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EXISTENCE RESULTS FOR FRACTIONAL ORDER SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, we establish sufficient conditions for the existence and uniqueness of mild solutions for some densely defined semilinear functional differential equations involving the Riemann-Liouville derivative.

რეზიუმე. ნაშრომში დადგენილია ინ ნახევრადწრფივი ფუნქციონალურ-დიფერენციალური განტოლების სუსტი ამონახსნების არსებობისა და ერთადერთობის საკმარისი პირობები, რომელიც შეიცავს რიმან-ლიუვილის წარმოებულს.

1. Introduction

This paper is concerned with existence of mild solutions defined on a compact real interval for fractional order semilinear functional differential equations of the form

$$D^{\alpha}y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b] \tag{1}$$

$$y(t) = \phi(t), \quad t \in [-r, 0], \tag{2}$$

where D^{α} is the standard Riemann-Liouville fractional derivative, $f : J \times C([-r, 0], E) \to E$ is a given function, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$, $\phi : [-r, 0] \to E$ a given continuous function with $\phi(0) = 0$ and $(E, |\cdot|)$ a real Banach space. For any function y defined on $[-r, b]$ and any $t \in J$ we denote by y_t the element of $C([-r, 0], E)$ defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$, up to the present time t. Functional differential and partial differential equations arise in many areas of applied mathematics and such equations have received much attention in recent years. A good guide to the literature for functional

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differential equations is the books by Hale [19] and Hale and Verduyn Lunel [20], Kolmanovskii and Myshkis [27] and Wu [41] and the references therein.

Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena in various fields of science and engineering. Indeed we can find numerous applications in viscoelasticity, electrochemistry, electromagnetism, etc. For details, see the monographs of Kilbas *et al* [25], Kiryakova [26], Miller and Ross [33], Podlubny [37] and Samko *et al* [40], and the papers of Diethelm *et al* [8, 9, 10], El-Sayed [13, 14, 15], Gaul *et al* [16], Glockle and Nonnenmacher [17], Mainardi [31], Metzler *et al* [32], Momani and Hadid [34], Momani *et al* [35], Podlubny *et al* [39], Yu and Gao [42] and the references therein. Some classes of evolution equations have been considered by El- Borai [11, 12], Jaradat *et al* [23] studied the existence and uniqueness of mild solutions for a class of initial value problem for a semilinear integrodifferential equation involving the Caputo's fractional derivative. Very recently some basic theory for the initial value problems of ordinary fractional differential equations involving Riemann-Liouville differential operator of order $0 < \alpha \leq 1$ has been discussed by Lakshmikantham and Vatsala [28, 29, 30]. In a series of papers (see [1, 2, 3]) the authors considered some classes of initial value problems for functional differential equations involving the Riemann-Liouville and Caputo fractional derivatives of order $0 < \alpha \leq 1$. For more details on the geometric and physical interpretation for fractional derivatives of both the Riemann-Liouville and Caputo types see [22, 38]. This paper is organized as follows: in Section 2, we will recall briefly some basic definitions and preliminary facts which will be used throughout the following sections. In Section 3, we give two existence results of mild solutions for problem (1)–(2). In Section 4 we indicate some generalizations to nonlocal functional differential equations. The results of the present paper can be considered as a contribution to this emerging field.

2. Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper.

By $C(J, E)$ we denote the Banach space of all continuous functions from J into E with the norm

$$\|y\|_\infty =: \sup\{|y(t)| : t \in J\}.$$

$C([-r, 0], E)$ is endowed with norm defined by

$$\|\phi\|_C =: \sup\{|\phi(\theta)| : -r \leq \theta \leq 0\}.$$

$B(E)$ denotes the Banach space of bounded linear operators from E into E, with norm

$$\|N\|_{B(E)} = \sup\{|N(y)| : |y| = 1\}.$$

$L^1(J, E)$ denotes the Banach space of measurable functions $y : J \longrightarrow E$ which are Bochner integrable normed by

$$\|y\|_{L^1} = \int_0^b |y(t)|dt.$$

Definition 2.1. ([37, 40]). The Riemann-Liouville fractional primitive of order α of a function $h : (0, b] \to E$ of order $\alpha \in \mathbb{R}^+$ is defined by

$$I_0^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds,$$

provided the right side is pointwise defined on $(0, b]$, and where Γ is the gamma function.

Definition 2.2. ([37, 40]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $h : (0, b] \to E$ is defined by

$$\frac{d^\alpha h(t)}{dt^\alpha} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} h(s)ds = \frac{d}{dt} I_0^{1-\alpha} h(t).$$

3. Existence of Mild Solutions

In this section we give our main existence result for problem (1)–(2). Before stating and proving this result, we give the definition of its mild solution.

