Proceedings of A. Razmadze
Mathematical Institute
Vol. **141** (2006), 75–85

ON THE ABSOLUTE CONVERGENCE OF SERIES OF FOURIER-HAAR COEFFICIENTS

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Abstract. In the present work the absolute convergence of the series of Fourier-Haar coefficients is considered in terms of the modulus of δ-variation of a function, and the sufficient conditions for the absolute convergence are established. We prove that these conditions are unimprovable in a certain sense.

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Let the Haar system be give as follows: $\chi_1(t) \equiv 1$ if $n > 1$, then

$$\chi_n(t) = \begin{cases} \sqrt{2^p}, & t \in \Big[\dfrac{2k-2}{2^{p+1}}, \dfrac{2k-1}{2^{p+1}}\Big), \\ -\sqrt{2^p}, & t \in \Big[\dfrac{2k-1}{2^{p+1}}, \dfrac{2k}{2^{p+1}}\Big), \\ 0, & \text{at the remaining points of the segment} \quad [0,1], \end{cases}$$

where $n = 2^p + k$, $p = 0, 1, \dots$, $k = 1, 2, \dots, 2^p$.

We denote the Fourier-Haar coefficients of the function $f \in L(0,1)$ by $a_n(f)$, i.e.,

$$a_n(f) = \int_0^1 f(t)\,\chi_n(t)\,dt.$$

The present work is devoted to the investigation of convergence of the series

$$\sum_{n=1}^{\infty} \big|a_n(f)\big|^{\gamma}. \tag{1}$$

2000 *Mathematics Subject Classification.* 42A16.

Key words and phrases. Absolute convergence, modulus of variation of a function, modulus of δ-variation.

The problems of convergence of the series (1) for various classes of functions have been studied in the works due to V. Orlicz [11], Z. Ciesielski and J. Musielak [10], P. Ul'yanov [7], V. Golubov [5] and Z. Chanturia [2].

First of all, we cite some notations and definitions.

$M(0,1)$ is a class of bounded functions on the interval $[0,1]$.

The modulus of variation of the function $f \in M(0,1)$ is denoted by $\upsilon(n,f)$, a definition of that function has been introduced by Z. Chanturia ([2], p. 26).

Definition 1. $\upsilon(0,f)=0$, and for natural $n \geq 1$

$$\upsilon(n,f)=\sup_{\Pi_n}\bigg\{\sum_{k=0}^{n-1}\big|f(x_{2k+1})-f(x_{2k})\big|\bigg\},$$

where Π_n is an arbitrary division of the interval $[0,1]$ by n nonintersecting intervals (t_{2k},t_{2k+1}), $k=0,1,\dots,n-1$.

Let $\upsilon(n)$ be a nondecreasing convex function for $n \geq 0$ and $\upsilon(0)=0$. $V[\upsilon(n)]$ is a class of those functions f for which

$$\upsilon(n,f)=O(\upsilon(n)) \quad \text{as} \quad n\to\infty.$$

Definition 2. Let $f \in M(0,1)$,

$$\varphi(n;\delta;f)=\sup_{\Pi_{n,\delta}}\sum_{k=1}^{n}\omega(f;I_k),$$

where $\Pi_{n,\delta}$ is a system consisting of n nonintersecting intervals $\{I_k\}$ of the segment $[0,1]$. The length of each of the segment is equal to δ, and $\omega(f;I_k)$ is oscillation of the function f on I_k. $\varphi(n;\delta;f)$ is called the modulus of δ-variation of the function f.

Definition 3. Let $\varphi(k;\delta)$ be an arbitrary function of integer k and of nonnegative $\delta>0$, satisfying the following conditions:

$$\varphi(k;0)=\varphi(0;\delta)=0, \quad k=0,1,\dots, \quad \delta>0,$$

$\varphi(k;\delta)$ is continuous and nondecreasing with respect to δ, convex and nondecreasing with respect to k,

$$\varphi(k;\delta)\leq C\varphi\bigg(\Big[k\frac{\delta}{\eta}\Big];\eta\bigg), \quad \delta\geq\eta>0,$$

where C is some constant. The function $\varphi(k;\delta)$ is called the modulus of δ-variation.

