Proceedings of A. Razmadze
Mathematical Institute
Vol. **141** (2006), 59–66

COMMUTATIVE ALGEBRAIC OPERATIONS

V. KHUKHUNASHVILI

Abstract. In the present work we study commutative algebraic operations acting in the space of solutions of the differential equation.

რეზიუმე. ნაშრომში შეისწავლება ალგებრულ-ბინარული კომუტაციური ოპერაციები რომლებიც მოქმედებენ დიფერენციალურ განტოლებათა ამონახსნთა სივრცეში.

1. In the present work we continue investigation of algebraic properties of autonomous systems of ordinary differential equations

$$\frac{du^k}{dt} = F^k\left(u^1, \dots, u^N\right), \qquad (k = 1, \dots, N). \tag{1}$$

started in [1-2]. Here, $F(u)$ are smooth functions given in the Euclidean space Γ^N, and t is an independent real function. By J_N^1 we denote the space of solutions of equations (1).

In [1], we have found the defining equation (1.3) ([1]) for binary operations acting in the space of solutions J_N^1. Introduce now new variables $r_{(1)}^k = \varphi^k(u_1)$, $r_{(2)}^k = \varphi^k(u_2)$ where $\varphi^k(u)$ are characteristic functions of equations (1). Using equation (2.2) from [1], the defining equation (1.3) from [1] will take for $\varphi(u)$ the form

$$\left(\frac{\partial \varphi^k(\Phi)}{\partial r_{(1)}^n} + \frac{\partial \varphi^k(\Phi)}{\partial r_{(2)}^n}\right) b^n = b^k, \qquad (k = 1, \dots, N). \tag{2}$$

As is stated in [1], every solution Φ of equation (2), being the function of arbitrary solutions u_1 and u_2, establishes one or another binary operation in J_N^1. In particular, the solution represented in the form of an implicit

2000 *Mathematics Subject Classification.* 35F99.
Key words and phrases. Partial differential equations, first order, binary operations, commutative operations.

function

$$\exp\left[\varphi^k(u_1) - \varphi^k(\Phi)\right] + \exp\left[\varphi^k(u_2) - \varphi^k(\Phi)\right] = 1, \qquad (3)$$
$$(k = 1, \dots, N),$$

assigns a commutative binary operation in J^1_N. Note that a rather wide class of solutions of equation (2) has the form

$$\varphi^k(\Phi) = \ln\left(\exp r^k_1 + \exp r^k_2\right) + Q^k(r_1 - r_2),$$

where Q are arbitrary functions of N arguments.

We will say that M-ary operation is defined on a given set if any ordered subset of M elements of this set corresponds to a uniquely defined element of the set.

Without going into details, we will show that to find the M-ary operation ([3]) in the space of solutions J^1_N, equation (2) extends to the equation

$$\sum_{a=1}^{M} \frac{\partial \varphi^k(\Phi)}{\partial r^n_{(a)}} b^n = b^k, \qquad (4)$$
$$(k = 1, \dots, N),$$

where $r_{(a)} = \varphi(u_a)$.

Using the results obtained in [5], among some other solutions of equation (4) we can write out the particular solution

$$\Phi = \varphi^{-1}\left(\ln \sum_{a=1}^{M} \exp\varphi(u_a)\right), \qquad (5)$$

where φ and φ^{-1} are the inverse functions. This result shows that some M-ary operations can be reduced to binary operations.

Assuming now in (4) that $M = 1$, we obtain

$$\frac{\partial \varphi^k(\Phi)}{\partial r^n} b^n = b^k. \qquad (6)$$

A solution of this equation assigns unary operation in J^1_N. As is mentioned in ([1] §3), the unary operation can be interpreted as the mapping of the space J^1_N into itself.

Thus we arrive to the conclusion that equation (4) determines a set of combinations of algebraic operations, starting from unitary ones and ending by the M-ary operations, inclusive. Therefore there arises the question whether the M-ary operations are reducible or not.

However, our aim at this step is to investigate the problems connected with commutative binary operations. Therefore, as for the M-ary operations, we will restrict ourselves to the following remark.