Definition 3.1. We say that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (1)–(2) if $y(t) = \phi(t)$, $t \in [-r, 0]$, and

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s)ds, \quad t \in J.$$

Our first existence result for problem (1)–(2) is based on the Banach's contraction principle.

Theorem 3.1. *Let $f : J \times C([-r, 0], E) \to E$. Assume:*

(H) *There exists a nonnegative constant k such that*

$$|f(t, u) - f(t, v)| \leq k\|u - v\|_C, \text{ for } t \in J \text{ and every } u, v \in C([-r, 0], E).$$

If

$$\frac{Mkb^\alpha}{\Gamma(\alpha+1)} < 1, \tag{3}$$

where

$$M = \sup\{\|T(t)\|_{B(E)} : t \in J\}.$$

Then there exists a unique mild solution of problem (1)–(2) *on* $[-r, b]$.

Proof. Transform the IVP (1)–(2) into a fixed point problem. Consider the operator $F : C([-r, b], E) \to C([-r, b], E)$ defined by

$$F(y)(t) = \begin{cases} \phi(t), & t \in [-r, 0], \\ \dfrac{1}{\Gamma(\alpha)} \displaystyle\int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, & t \in [0, b]. \end{cases}$$

Let $y, z \in C([-r, b], E)$, then for each $t \in [-r, b]$,

$$\begin{aligned} |F(y)(t) - F(z)(t)| &\le \frac{M}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, y_s) - f(s, z_s)| ds \le \\ &\le \frac{Mk}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s - z_s\|_C ds \le \\ &\le \frac{Mk}{\Gamma(\alpha)} \|y - z\|_\infty \int_0^t (t-s)^{\alpha-1} ds \le \\ &\le \frac{Mkb^\alpha}{\alpha\Gamma(\alpha)} \|y - z\|_\infty. \end{aligned}$$

Taking the supremum over t,

$$\|F(y) - F(z)\|_\infty \le \frac{Mkb^\alpha}{\Gamma(\alpha+1)} \|y - z\|_\infty,$$

which implies by (3) that F is a contraction and hence F has a unique fixed point by the Banach's contraction principle, which gives rise to a unique mild solution to the problem (1)–(2). □

Next we give an existence result based upon the following nonlinear alternative of Leray-Schauder applied to completely continuous operators [18].

Theorem 3.2. *Let X a Banach space, and $C \subset X$ convex with $0 \in C$. Let $F : C \to C$ be a completely continuous operator. Then either*

(a) *F has a fixed point, or*
(b) *the set $\mathcal{E} = \{x \in C : x = \lambda F(x), \quad 0 < \lambda < 1\}$ is unbounded.*

Essential for the main results of this section, we state a generalization of Gronwall's lemma for singular kernels ([21], Lemma 7.1.1).

Lemma 3.1. *Let $v, w : [0, b] \to [0, \infty)$ be continuous functions. If $w(\cdot)$ is nondecreasing and there are constants $a > 0$ and $0 < \alpha < 1$ such that*

$$v(t) \le w(t) + a \int_0^t \frac{v(s)}{(t-s)^\alpha} ds,$$

then there exists a constant $k = k(\alpha)$ *such that*

$$v(t) \leq \omega(t) + ka \int_0^t \frac{\omega(s)}{(t-s)^\alpha} ds,$$

for every $t \in [0, b]$.

Our main result reads

Theorem 3.3. *Assume that the following hypotheses hold:*

(H1) *The semigroup* $\{T(t)\}_{t \in J}$ *is compact for* $t > 0$*;*
(H2) $f : J \times C([-r, 0], E) \to E$ *is a continuous function;*
(H3) *There exist functions* $p, q \in C(J, \mathbb{R}_+)$ *such that*

$$|f(t, u)| \leq p(t) + q(t)\|u\|_C, \quad \textit{for a.e. } t \in J, \textit{ and each } u \in C([-r, 0], E).$$

Then the problem (1)–(2) *has at least one mild solution on* $[-r, b]$.

Proof. Transform the IVP (1)–(2) into a fixed point problem. Consider the operator F defined in the proof of Theorem 3.1. We shall show that the operator F is continuous and completely continuous.

Step 1: F is continuous.

Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([-r, b], E)$. Then

$$\begin{aligned} |F(y_n)(t) - F(y)(t)| &\leq \\ &\leq \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s)[f(s, y_{n_s}) - f(s, y_s)] ds \right| \leq \\ &\leq \frac{Mb^\alpha}{\alpha\Gamma(\alpha)} \|f(., y_{n_.}) - f(., y_.)\|_\infty. \end{aligned}$$

Since f is a continuous function, then we have

$$\|F(y_n) - F(y)\|_\infty \quad \leq \quad \frac{Mb^\alpha}{\Gamma(\alpha+1)} \|f(., y_{n_.}) - f(., y_.)\|_\infty \to 0 \quad as \quad n \mapsto \infty.$$

Thus F is continuous. □

Step 2: F maps bounded sets into bounded sets in $C([-r, b], E)$.