Definition of $\varphi(n;\delta;f)$ and $\varphi(k;\delta)$ of functions has been introduced by T. Karchava ([3], p. 335).

By $M(\varphi)$ we denote the class of those functions f for which the relation

$$\varphi(k;\delta;f) \le C_0\,\varphi(k;\delta),$$

is fulfilled; here $\varphi(k;\delta)$ is the modulus of δ-variation and C_0 is some constant.
In the sequel, we will need the following lemmas.

Lemma 1 (I. Wik [8], p. 75)**.** *Let* $b_n \ge 0$, $\sum\limits_{n=1}^{\infty} b_n = \infty$ *and* $b_n \le Cn^{\lambda}$, $\lambda \ge -1$. *Then for every* $0 < \alpha < 1$ *and* $\beta > 1$ *there exists the sequence of natural numbers* q_ν, *such that*

$$\alpha^{q_{\nu+1}-q_\nu} \le \frac{b_{q_\nu}}{b_{q_{\nu+1}}} \le \beta^{q_{\nu+1}-q_\nu}$$

and

$$\sum_{\nu=1}^{\infty} b_{q_\nu} = +\infty.$$

Lemma 2 (V. Golubov [5], p. 1280)**.** *If* $c_n \ge 0$, $\sum\limits_{n=1}^{\infty} c_n < +\infty$ *and*

$$f(t) = \sum_{k=1}^{\infty} c_k \cos 2^{k+1}\pi t,$$

then for the Fourier-Haar coefficients of the function f *the relation*

$$\sum_{n=2^p+1}^{2^{p+1}} |a_n(f)| \ge \frac{1}{\pi}\, 2^{\frac{p}{2}}\, c_p$$

is valid.

Let us prove the following

Theorem 1. *If the modulus of variation of the function* f - $\varphi(n;\delta;f)$ *satisfies the condition*

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}\gamma} \left(\sum_{k=n+1}^{2n} \frac{\varphi\big(k;\frac{1}{2n};f\big)}{k} \right)^{\gamma} < +\infty \tag{2}$$

for $0 < \gamma < 2$, *then the series* (1) *is convergent.*

Proof. Let $2^p + 1 \le n < 2^{p+1}$.

It is clear that

$$a_n(f) = 2^{\frac{p}{2}} \int\limits_{\frac{2k-2}{2^{p+1}}}^{\frac{2k-1}{2^{p+1}}} \Big[f(t) - f\Big(t + \frac{1}{2^{p+1}}\Big)\Big] dt =$$

$$= 2^{\frac{p}{2}} \int\limits_{0}^{\frac{1}{2^{p+1}}} \Big[f\Big(t + \frac{2k-2}{2^{p+1}}\Big) - f\Big(t + \frac{2k-1}{2^{p+1}}\Big)\Big] dt.$$

The summation yields

$$a_n(f) = 2^{\frac{p}{2}} \int\limits_{0}^{\frac{1}{2^{p+1}}} \sum_{k=2^p+1}^{2^{p+1}} \Big[f\Big(t + \frac{2k-2}{2^{p+1}}\Big) - f\Big(t + \frac{2k-1}{2^{p+1}}\Big)\Big] dt. \tag{3}$$

Using the following T. Karchava's inequality ([3], p. 335)

$$\sum_{k=2^p+1}^{2^{p+1}} \Big| f\Big(t + \frac{2k-2}{2^{p+1}}\Big) - f\Big(t + \frac{2k-1}{2^{p+1}}\Big)\Big| \le \sum_{k=2^p+1}^{2^{p+1}} \frac{\varphi\big(k; \frac{1}{2^{p+1}}; f\big)}{k},$$

from the relation (3) we obtain

$$|a_n(f)| \le \frac{1}{2 \cdot 2^{\frac{3}{2}p}} \sum_{k=2^p+1}^{2^{p+1}} \frac{\varphi\big(k; \frac{1}{2^{p+1}}; f\big)}{k}.$$

The latter inequality results in

$$\sum_{n=2}^{\infty} |a_n(f)|^\gamma = \sum_{p=0}^{\infty} \sum_{n=2^p+1}^{2^{p+1}} |a_n(f)|^\gamma \le$$

$$\le 2^{-\gamma} \sum_{p=0}^{\infty} 2^{-\frac{3}{2}\gamma p} \Bigg(\sum_{n=2^p+1}^{2^{p+1}} \frac{\varphi\big(k; \frac{1}{2^{p+1}}; f\big)}{k} \Bigg)^\gamma \cdot 2^p, \tag{4}$$

for $\gamma > 0$.