In [1-2] we studied in detail the commutative binary operation, $u_1 \underset{\varphi}{\dot{+}} u_2$ which is defined from (3):

$$u_1 \underset{\varphi}{\dot{+}} u_2 = \varphi^{-1}\left(\ln\left[\exp\varphi\left(u_1\right) \dot{+} \exp\varphi\left(u_2\right)\right]\right), \tag{7}$$

which is defined from (3). This operation in the space of solutions J_N^1 of equations (1) forms a commutative group in which e and h are the neutral elements:

$$u \underset{\varphi}{\dot{+}} e = u, \quad u \underset{\varphi}{\dot{+}} h = h. \tag{8}$$

As is mentioned in [1], if the components e^k and h^k of elements e and h are finite, then e and h are the stationary points of equations (1).

Besides (7), as is shown in [1-2], there exists another alternative sum

$$u_1 \underset{\varphi}{\ddot{+}} u_2 = \varphi^{-1}\left(\ln\left[\exp\left(-\varphi\left(u_1\right)\right) + \exp\left(-\varphi\left(u_2\right)\right)\right]^{-1}\right), \tag{9}$$

which likewise forms a commutative group. Unlike (8), there takes place

$$u \underset{\varphi}{\ddot{+}} e = e, \quad u \underset{\varphi}{\ddot{+}} h = u. \tag{10}$$

Due to the existence of equalities (8) and (10), the above two alternative groups get tied into a whole algebraic object, i.e., into a dual commutative group.

As is noted in [1-2], the space of solutions J_N^1 of equations (1) is a discretely fiber space ([4]), and the base space W_N^1 is the space of solutions of the system

$$\frac{dw^k}{dt} = w^k, \quad (k = 1, \ldots, N). \tag{11}$$

which is, in fact, a union of N one-dimensional independent equations. $\exp\varphi : J_N^1 \to W_N^1$ is the projector. As is mentioned in [2], each of equations from (11) generates a dual commutative group (1.2-1.5) ([2]). But then, proceeding from (7) and (9), we can write

$$u_1 \underset{\varphi}{\dot{+}} u_2 = \varphi^{-1}\left(\ln\left[w_1 \dot{+} w_2\right]\right), \tag{12}$$
$$u_1 \underset{\varphi}{\ddot{+}} u_2 = \varphi^{-1}\left(\ln\left[w_1 \ddot{+} w_2\right]\right).$$

If each of the discrete layers of the space J_N^1 is assumed to be one element, then taking into account (1.17-1.18) ([1]), from (12) it follows that dual commutative groups of equations (1) and (11) are isomorphic.

2. By $J_N^{N_0}$ and $J_{N\alpha}^1$ we denote spaces of solutions of equations (1) and (2) from ([5]). The superscript in J indicates a number of independent variables, and the subscript a number of unknown functions appearing in the corresponding equation. Obviously, $J_{N\alpha}^1 \subset J_N^{N_0}$ hold for every $\alpha \in \Gamma_{N_0}$,

where Γ_{N_0} is the N_0-dimensional Euclidean space introduced in ([5] §1). As is seen in [1], there incidentally appears a trivially fiber space $P\left(\Gamma_{N_0}, J_\alpha, \pi\right)$ with the base space Γ_{N_0}, layers J_α, and a projector $\pi : P \to \Gamma_{N_0}$. A solution χ of equations (21) from ([5]) should be interpreted as the mapping of the fiber space P into the space $J_N^{N_0}$,

$$P \xrightarrow{\chi} J.$$

which we call the χ-mapping. In [1], we studied thoroughly algebraic properties of equations (1) ([5]) by means of the functions χ. However, this does not exhaust algebraic substance of equations (1) from ([5]). The space $J^1_{N\alpha}$ is, in fact, the space of solutions of ordinary equations (2) ([5]). Proceeding from section 1, there exist M_α-ary operations which act in the same space. But then the defining equation which assigns simultaneously the M_α-mapping and the χ-ary operations has the form

$$\sum_{\alpha\in\Omega}\sum_{a=1}^{M_\alpha}\frac{\partial\varphi^k_\alpha\left(\chi\right)}{\partial r^n_{(a)\alpha}}b^n = b^k, \qquad (13)$$
$$(k = 1, \dots, N).$$

It is not difficult to write out a particular solution of equation (13) implicitly with regard for (7) and (40) from ([5]), and (5).

