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TO THE PROBLEM OF FULL TRANSITIVITY OF THE COTORSION HULL

T. KEMOKLIDZE

Abstract. We consider the problem of full transitivity of the cotorsion hull of abelian p-groups of type $T = A \oplus C$, where A is a separable group with certain properties containing elements with infinite carriers, and C is an unbounded direct sum of cyclic p-groups. It is shown that the cotorsion hull in the class of these groups is never fully transitive.

რეზიუმე. განიხილება $T = A \oplus C$ სახის აბელური p-ჯგუფის კოპერიოდული გარსის სრული ტრანზიტულობის საკითხი, სადაც A გარკვეული თვისების მქონე უსასრულო მზიდის ელემენტების შემცველი სეპარაბელური ჯგუფია, ხოლო C ციკლური p-ჯგუფების შემოუსაზღვრელი პირდაპირი ჯამი. ნაჩვენებია, რომ ამ ჯგუფთა კლასში კოპერიოდული გარსი არ არის სავსებით ტრანზიტული.

In the sequel, under "group" we mean an additively written abelian group. In the present work the use will be made of the notation and terminology adopted from the monographs [1] and [2].

By p we denote a prime number. $\mathbf{Z}$ and $\mathbf{Q}$ denote the groups of integers and rational numbers, respectively. A group A is said to be cotorsion, if its extension by any torsion free group C splits: $\operatorname{Ext}(C, A) = 0$.

Importance of the class of cotorsion groups is explained by two facts:

(1) for any groups A, B the group $\operatorname{Ext}(A, B)$ is cotorsion;

(2) every reduced group G can be isomorphically embedded into the cotorsion group $G^\bullet = \operatorname{Ext}(Q/Z, G)$ called the cotorsion hull of the group G, and moreover $G^\bullet/G$ is a torsion free divisible group. Every reduced cotorsion group C decomposes into direct sum $C = A \oplus G$, where A is a torsion free algebraically compact group and $G \cong \operatorname{Ext}(Q/Z, \mathrm{t}G)$, where $\mathrm{t}G$ is the torsion part of the group G. If $\mathrm{t}G = \bigoplus_p T_p$ is the expansion into the direct sum of primary components, then

$$\operatorname{Ext}(Q/Z, \mathrm{t}G) \cong \prod_p \operatorname{Ext}(Z(p^\infty), T_p).$$

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Thus the study of cotorsion groups is, to a considerable extent, reduced to the study of groups of type $\mathrm{Ext}(Z(p^\infty), T)$, where T is a p-group.

The *indicator* or the *Ulm sequence* of an element a of a group A is the increasing sequence of ordinal numbers and symbols ∞,

$$H(a) = (h(a), h(pa), \dots, h(p^n a), \dots),$$

where h is the generalized p-height of the element, i. e. $h(a) = \sigma$, if $a \in p^\sigma A \backslash p^{\sigma+1} A$, and $h(0) = \infty$. In the set of indicators we introduce the ordering

$$H(a) \leq H(b) \Longleftrightarrow r(p^i a) \leq h(p^i b), \quad i = 0, 1, 2, \dots.$$

A reduced group A is said to be fully transitive, if for its arbitrary elements a and b when $H(a) \leq H(b)$ for any prime p, there exists an endomorphism φ of the group A such that $\varphi a = b$.

Completion of a p-group T in its p-adic topology will be denoted by $\widehat{T}$. Its torsion part $\overline{T}$ is a torsion complete group. It is shown in [3] and [4] that if T is a torsion complete group or a direct sum of cyclic p-groups, then its cotorsion hull $T^\bullet = \mathrm{Ext}(Z(p^\infty), T)$ is fully transitive. In fully transitive groups, by means of indicators one can describe the lattice of fully invariant subgroups.

In particular, I. Kaplansky described fully invariant subgroups of any fully transitive p-group A. Every such subgroup has form

$$A(u) = \{a \in A | H(a) \geq u\},$$

where $u = (\sigma_0, \sigma_1, \dots, \sigma_n, \dots)$ is an increasing sequence of ordinal numbers and symbols ∞, satisfying the following condition: if there is a gap between σ_n and σ_{n+1}, then A has an element of order p and of height σ_n (see [2], Theorem 67.1).

Having analyzed I. Kaplansky's proof, A. Mader [3] has formulated the following theorem.