Introduce the notation

$$U_n = \sum_{k=n+1}^{2n} \frac{\varphi\big(k; \frac{1}{2n}; f\big)}{k}$$

and show that the sequence $\frac{U_n}{n}$ is nonincreasing. Indeed, since $\frac{\varphi(n;\delta;f)}{n}$ decreases with respect to n, we obtain

$$U_{n+1}=\sum_{k=n+2}^{2n+2}\frac{\varphi\big(k;\frac{1}{2n+2};f\big)}{k}\le U_n+\frac{\varphi\big(2n+1;\frac{1}{2n+1};f\big)}{2n+1}\le$$
$$\le U_n+\frac{\varphi\big(2n;\frac{1}{2n};f\big)}{2n}\,. \tag{5}$$

Notice that

$$\frac{\varphi\big(2n;\frac{1}{2n};f\big)}{2n}\le\frac{U_n}{n}\,. \tag{6}$$

Taking into account (6), from inequality (5) it follows that $\frac{U_n}{n}$ is nonincreasing. Therefore we can use the Cauchy theorem on the number series ([9], p. 21), and taking into account inequality (4), from the condition (2) we can conclude that

$$\sum_{n=1}^{\infty}\big|a_n(f)\big|^{\gamma}<+\infty.$$

It can be easily verified that from Theorem 1 we obtain Z. Chanturia's theorem ([2], p. 27).

If $f\in M(0,1)$ and

$$\sum_{n=1}^{\infty}n^{-\frac{3}{2}\gamma}\,\upsilon^{\gamma}(n;f)<+\infty, \tag{7}$$

where $0<\gamma<2$, then the series (1) converges.

Let us now construct an example of a function f_0 for which the series (7) diverges and the series (2) converges.

We take the numbers $C_k=\frac{1}{\ln\ln k}$, $n_k=2^{2^k}$ and the intervals

$$E_k=\Big[\frac{1}{n_k}-\frac{1}{4n_k}\,,\frac{1}{n_k}+\frac{1}{4n_k}\Big],\qquad \frac{1}{n_{k+1}}+\frac{1}{4n_{k+1}}<\frac{1}{n_k}-\frac{1}{4n_k}\,.$$

Find a sequence of the function

$$f_k(x)=\begin{cases}C_k & x=\dfrac{1}{n_k}\,,\\ 0, & x=\dfrac{1}{n_k}\pm\dfrac{1}{4n_k}\,,\\ \text{linearly on} & E_k,\\ 0, & \text{in the remaining}\quad x\quad\text{from}\quad[0,1]\end{cases}$$

and assume

$$f_0(x)=\sum_{k=1}^{\infty}f_k(x).$$

Suppose $n_{k-1} \le n < n_k$. It is not difficult to find that

$$\varphi\Big(i;\frac{1}{2n};f_0\Big) \le \frac{2iC_{k-2}\,n_{k-2}}{n} + 2C_{k-1} \tag{8}$$

while

$$\upsilon(n;f_0) \ge \sum_{k=1}^{n} C_k > n\,C_n. \tag{9}$$

Using inequalities (8) and (9), we can show that if $\gamma > 0$, then

$$\sum_{n=4}^{\infty} n^{-\frac{3}{2}\gamma}\bigg(\sum_{i=n+1}^{2n}\frac{\varphi\big(i;\frac{1}{2n};f_0\big)}{i}\bigg)^{\gamma} < +\infty \tag{10}$$

and the series

$$\sum_{n=1}^{\infty} 2^{-\frac{3}{2}\gamma}\upsilon^{\gamma}(n;f_0) = +\infty \quad (0 < \gamma \le 2).$$

By Theorem 1, from (10) it follows that

$$\sum_{n=1}^{\infty} \big|a_n(f_0)\big|^{\gamma} < +\infty. \qquad \square$$

Let us show that Theorem 1 is unimprovable in a certain sense. In particular, the following theorem is valid.