3. For the sake of simplicity, in what follows, it will be assumed that the neutral elements of the dual commutative group (2) ([5]) do not depend on α, i.e.,

$$e_\alpha = e, \quad h_\alpha = h, \qquad (14)$$

holds for every $\alpha \in \Gamma_{N_0}$.

As is shown in [2], the characteristic functions on neutral elements tend to infinity. By virtue of (6.6) from ([2]) we conclude that

$$\varphi_\alpha\left(e\right) = -\infty, \quad \varphi_\alpha\left(h\right) = +\infty. \qquad (15)$$

Taking now into account (14), if in (7-10) we replace $\varphi \to \varphi_\alpha$, then we will get a dual commutative group with neutral elements e, h, acting in the space $J^1_{N\alpha}$.

4. Consider now an implicit function (35) from ([5]) and assume that $q_\alpha = p^k_\alpha = 1$ for every $\alpha \in \Omega$, $k = 1, \dots, N$. Then taking into account (7) ([5]), we have

$$\sum_{\alpha\in\Omega}\exp\left[\varphi^k_\alpha\left(u_\alpha\right) - \varphi^k_\alpha\left(\chi\right)\right] = 1, \qquad (16)$$
$$(k = 1, \dots, N).$$

Here we cite some properties of the χ-mapping defined from (16).

(a) It follows directly from (6) ([5]) that the characteristic functions φ_α depend explicitly on $\alpha \in \Gamma_{N_0}$.

Consider in (16) two summands with indices α and β.

If in equality (16) we replace α and β and, respectively, u_α and u_β, then obviously (16) remains unchanged. This means that the function χ defined from (16) has the form

$$\chi = \chi\left(\ldots; \alpha, u_\alpha; \ldots\right) \tag{17}$$

and is a symmetric functions of the blocks (α, u_α), where α ranges over the set Ω. In the sequel, instead of (17) the use will be made of a shortened writing

$$\dot{\chi} = \dot{\chi}\left(\ldots, u_\alpha, \ldots\right).$$

(b) Let the solution $u_\beta = e$ for some $\beta \in \Omega$. Taking into account (15), the summand with the index β in the sum (16) vanish, and we obtain

$$\sum_{\alpha \in \Omega \setminus \beta} \exp\left[\varphi_\alpha^k\left(u_\alpha\right) - \varphi_\alpha^k\left(\dot{\chi}_\Omega\right)\right] = 1.$$

As is mentioned in [1], if all $u_\alpha = e$, as α ranges the set $\Omega \setminus \gamma$, then for $\dot{\chi}$ we obtain $\dot{\chi}_\Omega = u_\gamma$. In particular, if $u_\alpha = e$ for all $\alpha \in \Omega$, we have

$$\dot{\chi}_\Omega\left(\ldots, e, \ldots, e, \ldots\right) = e. \tag{18}$$

However, if for any $\beta \in \Omega$ the solution $u_\beta = h$, then by virtue of (15) equality (15) will be fulfilled as soon as $\dot{\chi}_\Omega = h$, i.e.

$$\dot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots, h, \ldots, u_\gamma, \ldots\right) = h. \tag{19}$$

5. In (35) from ([5]) we now put $q_\alpha = 1$, $p_\alpha^k = -1$. By analogy with (16), we obtain

$$\sum_{\alpha \in \Omega} \exp\left[\varphi_\alpha^k\left(\ddot{\chi}\right) - \varphi_\alpha^k\left(u_\alpha\right)\right] = 1, \tag{20}$$

$$(k = 1, \ldots, N).$$

The function $\ddot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots\right)$ possesses the same properties as $\dot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots\right)$, but unlike $\dot{\chi}_\Omega$, in $\ddot{\chi}_\Omega$ the neutral elements e and h show opposite properties. More exactly, instead of (18), we have

$$\ddot{\chi}_\Omega\left(\ldots, h, \ldots, h, \ldots\right) = h, \tag{21}$$

and (19) is replaced by the equality

$$\ddot{\chi}_\Omega\left(\ldots, u_\alpha, \ldots, e, \ldots, u_\gamma, \ldots\right) = e. \tag{22}$$

On the basis of the properties (18-19) and (21-22), in the sequel, $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ will be called an alternative mappings of the trivially fiber space $P\left(\Gamma_{N_0}, J^1_{N\alpha}, \pi\right)$ into the space $J_N^{N_0}$, or either $\dot{\chi}_\Omega$ and $\ddot{\chi}$ can be called alternative expansions of solutions of equations (1) from ([5]) in plane waves u_α.