It is enough to show that for any $\rho > 0$ there exists a positive constant δ such that for each $y \in B_\rho = \{y \in C([-r, b], E) : \|y\|_\infty \leq \rho\}$ we have $F(y) \in B_\delta$.

Then we have for each $t \in J$

$$|F(y)(t)| \quad = \quad \left| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds \right| \leq$$

$$\leq \frac{M\|p\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds + \frac{M\rho\|q\|_\infty}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}ds \leq$$

$$\leq \frac{Mb^\alpha}{\Gamma(\alpha+1)}(\|p\|_\infty + \rho\|q\|_\infty) =: \delta.$$

Step 3: F maps bounded sets into equicontinuous sets of $C([-r,b],E)$.

We consider B_q as in Step 2 . Let $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. thus if $\epsilon > 0$ and $\epsilon \leq \tau_1 \leq \tau_2$ we have

$$|F(y)(\tau_2) - F(y)(\tau_1)| \leq \frac{1}{\Gamma(\alpha)}\left|\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1}T(\tau_2-s)-\right.\right.$$

$$\left.\left. -(\tau_1-s)^{\alpha-1}T(\tau_1-s)\right]f(s,y_s)ds\right|+$$

$$+\frac{1}{\Gamma(\alpha)}\left|\int_{\tau_1-\epsilon}^{\tau_1}\left[(\tau_2-s)^{\alpha-1}T(\tau_2-s)-\right.\right.$$

$$\left.\left. -(\tau_1-s)^{\alpha-1}T(\tau_1-s)\right]f(s,y_s)ds\right|+$$

$$+\frac{1}{\Gamma(\alpha)}\left|\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}T(\tau_2-s)f(s,y_s)ds\right| \leq$$

$$\leq M\frac{\|p\|_\infty + \rho\|q\|_\infty}{\Gamma(\alpha)}\left(\left|\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1}-\right.\right.\right.$$

$$\left.\left. -(\tau_1-s)^{\alpha-1}\right]T(\tau_1-s)ds\right|+$$

$$+\left|\int_0^{\tau_1-\epsilon}(\tau_2-s)^{\alpha-1}T(\tau_1-\epsilon-s)\left(T(\tau_2-\tau_1-\epsilon)-T(\epsilon)\right)ds\right|+$$

$$+\int_{\tau_1-\epsilon}^{\tau_1}\left((\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)ds+$$

$$\left.+\int_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\right) \leq$$

$$\leq M\frac{\|p\|_\infty + \rho\|q\|_\infty}{\Gamma(\alpha)}\left(\int_0^{\tau_1-\epsilon}\left[(\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right]ds+\right.$$

$$+\|T(\tau_2-\tau_1-\epsilon)-T(\epsilon)\|_{B(E)}\int\limits_0^{\tau_1-\epsilon}(\tau_2-s)^{\alpha-1}ds+$$
$$+\int\limits_{\tau_1-\epsilon}^{\tau_1}\left((\tau_2-s)^{\alpha-1}-(\tau_1-s)^{\alpha-1}\right)ds+$$
$$+\int\limits_{\tau_1}^{\tau_2}(\tau_2-s)^{\alpha-1}ds\Big).$$

As $\tau_1 \to \tau_2$ and ϵ sufficiently small, the right-hand side of the above inequality tends to zero, since $T(t)$ is a strongly continuous operator and the compactness of $T(t)$ for $t>0$ implies the continuity in the uniform operator topology ([36]). As a consequence of steps 1 to 3 together with Arzelá-Ascoli theorem it suffices to show that F maps B_ρ into a precompact set in E.

Let $0<t<b$ be fixed and let ϵ be a real number satisfying $0<\epsilon<t$. For $y\in B_\rho$ we define

$$F_\epsilon(y)(t)=\frac{T(\epsilon)}{\Gamma(\alpha)}\int\limits_0^{t-\epsilon}(t-s-\epsilon)^{\alpha-1}T(t-s-\epsilon)f(s,y_s)ds.$$

Since $T(t)$ is a compact operator for $t>0$, the set

$$Y_\epsilon(t)=\{F_\epsilon(y)(t):\quad y\in B_\rho\}$$

is precompact in E for every ϵ, $\quad 0<\epsilon<t$. Moreover

$$\begin{aligned}|F(y)(t) \quad &- \quad F_\epsilon(y)(t)|\le M\frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha)}\times\\ &\times \quad \Big(\int\limits_0^{t-\epsilon}\Big[(t-s)^{\alpha-1}-(t-s-\epsilon)^{\alpha-1}\Big]ds+\\ &+ \quad \int\limits_{t-\epsilon}^{t}(t-s)^{\alpha-1}ds\Big)\le M\frac{\|p\|_\infty+\rho\|q\|_\infty}{\Gamma(\alpha)}\Big(t^\alpha-(t-\epsilon)^\alpha\Big).\end{aligned}$$

Therefore, the set $Y(t)=\{F(y)(t): y\in B_\rho\}$ is precompact in E. Hence the operator F is completely continuous.

