{N}^l)\overline{\iint\limits_{A\times B}}\mu(dA,dB)\end{aligned}\tag{1}$$

hold.

Proof. We prove the inequality

$$(\mathfrak{N}_1)\overline{\int\limits_{A}}\Bigl((\mathfrak{N}_2)\overline{\int\limits_{B}}\mu(dA,dB)\Bigr)\le(\mathfrak{N}^r)\overline{\iint\limits_{A\times B}}\mu(dA,dB).$$

The rest inequalities can be proved analogously.

If

$$(\mathfrak{N}^r)\overline{\iint\limits_{A\times B}}\mu(dA,dB)=+\infty,$$

then we have nothing to prove.

Thus we first assume that

$$-\infty<(\mathfrak{N}^r)\overline{\iint\limits_{A\times B}}\mu(dA,dB)<+\infty.$$

Then for every number $\varepsilon>0$ there is the right generalized-netting partition $\{\overline{A}_i\times\overline{B}{}_j^i\}$ $(i,j=1,2,\dots)$ of the rectangle $A\times B$, such that for any choice of its from the right generalized-netting continuation $\{A_i\times B_j^i\}$ $(i,j=1,2,\dots)$, for any choice of values $\mu\{A_i\times B_j^i\}$ $(i,j=1,2,\dots)$ the inequality

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\mu(A_i,B_j^i)<(\mathfrak{N}^r)\overline{\iint\limits_{A\times B}}\mu(dA,dB)+\varepsilon \tag{2}$$

holds.

Let us now show that for every set $\underline{A}\in\mathfrak{N}_1\overline{A}_i$ $(i=1,2,\dots)$ and for any choice of values of the multi-valued function μ

$$(\mathfrak{N}_2)\overline{\int\limits_B}\mu(\underline{A},dB)<+\infty.$$

Assume to the contrary that for some set $\underline{A}\in\mathfrak{N}_1\overline{A}_1$ and for any choice of values of the multi-valued function μ

$$(\mathfrak{N}_2)\overline{\int\limits_B}\mu(\underline{A},dB)=+\infty,$$

Since $\underline{A}\in\mathfrak{N}_1\overline{A}_1 i$, there exists the partition of the set $\overline{A}_1$, having the form $\{\underline{A},A_{11},A_{12},\dots\}$. Then for the number

$$(\mathfrak{N}^r)\overline{\iint\limits_{A\times B}}\mu(dA,dB)+\varepsilon-\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\mu\big(\overline{A}_i\times\overline{B}{}_j^i\big)-\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu\big(A_{1k}\times\widetilde{B}{}_j^1\big),$$

where $\mu(\overline{A}_i,\overline{B}{}_k^i)$ $(i=2,3,\dots;\ k=1,2,\dots)$ and $\mu(A_{1j},\widetilde{B}{}_k^1)$ $(j,k=1,2,\dots)$ is one of the values of the multi-valued function μ, there exists the continuation $\{B_1^1,B_2^1,\dots\}$ of the partition $\{\overline{B}{}_j^1\}$ of the set B, such that for any choice of values $\mu(\underline{A}\times B_k^1)$ $(k=1,2,\dots)$ the inequality

$$\sum_{j=1}^{\infty}\mu(\underline{A},B_j^1)>(\mathfrak{N}^r)\overline{\int\limits_{A\times B}}\mu(dA,dB)+\varepsilon-\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\mu\big(\overline{A}_i,\overline{B}{}_j^i\big)-\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu\big(A_{1j},\overline{B}{}_k^1\big)$$

holds, whence

$$\sum_{j=1}^{\infty}\mu(\underline{A},B_j^1)+\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\mu\big(\overline{A}_i,\overline{B}\,{}_j^i\big)+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu\big(A_{1j},\overline{B}\,{}_k^1\rangle(\mathfrak{N}^r)\overline{\int\limits_{A\times B}}\mu(dA,dB)+\varepsilon,$$

which contradicts inequality (2) because the right generalized-netting partition

$$\big\{\{\underline{A}\times B_j^1\}\,(j=1,2,\dots),\ \{A_{1j}\times\overline{B}\,{}_k^1\}\,(j,k=1,2,\dots),\\ \{\overline{A}_i\times\overline{B}\,{}_j^i\}\,(i=2,3,\dots;\ j=1,2,\dots)\big\}$$

of the rectangle $A\times B$ is the continuation of the right generalized-netting partition $\{\overline{A}_i\times\overline{B}\,{}_j^i\}\,(i,j=1,2,\dots)$.