6. Let us consider equation (38) ([5]), when the coefficients a^ν are the constant values. The use is made of the results of calculations presented in

([5] §12, b). From the equalities $\varphi_\alpha(u_\alpha) = z_\alpha + c_\alpha$, $\varphi_\alpha(u_\alpha) = a_\alpha \mu(u_\alpha)$ we obtain

$$\dot{\chi}_\Omega = \mu^{-1}\left(\ln \sum_{\alpha\in\Omega} \exp \mu(u_\alpha)\right).$$

Performing analogous calculations for finding an alternative expansion, from (20) for $N = 1$ we get

$$\ddot{\chi}_\Omega = \mu^{-1}\left(\ln \left[\sum_{\alpha\in\Omega} \exp(-\mu(u_\alpha))\right]^{-1}\right).$$

Now we get back to equation (43) ([5]). As is repeatedly mentioned [1-2], the neutral elements in linear equations have the form

$$e_0,\ h_0,$$

where $e_0^k = 0$, $h_0^k = \infty$, $(k = 1, \dots, N)$. From (55) ([5]) it directly follows that equalities (18-19) are fulfilled.

Let us now find an alternative sum for (55) ([5]). Towards this end, in (35) ([5]) we assume that $q = 1$, $p_\alpha^k = -\frac{1}{\lambda_\alpha^k}$. Taking into account (43) and (35) from ([5]), we obtain

$$\sum_{\alpha\in\Omega} B_\alpha(u_\alpha)\, \ddot{\chi}_\Omega = b,$$

where B_α is the matrix (49) ([5]), and the vector b has the form (51) ([5]). From the above equality we easily find an alternative expansion e_0, h_0.

It is not difficult to verify that the expansion (23) for neutral elements

$$\ddot{\chi}_\Omega = \left[\sum_{\alpha\in\Omega} B_\alpha(u_\alpha)\right]^{-1} b \tag{23}$$

and e_0, h_0 satisfy the conditions (21-22).

Using equality (50) ([5]), we rewrite (55) ([5]) in the form

$$\dot{\chi}_\Omega = \sum_{\alpha\in\Omega} B_\alpha^{-1}(u_\alpha)\, b.$$

Obviously, $\dot{\chi}_\Omega$ and $\ddot{\chi}_\Omega$ are the alternative expansion of solutions of equations (43) ([5]).

7. Let $\omega \subset \Omega$. Consider now equality (16) on ω, i.e.

$$\sum_{\alpha\in\omega} \exp\left[\varphi_\alpha(u_\alpha) - \varphi_\alpha(\dot{\chi})\right] = 1. \tag{24}$$

Using the property (15), we add to equality (24) the summands of the type $\exp\left[\varphi_\beta(u_\beta) - \varphi_\beta(\dot{\chi})\right]$, when

$$u_\beta = e, \quad \beta \in \Omega \setminus \omega. \tag{25}$$

It is obvious that equality (24) remains unchanged. But then under the condition (25) we can write

$$\dot{\chi}_{\omega}(\ldots, u_{\alpha}, \ldots) = \dot{\chi}_{\Omega}(\ldots, u_{\alpha}, \ldots). \tag{26}$$

Analogously, we can write

$$\ddot{\chi}_{\omega}(\ldots, u_{\alpha}, \ldots) = \ddot{\chi}_{\Omega}(\ldots, u_{\alpha}, \ldots), \tag{27}$$

when

$$u_{\beta} = h, \quad \beta \in \Omega \setminus \omega.$$

8. As is known in ([5-6]), in the classical theory of linear partial differential equations we can, generally speaking, choose in the representation (55) from ([5]) u_{α} and the set Ω such that the given solution coincides with the sum (55) ([5]). On the basis of the above-said, at this step of our investigation, without proof we assume that for every given solution of equations (1) from ([5]) there exist a set Ω and a corresponding collection u_{α}, $\alpha \in \Omega$, such that the solution can be represented in the form

$$u = \dot{\chi}_{\Omega}(\ldots, u_{\alpha}, \ldots). \tag{28}$$

Note, since $J_N^{N_0}$ is the discretely fiber space, the solution $u(x)$ is a definite sheet of that space ([4]).