Let A be a module over a commutative ring R, let M be the lattice of all its fully invariant submodules, let S be some meet semilattice, and let $U : A \to S$ be a map with the following properties:

1. U is surjective.
2. $U(fa) \geq U(a)$ for any $a \in A$ and for any endomorphism f of the module A.
3. $U(a + b) \geq U(a) \wedge U(b)$.
4. If $U(a) \geq U(b)$, then there exists an endomorphism f of the module A, such that $f(b) = a$.
5. For $C \in M$ and any a, $b \in C$ there exists a $c \in C$ such that $U(c) = U(a) \wedge U(b)$.

Then the set S^* of all dual ideals of S ordered by inclusion is a lattice, and the mapping $\alpha : S^* \to M$ defined by the rule $\alpha(D) = \{a \in A | U(a) \in D\}$, is a lattice isomorphism.

In case T is a torsion complete group or a direct sum of cyclic p-groups, A. Mader [3] and A. I. Moskalenko [4] have used the indicator function $U = H$ to describe the lattice of fully invariant subgroups of the cotorsion hull $T^\bullet$ of T, since in these cases, as is mentioned above, $T^\bullet$ is fully transitive, and all conditions of A. Mader's theorem are fulfilled. The problem of full transitivity of the group $T^\bullet$, when T is a direct sum of torsion complete groups, has been studied by the author in [5].

For separable p-groups T, A. I. Moskalenko [4] represented elements of $T^\bullet$ in the form of countable sequences

$$T^\bullet = \{(a_0, a_1+T, \dots, a_i+T, \dots) | a_i \in \widehat{T},\ pa_{i+1}-a_i \in T;\ i = 0, 1, \dots\}. \quad (1)$$

Such representation makes easy calculation of the height and the indicator. In particular, let $a = (a_0, a_1 + T, \dots)$ and denote by $H_{T^\bullet}(a)$ the indicator of the element a in the group $T^\bullet$, then we have

$$H_{T^\bullet}(a) = \begin{cases} H_{\widehat{T}}(a_0), \quad \text{if the order} \quad O(a_0) = \infty; \\ (h_{\widehat{T}}(a_0), h_{\widehat{T}}(pa_0), \dots, h_{\widehat{T}}(p^{n-1}a_0), \\ \quad \omega + m, \omega + m + 1, \dots) \\ \text{if} \quad a_0 \in \widehat{T} \backslash T, O(a_0) = p^n, \quad O(a_0 + T) = p^{n-m}; \\ (h_{\widehat{T}}(a_0), h_T(pa_0), \dots, h_{\widehat{T}}(p^{n-1}a_0), \\ \quad \omega + n + k, \omega + n + k + 1, \dots) \\ \text{if} \quad O(a_0) = p^n,\ a_0, a_1, \dots, a_k \in T, a_{k+1} \notin T; \\ H_T(a_0), \quad \text{if} \quad a_i \in T \quad \text{for any} \quad i \end{cases} \quad (2)$$

where ω is the smallest infinite ordinal.

Let A be a separable p-group with the basic subgroup $B = \bigoplus_{n=1}^{\infty} B_n$, $B_n = \bigoplus_{m_n} Z(p^n)$, such that there exists a direct summand

$$B' = \bigoplus_{i=1}^{\infty} \langle b_i \rangle \quad (3)$$

of the group B, where $\langle b_i \rangle$ is a direct summand in B_i, $O(b_i) = p^i$, and for any subgroup $B'_\alpha = \bigoplus_{i=1}^{\infty} \langle b_{\alpha_i} \rangle$ of the group B', there exists in the group $\overline{B}'_\alpha$ an element with infinite carrier lying in A.

Consider a separable p-group

$$T = A \oplus C, \quad (4)$$

where C is an unbounded direct sum of cyclic p-groups, $C = \bigoplus_{n=1}^{\infty} C_n$, $C_n = \bigoplus_{S_n} Z(p^n)$. Obviously, $B \oplus C$ is the basic subgroup of the group T, and $\widehat{T} = \widehat{A} \oplus \widehat{C}$.

Note that aforementioned property of the group A is satisfied by groups from such an important class of separable p-groups as the class of direct sums of torsion complete groups. Moreover a particular case of groups of type (4) is the example of a group, presented by A. I. Moskalenko [3], whose cotorsion hull is not fully transitive.