Theorem 2. *Let the modulus of δ-variation $\varphi(k;\delta)$ satisfy the condition*

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}\gamma}\bigg(\sum_{i=n+1}^{2n}\frac{\varphi\big(i;\frac{1}{2n}\big)}{i}\bigg)^{\gamma} = +\infty \tag{11}$$

when $\frac{2}{3} \le \gamma < 2$, and if $\frac{2}{3} < \gamma < 1$, then $\varphi(k;\delta)$ has additionally the following property: for an arbitrary number b we can find $0 < d < 1$ and a natural number k_0, such that if $k > k_0$, then the inequality

$$\varphi(k;\delta) \ge b\,\varphi(dk;\delta),$$

holds.

Then in the class $M(\varphi)$ there exists the function f_0 for which

$$\sum_{n=1}^{\infty} \big|a_n(f_0)\big|^{\gamma} = +\infty.$$

Proof. Without losing generality, we can assume that

$$\varphi\Big(n;\frac{1}{n}\Big) \le n^{\frac{3}{2}-\frac{1}{\gamma}} \tag{12}$$

since, otherwise, instead of φ we would consider

$$\varphi_1\Big(n;\frac{1}{n}\Big) = \min\Big(\varphi\Big(n;\frac{1}{n}\Big);n^{\frac{3}{2}-\frac{1}{\gamma}}\Big).$$

Introduce the notation

$$B_n = \frac{1}{n} \sum_{i=n+1}^{2n} \frac{\varphi\left(i; \frac{1}{2n}\right)}{i}.$$

Notice that the sequence B_n is nonincreasing and, moreover,

$$B_n \le \frac{\varphi\left(n; \frac{1}{2n}\right)}{n}. \tag{13}$$

Consequently, taking into account the condition (11), we obtain

$$\sum_{n=1}^{\infty} n^{-\frac{1}{2}\gamma} B_n^{\gamma} = +\infty. \tag{14}$$

The sequence $B_n^{\gamma} n^{-\frac{1}{2}\gamma}$ for $\gamma > 0$ is nonincreasing. Using the Cauchy theorem, we can conclude that

$$\sum_{n=1}^{\infty} 2^{n\left(1-\frac{\gamma}{2}\right)} B_{2^n}^{\gamma} = +\infty. \tag{15}$$

Having fulfilled the conditions (12) and (13), we obtain

$$2^{n\left(1-\frac{\gamma}{2}\right)} B_{2^n}^{\gamma} \le 2^{n\left(1-\frac{\gamma}{2}\right)} \frac{\varphi^{\gamma}\left(2^n; \frac{1}{2^n}\right)}{2^{n\gamma}} \le 1. \tag{16}$$

Using Lemma 1, from the conditions (15) and (16) we find that for any numbers α and $\beta\left(2^{\frac{\gamma}{2}-1} < \alpha < 1,\ 1 < \beta < 2^{\frac{3}{2}\gamma-1}\right)$ there exists the sequence q_ν, such that

$$\sum_{\nu=1}^{\infty} B_{2^{q_\nu}}^{\gamma} 2^{q_\nu\left(1-\frac{\gamma}{2}\right)} = \infty \tag{17}$$

and

$$\alpha^{q_{\nu+1}-q_\nu} \le \frac{B_{2^{q_\nu}}^{\gamma}\ 2^{q_\nu(1-\frac{\gamma}{2})}}{B_{2^{q_{\nu+1}}}\ 2^{q_{\nu+1}\left(1-\frac{\gamma}{2}\right)}} \le \beta^{q_{\nu+1}-q_\nu},$$

or, what is the same thing,

$$\left(2^{\frac{1}{\gamma}-\frac{1}{2}} \alpha^{\frac{1}{\gamma}}\right)^{q_{\nu+1}-q_\nu} \le \frac{B_{2^{q_\nu}}}{B_{2^{q_{\nu+1}}}} \le \left(2^{\frac{1}{\gamma}-\frac{1}{2}} \beta^{\frac{1}{\gamma}}\right)^{q_{\nu+1}-q_\nu}. \tag{18}$$

