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ON ONE NONLINEAR ANALOGUE OF THE MEAN VALUE PROPERTY AND ITS APPLICATION TO THE INVESTIGATION OF THE NONLINEAR GOURSAT PROBLEM

M. KLEBANSKAYA

Abstract. On the basis of an analogue of the mean value property we investigate the characteristic Goursat problem for the quasilinear hyperbolic equation of second order with admissible parabolic degeneracy. Its general solution is representable by superposition of two arbitrary functions. The structure of the domain of definition of a solution is described. The case of characteristic parabolic degeneracy of the equation is considered.

რეზიუმე. საშუალო მნიშვნელობის თვისების არაწრფივი ანალოგის საფუძველზე გამოკვლეულია გურსას მახასიათებელი ამოცანა მეორე რიგის პარაბოლურად გადაგვარებადი კვაზიწრფივი ჰიპერბოლური განტოლებისათვის, რომლის ზოგადი ამონახსნის წარმოდგენა შესაძლებელია ორი ნებისმიერი ფუნქციის სუპერპოზიციის სახით. აღწერილია ამოცანის ამონახსნის განსაზღვრის არე. მოყვანილია მოცემული განტოლების მახასიათებელი პარაბოლური გადაგვარების შემთხვევა.

In the present paper we consider a nonlinear version of the characteristic Goursat problem for the equation

$$2y(u_y - 2y)u_{xx} + (u_y - 2yu_x - 2y)u_{xy} - u_x u_{yy} = 2u_x(u_x - 1), \tag{1}$$

whose general solution is given by superposition of two arbitrary smooth functions ([1])

$$u = y^2 + f\big[y + g(x - y^2)\big]. \tag{2}$$

It should be noted that this equation is equivalent to the following nonlinear conservation law

$$\begin{cases} \dfrac{u_y - 2y}{u_x} + 2y = v, \\ 2yv_x + v_y = 0. \end{cases}$$

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The characteristic directions of equation (1) are defined by the relations

$$2y\,dy - dx = 0, \quad (u_y - 2y)dy + u_x\,dx = 0. \tag{3}$$

Depending on the derivatives of the first order of an unknown solution, when the condition

$$u_y - 2y(1 - u_x) = 0 \tag{4}$$

is fulfilled, it follows from the relations (3) that the given equation along the corresponding solutions degenerates parabolically. Therefore equation (1) belongs to a class of mixed type quasilinear hyperbolic equations ([2–4]). Following the classical theory of characteristics ([5]), the relations (3) together with equation (1) define a set of four characteristic invariants. One pair of invariants $\xi = x - y^2$ and $\xi_1 = \frac{q-2y}{p} + 2y$ is constant along all characteristics which correspond to the first of the relations (3). A set of these characteristics will be in the sequel called a ξ-family. Another pair of invariants $\eta = u - y^2$ and $\eta_1 = 2y(p-1) + q$ is defined by means of the second of relations (3), and the corresponding set of characteristics will be called an η-family ([1]).

Using the above-mentioned characteristic invariants, we constructed the general solution (2) of equation (1). This class of quasi-linear equations with general solutions representable by superposition of arbitrary functions one of which is contained in the argument of the other, has been considered in [6–8]. In some particular cases both the initial and the characteristic problems are investigated ([8–12]).

On the basis of the general representation of the solution (2), the initial Cauchy problem for the above equation is easily solvable (see [1]). As for the characteristic problems, there arises, unlike the linear case, a number of essential difficulties which are caused due to the dependence of the η-family of characteristics on an unknown solution.

1. The Nonlinear Goursat Problem. As is known, the solution of the Goursat problem is defined by its values prescribed on the arcs of two different characteristics coming out of the general point, say, from the origin. Thus one characteristic is known and given explicitly by the equality $x = y^2$. Therefore just as in the linear case, we can assign on it any values of a solution. Another characteristic depends on an unknown solution, and we choose it arbitrarily. Suppose that it is representable explicitly by the equation

$$x = \psi(y), \quad 0 \le y \le b, \quad \psi(0) = 0, \tag{5}$$

where $\psi \in C^2[0,b]$ is the given function. We consider that the arc of the characteristic (5) intersects the characteristics of the ξ-family not more than once, and its direction nowhere coincides with the ξ-characteristic direction. In other words, the equation

$$\psi(y) - y^2 = c \tag{6}$$

for an arbitrary constant $c > 0$ with respect to y fails to have more than one solution, and everywhere in the interval $[0, b]$ the inequality

$$\psi'(y) - 2y \neq 0 \tag{7}$$

is fulfilled.