Analogously, choosing $\tilde{\Omega}$ and u_{α}, the same solution $u \in J_N^{N_0}$ can be represented in an alternative form

$$u = \ddot{\chi}_{\tilde{\Omega}}(\ldots, u_{\alpha}, \ldots). \tag{29}$$

Thus we can conclude that the existence of $\dot{\chi}_{\Omega}$ and $\ddot{\chi}_{\Omega}$ generates dual representation of solutions of equations (1) ([5]).

9. Consider equality (35) from [5]. Using (7) ([5]), for $\dot{\chi}_{\Omega}$ and $\ddot{\chi}_{\Omega}$ we find the corresponding equations

$$\dot{\sum_{\alpha \in \Omega}} q_{\alpha} \exp p_{\alpha}^{k} \left[\varphi_{\alpha}^{k}(u_{\alpha}) - \varphi_{\alpha}^{k}(\dot{\chi})\right] = 1, \tag{30}$$

$$(k = 1, \ldots, N)$$

and

$$\ddot{\sum_{\alpha \in \Omega}} q_{\alpha} \exp p_{\alpha}^{k} \left[\varphi_{\alpha}^{k}(u_{\alpha}) - \varphi_{\alpha}^{k}(\ddot{\chi})\right] = 1, \tag{31}$$

$$(k = 1, \ldots, N).$$

It should be recalled that $\dot{\sum}$ denotes the summation by the standard rule $\left(a \dot{+} b = a + b\right)$, and $\ddot{\sum}$ is the alternative summation $\left(a \ddot{+} b = \left(\frac{1}{a} + \frac{1}{b}\right)^{-1}\right)$. It should also be noted that (31) is obtained from (30) by means of the substitution $q_{\alpha} \to 1/q_{\alpha}$, $p_{\alpha}^{k} \to 1/p_{\alpha}^{k}$.

If equalities (14)–(15) are fulfilled, then if p_α^k are reals and change their sign for different values $\alpha \in \Omega$, then equalities (18)–(19) and (21)–(22) fail to be fulfilled. In this case, the restriction (14) should be neglected, and we have to require

$$p_\alpha^k \varphi^k (e_\alpha) = -\infty,\ p_\alpha^k \varphi^k (h_\alpha) = +\infty. \tag{32}$$

This means that depending on p_α^k, the neutral elements change their representation.

References

1. Z. V. Khukhunashvili and Z. Z. Khukhunashvili, Algebraic structure of space and field. *Electron. J. Qual. Theory Differ. Equat.*, 2001, No. 6, 1–52.
2. Z. Z. Khukhunashvili and V. Z. Khukhunashvili, Alternative analysis generated by a differential equation. *Electron. J. Qual. Theory Differ. Equat.* 2003, No. 2, 1–31.
3. A. G. Kurosh, Lectures in general algebra. *Translated by Ann Swinfen; translation edited by P. M. Cohn. International Series of Monographs in Pure and Applied Mathematics*, vol. 70 *Pergamon Press, Oxford-Edinburgh-New York*, 1965.
4. B. A. Dubrovin and S. R. Novikov, and A. T. Fomenko, Modern geometry—methods and applications. Part I. The geometry of surfaces, transformation groups, and fields. *Translated from the Russian by Robert G. Burns. Graduate Texts in Mathematics*, 93. *Springer-Verlag, New York*, 1984.
5. V. Z. Khukhunashvili, On nonlinear expansion of solutions of a quasilinear system, *Proc. A. Razmadze Math. inst.* **140**(2006), 109–119.
6. R. Courant and D. Hilbert, Methods of mathematical physics, v. II, *Translated from the German, Gos. Tex-Teor. Izd., Moscow,* 1945.

(Received 17.01.2006)

Author's address:

I. Javakhishvili Tbilisi State University
Faculty of Exact and Natural Sciences
2, University St., Tbilisi 0143
Georgia