Note that $\theta = 2^{\frac{1}{\gamma}-\frac{1}{2}} \alpha^{\frac{1}{\gamma}} > 1$, $1 < \mu = 2^{\frac{1}{\gamma}-\frac{1}{2}} \beta^{\frac{1}{\gamma}} < 2$, therefore from (18) we can get

$$\sum_{1}^{\infty} B_{2^{q_\nu}} < +\infty \quad \text{and} \quad \sum_{\nu=1}^{\nu(n)} B_{2^{q_\nu}} 2^{q_\nu} \le C B_{2^{q_{\nu(n)}}} 2^{q_{\nu(n)}},$$

where C is some constant.

Suppose

$$f_0(t) = \sum_{\nu=1}^{\infty} \pi B_{2^{q_\nu}} \cos 2^{q_\nu+1} \pi t.$$

Let $E_n(f_0)$ denote the best approximation of the function f_0 by the n-power trigonometric polynomials. From the condition (18) we can conclude that if $2^{q_\nu-1}\le n<2^{q_\nu}$, then the relation

$$E_n(f_0)\le \pi\sum_{j=\nu}^{\infty}B_{2^{q_j}}\le CB_{2^{q_\nu}} \tag{19}$$

holds.

Assume $\nu(n)=\max\{\nu;2^{q_\nu}\le n\}$, $q_0=0$. Let $\omega(\delta;f)$ denote the modulus of continuity of the function f. Using Stechkin's inequality ([6], p. 234)

$$\omega\Big(\frac{1}{n};f\Big)\le\frac{C}{n}\sum_{k=0}^{n}E_k(f).$$

we obtain

$$\begin{aligned}\omega\Big(\frac{1}{n};f_0\Big)&\le\frac{C}{n}\sum_{k=1}^{n}E_k(f_0)=\frac{C}{n}\bigg\{\sum_{\nu=1}^{\nu(n)}\sum_{2^{q_\nu-1}+1}^{2^{q_\nu}}E_k(f_0)+\sum_{k=2^{\nu(n)}+1}^{n}E_k(f_0)\bigg\}\le\\&\le\frac{C}{n}\bigg\{\sum_{\nu=1}^{\nu(n)}B_{2^{q_\nu}}\cdot 2^{q_\nu}+B_{2^{q_{\nu(n)}}}n\bigg\}\le CB_{2^{q_{\nu(n)}}}\le\\&\le C\frac{\varphi\big(2^{q_{\nu(n)}};\frac{1}{2^{q_{\nu(n)+1}}}\big)}{2^{q_{\nu(n)}}}\le C\frac{\varphi\big(n;\frac{1}{n}\big)}{n}.\end{aligned} \tag{20}$$

It is not difficult to verify that

$$\varphi(k;\delta;f)\le k\,\omega(\delta;f).$$

Consequently, from (20) it follows that

$$\begin{aligned}\varphi\Big(k;\frac{1}{n};f_0\Big)&\le k\,\omega\Big(\frac{1}{n};f_0\Big)\le k\,C\frac{\varphi\big(n;\frac{1}{n}\big)}{n}\le\\&\le k\,C\frac{\varphi\big(k;\frac{1}{n}\big)}{k}=C\,\varphi\Big(k;\frac{1}{n}\Big)\quad(k\le n).\end{aligned} \tag{21}$$

Let $\delta>0$ and $\frac{1}{n+1}\le\delta<\frac{1}{n}$, then taking into account the fact that the function $\varphi(k;\delta;f)$ nondecreasing with respect to δ, with regard for inequality (21), we have

$$\varphi(k;\delta;f_0)\le\varphi\Big(k;\frac{1}{n};f_0\Big)\le\varphi\Big(k;\frac{2}{n+1};f_0\Big)\le2\varphi\Big(k;\frac{1}{n+1};f_0\Big)\le C_1\,\varphi(k;\delta),$$

that is, $f_0\in M(\varphi)$.