All the above assumptions regarding the function ψ do not contradict one more condition that equation (6) for any constant $c \in [0, \psi(b) - b^2]$ has a unique solution

$$y = \Psi(c), \quad \Psi(0) = 0, \tag{8}$$

which continuously differentiable in a closed interval. Moreover, the functions $\psi(y) - y^2$ and $\Psi(c)$ must be inverse.

The Goursat Problem. *Find a solution of equation* (1) *together with its domain of definition, if it satisfies the condition*

$$u\big|_{x=y^2} = \varphi(y), \quad 0 \leq y \leq a, \quad \varphi \in C^2[0, a], \tag{9}$$

and the arc of the curve given by the relation (5) *is the characteristic of the η-family.*

This problem has been investigated in [1] and [12] on the basis of the general solution (2), where under certain conditions for the functions φ and ψ has been established its solvability in the class of strictly hyperbolic solutions, as well as in the singular case when the directions of characteristics (the data supports) coincide at a general point. In the present paper we endeavored to study this problem under less rigid requirements. Our investigation will be carried out on the basis of the nonlinear analogue of the well-known mean value property for equation (1). This property allows one to prove the solvability of the problem in a class both of regular and of generalized solutions.

The mean value properties for hyperbolic equations have been established in [13]–[15]. In the simplest case, for the string equation, it consists in the following: *the sums of values of a solution at the opposite vertices of any characteristic quadrangle are equal.* On the basis of the general representation of the solution (2), the above-mentioned property for equation (1) can be rephrased as follows: *the sums of ordinates at the opposite vertices of the characteristic quadrangle are equal.*

Below, it will be stated that for the given arc of the characteristic (5) we can represent explicitly all arcs of characteristics of the η-family, coming out of the points of the parabola $x = y^2$. Thus we will be able to describe completely the structure of the domain of definition of a solution of the posed problem.

Indeed, through an arbitrary point (x_0, y_0) we draw the characteristic of the ξ-family: $x - y^2 = x_0 - y_0^2$. According to (6), this parabola either does not intersect the arc of the characteristic (5), or has only one point of intersection. Naturally, the first group of points does not fall into the domain of definition of a solution. We consider only the second one. The coordinates of a set of points of this group must satisfy the inequalities

$$y_0^2 \le x_0 \le y_0^2 + \psi(b) - b^2, \tag{10}$$

which allow one to determine an infinite parabolic strip D. Assuming that the point (x_0, y_0) lies in the strip D, by the conditions (6) and (7) the parabola $x - y^2 = x_0 - y_0^2$ intersects the arc (5) at only one point, and equation (6) for the right-hand side $c = x_0 - y_0^2$ has a unique solution which is, according to (8), represented by the formula

$$y = \Psi(x_0 - y_0^2). \tag{11}$$

This relation allows us to find the ordinate of the point of intersection of two characteristic arcs. Thus the point of intersection of these characteristics is defined completely: $x = \psi[\Psi(x_0 - y_0^2)]$, $y = \Psi(x_0 - y_0^2)$.

In the strip D, on the arc of the characteristic (5) we take an arbitrary point with the ordinate $y_2 \in [0, b]$, through which we likewise draw the characteristic of the ξ-family: $x - y^2 = \psi(y_2) - y_2^2$.

Two characteristics, the parabola $x - y^2 = x_0 - y_0^2$ and an unknown characteristic of the η-family are bound to pass through the point (x_0, y_0). The latter one must intersect the parabola $x - y^2 = \psi(y_2) - y_2^2$ at some point with the coordinates (x_3, y_3). Thus we formally close the characteristic quadrangle with ordinates y_0, $\Psi(x_0 - y_0^2)$, y_2, y_3 at the vertices. Relying on the above-mentioned mean value property, we have $y_0 + y_2 = y_1 + y_3$, which allows one to find the ordinate of the point (x_3, y_3),

$$y_3 = y_0 + y_2 - \Psi(x_0 - y_0^2). \tag{12}$$

The abscissa of that point can be found from the equation of the parabola $x - y^2 = \psi(y_2) - y_2^2$:

$$x_3 = y_3^2 + \psi(y_2) - y_2^2. \tag{13}$$

The value y_2 in the relation (12) is assumed to be the parameter running through the values from the interval $[0, b]$. Consider one-parametric family y_3 which depends both on the parameter y_2 and on arguments x_0, y_0. The relations (12–13) is in fact the parametric representation of the characteristic of the η-family, passing through the point (x_0, y_0). To obtain its equation explicitly, we exclude from these equalities the parameter y_2:

$$x = y^2 + \psi\big\{y - y_0 + \Psi(x_0 - y_0^2)\big\} - \big\{y - y_0 + \Psi(x_0 - y_0^2)\big\}^2. \tag{14}$$