Step 5: A priori bounds on solutions.

Now, it remains to show that the set

$$\mathcal{E}=\{y\in C([-r,b],E): y=\lambda F(y) \text{ for some } 0<\lambda<1\}$$

is bounded.

Let $y \in \mathcal{E}$ be any element. Then, for each $t \in J$,

$$y(t) = \lambda \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds.$$

Then

$$|y(t)| \leq M \|I^\alpha p\|_\infty + \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \|y_s\| ds. \tag{4}$$

We consider the function defined by

$$\mu(t) = \max\{|y(s)| : \quad -r \leq s \leq t\}, \quad t \in J.$$

Let $t^* \in [-r, t]$ such that $\mu(t) = |y(t^*)|$, If $t^* \in [0, b]$ then by (4) we have, for $t \in J$, (note $t^* \leq t$)

$$\mu(t) \leq M \|I^\alpha p\|_\infty + \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \mu(s) ds.$$

If $t^* \in [-r, 0]$ then $\mu(t) = \|\phi\|_C$ and the previous inequality holds.
By the Lemma 3.1 we have

$$\begin{aligned}
\mu(t) \quad &\leq \quad M \|I^\alpha p\|_\infty + k \frac{M\|q\|_\infty}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} M \|I^\alpha p\|_\infty ds \leq \\
&\leq \quad M \|I^\alpha p\|_\infty + \frac{kM^2 b^\alpha \|q\|_\infty \|I^\alpha p\|_\infty}{\Gamma(\alpha+1)} =: \Lambda
\end{aligned}$$

Hence

$$\|y\|_\infty \leq \max\{\|\phi\|_C, \Lambda\} \quad \text{for all} \quad y \in \mathcal{E}.$$

This shows that the set $\mathcal{E}$ is bounded. As a consequence of the Lemma 3.2, we deduce that the operator F has a fixed point which is a mild solution of the problem (1)–(2). $\square$

4. Nonlocal Problems

In this section we shall prove existence results for the following class of nonlocal problem

$$D^\alpha y(t) = Ay(t) + f(t, y_t), \quad t \in J := [0, b] \tag{5}$$

$$y(t) + h_t(y) = \phi(t), \ t \in [-r, 0], \tag{6}$$

where $h_t : C([-r, b], E) \to E$ is given function. The non-local condition can be applied in physics with better effect than the classical initial condition

$y(0) = y_0$. For example, $h_t(y)$ may be given by

$$h_t(y) = \sum_{i=1}^{p} c_i y(t_i + t), \quad t \in [-r, 0]$$

where c_i, $i = 1, \ldots, p$, are given constants and $0 < t_1 < \cdots < t_p \leq b$. At time $t = 0$, we have

$$h_0(y) = \sum_{i=1}^{p} c_i y(t_i).$$

Non-local conditions were initiated by Byszewski [4] (see also [5, 6, 7]) in which we refer for motivation and other references. Nonlocal conditions were initiated by Byszewski [4] when he proved the existence and uniqueness of mild and classical solutions of non-local Cauchy problems. As remarked by Byszewski [5], the nonlocal condition can be more useful than the standard initial condition to describe some physical phenomena.

Definition 4.1. We say that a continuous function $y : [-r, b] \to E$ is a mild solution of problem (5)–(6) if $y(t) = \phi(t)$, $t \in [-r, 0]$, and

$$y(t) = T(t)(\phi(0) - h_0(y)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} T(t-s) f(s, y_s) ds, \quad t \in J.$$

Theorem 4.1. *Assume that* (H) *holds and, moreover there exists a nonnegative constant* k^* *such that*

$$\|h_0(u) - h_0(v)\| \leq k^* \|u - v\|_C, \text{ for every } u, v \in C([-r, 0], E).$$

If

$$Mk^* + \frac{Mkb^\alpha}{\Gamma(\alpha+1)} < 1,$$

then the problem (5)–(6) *has a unique mild solution on* $[-r, b]$.

Theorem 4.2. *Assume that hypotheses* $(H1)$–$(H3)$ *hold and, moreover the function* h *is continuous with respect to* t, *and there exists a constant* $\beta > 0$ *such that*

$$|h_t(u)| \leq \beta, \text{ for each } u \in C([-r, b], E)$$

and for each $k > 0$ *the set*

$$\{\phi(0) - h_0(y),\ y \in C([-r, b], E), \|y\|_\infty \leq k\}$$

is precompact in E, *then the problem* (5)–(6) *has at least one mild solution on* $[-r, b]$.

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