Using Hölder's inequality ([10], p. 26), we obtain

$$\bigg(\sum_{n=2^p+1}^{2^{p+1}}|a_n|^\gamma\bigg)^{\frac{1}{\gamma}}\ge 2^{p\left(\frac{1}{\gamma}-1\right)}\sum_{n=2^p+1}^{2^{p+1}}|a_n|,$$

for $1 \le \gamma < 2$, or, what is the same thing,

$$\sum_{n=2^p+1}^{2^{p+1}} |a_n|^\gamma \ge 2^{p(1-\gamma)} \bigg(\sum_{n=2^p+1}^{2^{p+1}} |a_n| \bigg)^\gamma . \tag{22}$$

Using now Lemma 2, inequality (20) yields

$$\sum_{n=2^{q_\nu}+1}^{2^{q_\nu+1}} \big|a_n(f_0)\big|^\gamma \ge 2^{q_\nu(1-\gamma)} \bigg(\sum_{n=2^q+1}^{2^{q_\nu+1}} \big|a_n(f_0)\big| \bigg)^\gamma \ge$$
$$\ge 2^{q_\nu(1-\gamma)} \bigg(2^{\frac{q_\nu}{2}} B_{2^{q_\nu}} \bigg)^\gamma = B^\gamma_{2^{q_\nu}} 2^{q_\nu\left(1-\frac{\gamma}{2}\right)}.$$

Taking into account (17), the last inequality results in

$$\sum_{n=1}^{\infty} \big|a_n(f_0)\big|^\gamma = +\infty,$$

for $1 \le \gamma < 2$.

Consider the case $\frac{2}{3} < \gamma < 1$. Note that the condition (11) implies

$$\sum_{n=1}^{\infty} n^{-\frac{3}{2}\gamma} \varphi^\gamma \Big(n; \frac{1}{n} \Big) = +\infty,$$

and therefore $\varphi\big(n; \frac{1}{n}\big) \ne O(1)$.

Let E'_ν be the set of those numbers k, $1 \le k \le 2^{q_\nu}$ for which the inequality

$$\Big|a_{2^{q_\nu}+k}(f_0)\Big| \ge \frac{1}{2} 2^{-\frac{q_\nu}{2}} B_{2^{q_\nu}}$$

is fulfilled, and let E'_ν be the set of the rest integers from the interval $[2^{q_\nu}+1, 2^{q_\nu+1}]$.

On the basis of Lemma 2, we have

$$2^{\frac{q_\nu}{2}} B_{2^{q_\nu}} \le \sum_{n=2^{q_\nu}+1}^{2^{q_\nu+1}} \big|a_n(f_0)\big| = \sum_{k\in E'_\nu} \big|a_{2^{q_\nu}+k}(f_0)\big| + \sum_{k\in E''{}_\nu} \big|a_{2^{q_\nu}+k}(f_0)\big| \le$$
$$\le \frac{1}{2\sqrt{2^{q_\nu}}} \varphi\bigg(|E'_\nu|; \frac{1}{2^{q_\nu+1}}; f_0 \bigg) + \frac{2^{\frac{q_\nu}{2}}}{2} B_{2^{q_\nu}} . \tag{23}$$

Here we have used the fact that for the Fourier-Haar coefficients the estimate

$$\sum_{k\in\sigma} \big|a_n(f)\big| \le \frac{1}{2\sqrt{2^{q_\nu}}} \varphi\bigg(|\sigma|; \frac{1}{2^{q_\nu+1}}; f \bigg)$$

is valid when $2^{q_\nu} + 1 \le n < 2^{q_\nu+1}$, and σ is the subset of the set $\{2^{q_\nu}+1, \dots, 2^{q_\nu+1}\}$; $|\sigma|$ denotes a number of elements σ. Inequality (23) implies that

$$\varphi\left(|E'_\nu|;\frac{1}{2^{q_{\nu+1}}};f_0\right)\geq 2^{q_\nu}\,B_{2^{q_\nu}}\,. \tag{24}$$