Let us now prove that

$$(\mathfrak{N}_1)\overline{\int\limits_{A}}\bigg((\mathfrak{N}_2)\overline{\int\limits_{B}}\mu(dA,dB)\bigg)<+\infty.$$

Suppose to the contrary that

$$(\mathfrak{N}_1)\overline{\int\limits_{A}}\bigg((\mathfrak{N}_2)\overline{\int\limits_{B}}\mu(dA,dB)\bigg)=+\infty.$$

Then for the number

$$(\mathfrak{N}^r)\overline{\iint\limits_{A\times B}}\mu(dA,dB)+2\varepsilon$$

there exists the continuation $\{A_1',A_2',\dots\}$ of the partition $\{\overline{A}_1,\overline{A}_2,\dots\}$ of the set A, such that for some choice of values of the integrals

$$(\mathfrak{N}_2)\overline{\int\limits_{B}}\mu(A_i',dB)\ (i=1,2,\dots),$$

the inequality

$$(\mathfrak{N}_2)\overline{\int\limits_{B}}\mu(A_i',dB)>(\mathfrak{N}^r)\overline{\iint\limits_{A\times B}}\mu(dA,dB)+2\varepsilon \tag{3}$$

holds.

On the other hand, according to the above-proven, for every A_i' $(i=1,2,\dots)$ and any choice of values of the multi-valued function μ,

$$(\mathfrak{N}_2)\overline{\int\limits_{B}}\mu(A_i',dB)<+\infty.$$

Thus for the number $\frac{\varepsilon}{2^i}$ there exists the continuation $\{B'_1, B'_2, \dots\}$ of the partition $\{\overline{B}'_1, \overline{B}'_2, \dots\}$, such that for any choice of values $\mu(A'_i, B'_k)$ $(k = 1, 2, \dots)$ the inequality

$$(\mathfrak{N}_2)\overline{\int_B}\mu(A'_i, dB) < \sum_{k=1}^{\infty}\mu(A'_i, B^i_k) + \frac{\varepsilon}{2^i} \tag{4}$$

holds.

From inequalities (3) and (4) we find that for some choice of values $\mu(A'_i, \widetilde{B}^i_k)$ $(k = 1, 2, \dots)$,

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\mu(A'_i, B^i_k) > (\mathfrak{N}^r)\overline{\iint_{A\times B}}\mu(dA, dB) + \varepsilon,$$

which contradicts inequality (2), since the right generalized-netting partition $\{A'_i \times B^i_k\}$ $(i, k = 1, 2, \dots)$ is the continuation of the right generalized-netting partition $\{\overline{A}_i \times \overline{B}{}^i_k\}$ $(i, k = 1, 2, \dots)$.

Now for the chosen number $\varepsilon > 0$ there exists the continuation $\{A'_1, A'_2, \dots\}$ of the partition $\{\overline{A}_1, \overline{A}_2, \dots\}$, such that the inequality

$$(\mathfrak{N}_1)\overline{\int_A}\left((\mathfrak{N}_2)\overline{\int_B}\mu(dA, dB)\right) < \sum_{i=1}^{\infty}(\mathfrak{N}_2)\overline{\int_B}\mu(A'_i, dB) + \frac{\varepsilon}{2} \tag{5}$$

holds.

On the other hand, for the number $\frac{\varepsilon}{2^{i+1}}$ there exists the continuation $\{B^i_1, B^i_2, \dots\}$ of the partition $\{\overline{B}^i_1, \overline{B}^i_2, \dots\}$, such that for some choice of values $\mu(A'_i, B^i_k)$ $(k = 1, 2, \dots)$ the inequality

$$(\mathfrak{N}_2)\overline{\int_B}\mu(A'_i, dB) < \sum_{k=1}^{\infty}\mu(A'_i, B^i_k) + \frac{\varepsilon}{2^{i+1}}$$

holds.