Equation (14) provides us with the characteristics of the η-family, passing through any point (x_0, y_0). All these characteristics intersect without fail the parabola $x = y^2$, $y \in [0, a]$. Since the function $\psi(z) - z^2$ has only one zero $z = 0$, the points of intersection of the parabola and the characteristics of the type (14) can be calculated from the condition

$$\psi\big\{y - y_0 + \Psi(x_0 - y_0^2)\big\} - \big\{y - y_0 + \Psi(x_0 - y_0^2)\big\}^2 = 0,$$

which, according to (8), allows one to determine uniquely the ordinate y,

$$y = y_0 - \Psi(x_0 - y_0^2).$$

Therefore there is no one point (x_0, y_0) at which the characteristic of the η-family would not intersect the parabola $x = y^2$.

Lemma. *The arcs of characteristics of the η-family, coming out of the points of the parabola $x = y^2$ cover the whole strip D and do not intersect each other if the conditions* (6), (7), (8) *are fulfilled.*

Indeed, under the assumption that the above-mentioned characteristics leave any lacuna in the strip D, we obtain contradiction as far as one can draw from any point of that lacuna the characteristic of type (14) which reaches the characteristic parabola $x = y^2$. Thus we can conclude that the two-parametric family of characteristics (14) is equivalent to the one-parametric family of curves represented by the relations

$$x - y^2 = \psi(y - y_0) - (y - y_0)^2, \tag{15}$$

where y_0 is the parameter with values from the interval $[0, a]$.

One can easily prove the second part of the lemma by assuming the contrary. If any two characteristics of the η-family intersect at the point $(x^*, y^*) \in D$, then we find that two characteristics of the above-mentioned family pass through that point. But this contradicts the above-proven fact that through any point of the strip D can pass not more than one η-characteristic. Thus the lemma is complete.

Consequently, all the characteristics of the ξ-family, lying in the strip D, are known. They are parabolas $x = y^2$+const. Moreover, we have constructed all characteristics of the η-family which come out of the points of the parabola-support $x = y^2$, and are represented by the one-parametric family (15).

Having all the characteristics of both families at hand, it is not difficult to represent the solution of the Goursat problem (1), (5), (9) explicitly. As is said, the combination $\eta = u - y^2$, being the characteristic invariant, retains constant value along every curve of the η-family.

In other words, the curves (15) are, in fact, a set of level curves of the difference $u - y^2$. Along every curve (15), the values of an unknown solution are also known on the basis of the fact that

$$\big(u(x,y) - y^2\big)\big|_{x=y^2+\psi(y-y_0)-(y-y_0)^2} = \big(u(x,y) - y^2\big)\big|_{(y_0^2, y_0)}.$$

This implies that

$$u = y^2 + \varphi(y_0) - y_0^2. \tag{16}$$

But this is, in fact, the value of the solution along the η-characteristic, coming out of the point of the parabola $x = y^2$ with the ordinate $y = y_0$, while we have to find a solution at an arbitrarily taken point (x', y') of the strip D. The value of the invariant $\eta = u - y^2$ at this point is the same as in the Γ-characteristic of the η-family, passing through that point. According to (14), this characteristic intersects the parabola-support at the point having the ordinate $y' - \Psi(x' - y'^2)$, and a value of the above-mentioned invariant along the whole characteristic is determined by its value at the given point. Therefore

$$u(x,y)\big|_\Gamma = y^2 + \varphi\big(y' - \Psi(x' - y'^2)\big) - \big(y' - \Psi(x' - y'^2)\big)^2.$$

Since the point (x', y') is taken arbitrarily, we can conclude that

$$u(x,y) = y^2 + \varphi\big\{y - \Psi(x - y^2)\big\} - \big\{y - \Psi(x - y^2)\big\}^2. \tag{17}$$

Naturally, the smoothness of the solution (17) is defined by that of the functions φ, Ψ. In our above reasoning there was no need to require that the function φ to be differentiable, it was sufficient only to require that it be continuous. Therefore the solution constructed by us is generalized one for equation (1). Under the requirement for the functions φ, Ψ to be twice continuously differentiable, formula (17) allows one to obtain regular solutions.