It is clear from the expression B_n that

$$n\,B_n=\sum_{k=n+1}^{2n}\frac{\varphi\big(k;\frac{1}{2n}\big)}{k}\geq\frac{\varphi\big(2n;\frac{1}{2n}\big)}{2n}\,n.$$

Thus we obtain

$$2^{q_\nu}\,B_{2^{q_\nu}}\geq\frac{1}{2}\,\varphi\left(2^{q_\nu+1};\frac{1}{2^{q_\nu+1}}\right). \tag{25}$$

Since $f_0\in M(\varphi)$, there exists the number C_0, such that for any natural n and $\delta>0$ the inequality

$$\varphi\big(n;\delta;f_0\big)\leq C_0\,\varphi(n;\delta) \tag{26}$$

holds.

From the relations (24), (25) and (26) we obtain

$$\varphi\left(|E'_\nu|;\frac{1}{2^{q_\nu+1}}\right)\geq\frac{1}{C_0}\,\varphi\left(|E'_\nu|;\frac{1}{2^{q_\nu+1}};f_0\right)\geq\frac{1}{C_0}\,2^{q_\nu}\,B_{2^{q_\nu}}\geq$$

$$\geq\frac{1}{2C_0}\,\varphi\left(2^{q_\nu+1};\frac{1}{2^{q_\nu+1}}\right). \tag{27}$$

It follows from the condition of the theorem that for any number $2C_0$ there exist the natural number n_0 and $0<d<1$, such that

$$\varphi(n;\delta)\geq 2C_0\,\varphi\big([dn];\delta\big),\quad n>n_0.$$

Using the last relation, inequality (27) results in

$$\varphi\left(|E'_\nu|;\frac{1}{2^{q_\nu+1}}\right)\geq\varphi\left(\big[d\,2^{q_\nu+1}\big];\frac{1}{2^{q_\nu+1}}\right)\quad\text{for}\quad\nu>\nu_1.$$

Thus we can conclude that

$$|E'_\nu|\geq\big[d\,2^{q_\nu+1}\big]$$

that is,

$$|E'_\nu|\geq d\cdot 2^{q_\nu+1}-1>d\,2^{q_\nu},\quad\nu>\nu_1. \tag{28}$$

Let us now estimate the sum

$$\sum_{n=2^{q_\nu}+1}^{2^{q_\nu+1}}\big|a_n(f_0)\big|^\gamma\geq\sum_{k\in E'_\nu}\big|a_n(f_0)\big|^\gamma\geq 2^{-\gamma}\,2^{-\frac{q_\nu\gamma}{2}}\,B^\gamma_{2^{q_\nu}}|E'_\nu|\geq$$

$$\geq C\,2^{q_\nu\left(1-\frac{\gamma}{2}\right)}\,B^\gamma_{2^{q_\nu}}\quad(\nu>\nu_1).$$

Here we have used the definition of the set E'_ν and inequality (28).

If we take into account the condition (15), then we will get

$$\sum_{n=1}^{\infty} \big|a_n(f_0)\big|^{\gamma} \geq \sum_{\nu\geq\nu_1} \sum_{n=2^{q_\nu}+1}^{2^{q_\nu+1}} \big|a_n(f_0)\big|^{\gamma} \geq \sum_{\nu\geq\nu_1} C\, 2^{q_\nu\left(1-\frac{\gamma}{2}\right)}\, B_{2^{q_\nu}}^{\gamma} = +\infty.$$

It remains to consider the case $\gamma = \frac{2}{3}$. As P. L. Ul'ianov ([7], p. 373) has shown, the function $f(t) = 1 - 2t \in V$, and for this function

$$\sum_{n=1}^{\infty} \big|a_n(f)\big|^{\frac{2}{3}} = +\infty.$$

But $f(t) \in M(\varphi)$ for any modulus of δ-variation. Thus the theorem is complete. □

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(Received 21.04.2006)

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