From inequalities (2) and (5) we find that for some choice of values $\mu(A'_i, B^i_k)$ $(k = 1, 2, \dots)$,

$$(\mathfrak{N}_1)\overline{\int_A}\left((\mathfrak{N}_2)\overline{\int_B}\mu(dA, dB)\right) < \sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\mu(A'_i, B^i_k) + \varepsilon.$$

From the obtained inequality, since the right generalized-netting partition $\{A'_i \times B^i_k\}$ $(i, k = 1, 2, \dots)$ is the continuation of the right generalized-netting partition $\{\overline{A}_i \times \overline{B}^i_k\}$ $(i, k = 1, 2, \dots)$, in view of inequalities (2) we obtain

$$(\mathfrak{N}_1)\overline{\int_A}\left((\mathfrak{N}_2)\overline{\int_B}\mu(dA, dB)\right) \leq (\mathfrak{N}^r)\overline{\iint_{A\times B}}\mu(dA, dB) + 2\varepsilon,$$

whence, owing to the fact that the number $\varepsilon > 0$ is arbitrary, we obtain

$$(\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big) \le (\mathfrak{N}^r)\overline{\iint_{A\times B}}\mu(dA,dB).$$

Let us now prove that

$$(\mathfrak{N}^r)\overline{\iint_{A\times B}}\mu(dA,dB) = -\infty$$

and

$$-\infty < (\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big)$$

are incompatible.

We first assume that

$$(\mathfrak{N}^r)\overline{\iint_{A\times B}}\mu(dA,dB) = -\infty$$

and

$$-\infty < (\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big) < +\infty.$$

Then on the one hand, for the number

$$(\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big) - \varepsilon$$

there exists the right generalized-netting partition $\{\overline{A}_i\times\overline{B}{}^i_k\}$ $(i,k=1,2,\dots)$ of the rectangle $A\times B$, such that for any its right generalized-netting continuation $\{A_i\times B^i_k\}$ $(i,k=1,2,\dots)$, for any choice of values $\mu(A_i\times B^i_k)$ $(i,k=1,2,\dots)$ the inequality

$$\sum_{i=1}^{\infty}\sum_{k=1}^{\infty}\mu(A_i,B^i_k) < (\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big) - \varepsilon \tag{6}$$

holds.

On the other hand, for the number $\frac{\varepsilon}{2}$ there exists the continuation $\{A'_i, A'_2,\dots\}$ of the partition $\{\overline{A}_1,\overline{A}_2,\dots\}$, such that for some choice of values of the integrals

$$(\mathfrak{N}_2)\overline{\int_B}\mu(A'_i,dB)\ (i=1,2,\dots)$$

the inequality

$$(\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big) < \sum_{i=1}^{\infty}(\mathfrak{N}_2)\overline{\int_B}\mu(A_i',dB)+\frac{\varepsilon}{2} \tag{7}$$

holds.

Let us now show that for any $\underline{A}\in\mathfrak{N}_1\overline{A}_i$ $(i=1,2,\dots)$ and for every choice of values of the multi-valued function μ,

$$(\mathfrak{N}_2)\overline{\int_B}\mu(\underline{A},dB)<+\infty.$$

Assume to the contrary that for some $\underline{A}\in\mathfrak{N}_1\overline{A}_1$ and for some choice of values of the multi-valued function μ,

$$(\mathfrak{N}_2)\overline{\int_B}\mu(\underline{A},dB)=+\infty.$$

Since $\underline{A}\in\mathfrak{N}\overline{A}_1$, there exists the partition of the set $\overline{A}_1$ having the form $\{\underline{A},A_{11},A_{12},\dots\}$. Then for the numbers

$$(\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big)-\varepsilon-\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\mu(\overline{A}_i,\overline{B}{}_j^{\,i})-\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu(A_{1k},\overline{B}{}_j^{\,1}),$$