It is not less important to describe the structure of the domain of definition of the solution (17). Obviously, this domain lies in the strip D and is defined by the characteristics of the η-family, coming out of the points of the parabola-support $x = y^2$ for $y \in [0, a]$. But this family is represented completely by the equalities of the type (14) or (15). They cover continuously some part of the strip D free from lacunas. Therefore this domain will be bounded by the characteristics (15) for the parameter values $y_0 = 0$ and $y_0 = a$. For $y_0 = 0$, (15) provides us with the equation of arc of the curve (5). Another part of the domain represents the characteristic parabola $x = y^2 + \psi(b) - b^2$. The domain bounded by these curves we denote by D_0. According to the above reasoning, this domain is simply connected.

Therefore the following theorem is valid.

Theorem 1. *If the conditions* (6) *and* (7) *are fulfilled, and the function $\psi(y) - y^2$ has the only one inverse Ψ, $\Psi(0) = 0$, then the problem* (1), (5), (9) *is uniquely solvable in a class of regular solutions; its solution is represented by formula* (17) *and defined in a curvilinear characteristic quadrangle bounded by the curves $x = y^2$, $x = \psi(y)$, $x = y^2 + \psi(b) - b^2$, $x = y^2 + \psi(y - a) + (y - a)^2$.*

2. The Case of Parabolic Degeneracy. As is seen from the above investigation, the condition (7) is of importance. It guarantees the existence of the function Ψ and excludes parabolic degeneracy of equation (1) on the curve, represented by equation (5). If this condition is violated for some b_1 of the argument y, then the tangential directions of both characteristics coincide at the point of their intersection $(\psi(b_1), b_1)$, i.e. at the given point we have parabolic degeneracy. In this case it is impossible to invert equation (6). Consequently, the existence of the function Ψ and hence the solvability of the problem itself call in question. It is necessary to state whether parabolic degeneracy spreads in any direction.

The following theorem is valid.

Theorem 2. *If for some* $y = b_1 \in (0, b]$

$$\psi'(b_1) = 2b_1, \tag{18}$$

and for $0<y<b_1$ *the condition* (7) *is fulfilled, then the whole* ξ*-characteristic*

$$x = y^2 + \psi(b_1) - b_1^2, \tag{19}$$

passing through the point $(\psi(b_1), b_1)$*, is the set of points of the characteristic parabolic degeneracy of equation* (1) ([4]).

Proof. Our above reasoning connected with the construction of the characteristic of the type (14) remains valid for $0 < y < b_1$. Therefore the characteristics of the η-family can be represented by formula (14), and hence by formula (15) at every point (x_0, y_0), $y_0 \in [0, a]$ of the strip

$$0 \leq x - y^2 \leq \psi(b_1 - \varepsilon) - (b_1 - \varepsilon)^2, \tag{20}$$

where $\varepsilon > 0$ is a given sufficiently small number. Each of these characteristics intersects with the characteristic of the ξ-family

$$x = y^2 + \psi(b_1 - \varepsilon) - (b_1 - \varepsilon)^2 \tag{21}$$

at the point with the ordinate $y = y_0 + \Psi\big[\psi(b_1 - \varepsilon) - (b_1 - \varepsilon)^2\big] = y_0 + b_1 - \varepsilon$.

The tangents of the characteristic curves (15) and (21) have different directions at this point and are equal, respectively, to

$$\frac{dy}{dx}\Big|_{y=y_0+b_1-\varepsilon} = \frac{1}{\psi'(b_1 - \varepsilon) + 2y_0}\,, \quad \frac{dy}{dx}\Big|_{y=y_0+b_1-\varepsilon} = \frac{1}{2(y_0 + b_1 - \varepsilon)}\,.$$

We now pass to the limit, as $\varepsilon \to 0$. In this case we go into the whole strip $0 \leq x - y^2 \leq \psi(b_1) - b_1^2$, in which the condition (18) is fulfilled, and the characteristic (15) and parabola (19) have at the point of their intersection a common tangent with the slope

$$\frac{dy}{dx}\Big|_{y=y_0+b_1} = \frac{1}{2(y_0 + b_1)}\,.$$

Consequently, all characteristics of the η-family meet the parabola (19) upon the intersection.

Thus we can conclude that the equation (1) degenerates parabolically at every point of the parabola (19). $\square$

If we assume that $b_1 = b$, then equation (1) degenerates parabolically along the whole boundary of the domain $x = y^2 + \psi(b) - b^2$, and Theorem 2 is valid in the whole strip D_0.

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Author's address:

Ministry of Education and Science of Georgia
52, D. Uznadze St., 0102, Tbilisi
Georgia