where $\mu(\overline{A}_i,\overline{B}{}_k^{\,i})$ $(i=2,3,\dots;\ k=1,2,\dots)$ and $\mu(A_{1k},\overline{B}{}_j^{\,1})$ $(j,k=1,2,\dots)$ are the values of the multi-valued function μ, there exists the continuation $\{B_1',B_2',\dots\}$ of the partition $\{\overline{B}{}_j^{\,1}\}$ of the set B, such that for some choice of values $\mu(\underline{A},B_j')$ $(j=1,2,\dots)$ the inequality

$$\sum_{j=1}^{\infty}\mu(\underline{A},B_j')>(\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big)-\varepsilon-$$
$$-\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\mu(\overline{A}_i,\overline{B}{}_j^{\,i})-\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu(A_{1i},B_j')$$

holds, whence

$$\sum_{j=1}^{\infty}\mu(\underline{A},B_j')+\sum_{i=2}^{\infty}\sum_{j=1}^{\infty}\mu(\overline{A}_i,\overline{B}{}_j^{\,i})+\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}\mu(A_{1i},B_j')>$$
$$>(\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA,dB)\Big)-\varepsilon.$$

But this contradicts inequality (6), since the right generalized-netting partition $\big\{\{\underline{A}\times B_j'\}$ $(j=1,2,\dots)$, $\{A_{1j}\times\overline{B}{}_k^{\,1j}\}$ $(j,k=1,2,\dots)$, $\{\overline{A}_i\times\overline{B}{}_j^{\,i}\}$

$(i = 2, 3, \dots; j = 1, 2, \dots)\big\}$ of the rectangle $A \times B$ is the continuation of the right generalized-netting partition $\{\overline{A}_i \times \overline{B}{}^i_j\}\,(i, j = 1, 2, \dots)$.

Therefore for the number $\frac{\varepsilon}{2^i}$ there exists the continuation $\{B^i_1, B^i_2, \dots\}$ of the partition $\{\overline{B}{}^i_1, \overline{B}{}^i_2, \dots\}$, such that for some choice of values $\mu(A'_i, B^i_k)$ $(k = 1, 2, \dots)$ the inequality

$$(\mathfrak{N}_2)\overline{\int_B} \mu(A'_i, dB) < \sum_{j=1}^{\infty} \mu(A'_i, B^i_j) + \frac{\varepsilon}{2^{i+1}} \tag{8}$$

holds.

It follows from inequalities (7) and (8) that for some choice of values $\mu(A'_j, B^j_k)$ $(j, k = 1, 2, \dots)$,

$$(\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(dA, dB)\Big) < \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\mu(\widetilde{A}_{1j} \times \widetilde{B}{}^{1j}_k) + \varepsilon,$$

which contradicts inequality (6), since the right generalized-netting partition $\{A'_i \times B^i_j\}\,(i, j = 1, 2, \dots)$ is the continuation of the right generalized=netting partition $\{\overline{A}_i \times \overline{B}{}^i_j\}\,(i, j = 1, 2, \dots)$.

It is also proved that the equalities

$$(\mathfrak{N}^r)\overline{\iint_{A\times B}}\mu(dA, dB) = -\infty$$

and

$$(\mathfrak{N}_1)\overline{\int_A}\Big((\mathfrak{N}_2)\overline{\int_B}\mu(\widetilde{A}, dB)\Big) = +\infty.$$

are incompatible. □

Since the inequalities

$$(\mathfrak{N})\underline{\iint_{A\times B}}\mu(dA, dB) \le (\mathfrak{N}^r)\underline{\iint_{A\times B}}\mu(dA, dB) \le (\mathfrak{N}^r)\overline{\iint_{A\times B}}\mu(dA, dB) \le$$
$$\le (\mathfrak{N})\overline{\iint_{A\times B}}\mu(dA, dB),$$
$$(\mathfrak{N})\underline{\iint_{A\times B}}\mu(dA, dB) \le (\mathfrak{N}^l)\underline{\iint_{A\times B}}\mu(dA, dB) \le (\mathfrak{N}^l)\overline{\iint_{A\times B}}\mu(dA, dB) \le$$
$$\le (\mathfrak{N})\overline{\iint_{A\times B}}\mu(dA, dB)$$

hold, the above-proven theorem immediately results in

Corollary 2. *Let $\mathfrak{N}_1$ and $\mathfrak{N}_2$ be the normal classes of the sets, $\mathfrak{N}=\mathfrak{N}_1\otimes\mathfrak{N}_2$ be their product, and let on the rectangle $A\times B\in\mathfrak{N}$ be assigned an arbitrary multi-valued function of the rectangle μ. Then the inequalities*

$$(\mathfrak{N})\underline{\iint\limits_{A\times B}}\mu(dA,dB)\le(\mathfrak{N}_1)\underline{\int\limits_{A}}\Big((\mathfrak{N}_2)\underline{\int\limits_{B}}\mu(dA,dB)\Big)\le$$
$$\le(\mathfrak{N}_1)\overline{\int\limits_{A}}\Big((\mathfrak{N}_2)\overline{\int\limits_{B}}\mu(dA,dB)\Big)\le(\mathfrak{N})\overline{\iint\limits_{A\times B}}\mu(dA,dB),$$
$$(\mathfrak{N})\underline{\iint\limits_{A\times B}}\mu(dA,dB)\le(\mathfrak{N}_2)\underline{\int\limits_{B}}\Big((\mathfrak{N}_1)\underline{\int\limits_{A}}\mu(dA,dB)\Big)\le$$
$$\le(\mathfrak{N}_2)\overline{\int\limits_{B}}\Big((\mathfrak{N}_1)\overline{\int\limits_{A}}\mu(dA,dB)\Big)\le(\mathfrak{N})\overline{\iint\limits_{A\times B}}\mu(dA,dB)$$

hold.

Theorem 3. *Let $\mathfrak{N}_1,\dots,\mathfrak{N}_n$ be the normal classes of the sets, $\mathfrak{N}=\mathfrak{N}_1\otimes\cdots\otimes\mathfrak{N}_n$ be their product, and let on the rectangle $A_1\times\cdots\times A_n\in\mathfrak{N}$ be assigned an arbitrary multi-valued function of the rectangle μ. Then there take place the $n!$ inequalities*

$$(\mathfrak{N})\underline{\iint\limits_{A_1\times\cdots\times A_n}}\mu(dA_1,\dots,dA_n)\le$$
$$\le(\mathfrak{N}_n)\underline{\int\limits_{A_n}}\Big(\cdots\Big((\mathfrak{N}_1)\underline{\int\limits_{A_1}}\mu(dA_1,\dots,dA_n)\Big)\cdots\Big)\le$$
$$\le(\mathfrak{N}_n)\overline{\int\limits_{A_n}}\Big(\cdots\Big((\mathfrak{N}_1)\overline{\int\limits_{A_1}}\mu(dA_1,\dots,dA_n)\Big)\cdots\Big)\le$$
$$\le(\mathfrak{N})\overline{\iint\limits_{A_1\times\cdots\times A_n}}\mu(dA_1,\dots,dA_n),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad (*)$$
$$(\mathfrak{N})\underline{\iint\limits_{A_1\times\cdots\times A_n}}\mu(dA_1,\dots,dA_n)\le$$
$$\le(\mathfrak{N}_1)\underline{\int\limits_{A_n}}\Big(\cdots\Big((\mathfrak{N}_n)\underline{\int\limits_{A_n}}\mu(dA_1,\dots,dA_n)\Big)\cdots\Big)\le$$

$$\leq (\mathfrak{N}_1)\overline{\int\limits_{A_1}}\bigg(\cdots\bigg((\mathfrak{N}_n)\overline{\int\limits_{A_n}}\mu(dA_1,\dots,dA_n)\bigg)\cdots\bigg) \leq$$

$$\leq (\mathfrak{N})\overline{\iint\limits_{A_1\times\cdots\times A_n}}\mu(dA_1,\dots,dA_n),$$

which can, respectively, be obtained from inequalities $(*)$ *by means of all possible permutations of the sets* $A_1,\dots,A_n$.

The proof of the above theorem follows directly from Corollary 2 by using the method of mathematical induction.

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Authors' address:

Iv. Javakhishvili Tbilisi State University
2, Javakhishvili Str. Tbilisi
Georgia