Every element of the p-adic completion $\widehat{T}$ of the group T can be written in the form of an infinite vector $t = (t_1, t_2, \dots, t_n, \dots)$, where $t_n \in B_n \oplus C_n$, and $h(t_n) \to \infty$ as $n \to \infty$. We will write this element also in the form

$$t = t_1 + t_2 + \cdots = \sum_{n=1}^{\infty} t_n, \tag{5}$$

since this sum converges in the p-adic topology to the element t. Note that when $t \in \overline{T}$ is of order $O(t) = p^m$, every summand in (5) will be p^m-bounded: $p^m b_i = O$, $i = 1, 2, \dots$.

Theorem. *If a separable p-group T is of the form* (4)*, then its cotorsion hull $T^\bullet$ is not fully transitive.*

Proof. By the condition, in the basic subgroup B of the group A there exists a direct summand B' of type (3). C is an unbounded direct sum of cyclic p-groups. Therefore we can choose in the groups B' and C the elements d_i and g_i, respectively, such that $d_i \in \langle b_{\alpha_i} \rangle$, $g_i \in \langle c_{\beta_i} \rangle$ and

$$\mathrm{e}(d_i) = \mathrm{e}(g_i), \quad h(d_i) \le h(g_i)$$
$$\mathrm{e}(d_i) < \mathrm{e}(g_{i+1}), \quad h(d_{i+1}) > h(g_i) + \mathrm{e}(g_i)$$

where e denotes the exponent, and $i = 1, 2, \dots$. Obviously, $h(d_i)$, $h(g_i) \to \infty$ as $i \to \infty$. Then

$$d_0 = d_1 + d_2 + \cdots \quad \text{and} \quad g_0 = g_1 + g_2 + \cdots$$

are elements of infinite order from the groups $\widehat{A}$ and $\widehat{C}$, respectively.

It is eqsy to see that

$$H_{\widehat{A}}(d_0) = H_{\widehat{A}}(d_0) \le H_{\widehat{C}}(g_0).$$

Using the representation (1) of the group $T^\bullet$, we consider its elements

$$d = (d_0', d_1' + T, \dots) \quad \text{and} \quad g = (g_0', g_1' + T, \dots),$$

where $d_0' = d_0$, $g_0' = g_0$. By equality (2), $H_{\widehat{T}}(d_0) = H_{T^\bullet}(d)$ and $H_{\widehat{T}}(g_0) = H_{T^\bullet}(g)$. Therefore $H_{\widehat{T}}(d) = H_{T^\bullet}(g)$.

Let us show that there exists no endomorphism of the group $T^\bullet$ which maps d onto g. This is equivalent to the proof that there is no endomorphism of the group $\widehat{T}$, inducing endomorphism on the subgroup T and mapping d_0 onto g_0.

Since $H_{\widehat{T}}(d_0) \le H_{T^\bullet}(g_0)$ and the algebraically compact group $\widehat{T}$ is fully transitive (see [3]), there indeed exists an endomorphism φ of the group $\widehat{T}$ with $\varphi d_0 = g_0$. Let us show that there exists an element $t \in T$ for which $\varphi t \notin T$.

On the basis of the properties of the group $\widehat{T}$,

$$\varphi d_0 = \varphi(d_1 + d_2 + \cdots) = \varphi d_1 + \varphi d_2 + \cdots = g_0 = g_1 + g_2 + \cdots .$$

We denote the projection of the group $\widehat{T}$ onto the direct summand $\widehat{T}$ by π and onto $\langle C_{\beta_i} \rangle$ by π_i. Then

$$\varphi d_0 = g_0 = \pi\varphi d_0 = \pi\varphi d_2 + \cdots = d_1 + g_2 + \cdots .$$

The element $d_i \in B' \subset T$, therefore if for some i the element $\pi\varphi d_i$ lying in $\overline{C}$ has infinite carrier, then $\pi\varphi d_i \notin T$. This implies that $\varphi d_i \notin T$, consequently, φ does not induce an endomorphism on the subgroup T.

Suppose now that for every i, the element $\pi\varphi d_i$ has finite carrier, i. e. lies in C. Since $h(d_{i+1}) > h(g_i) + e(g_i)$ and $g_i \in \langle C_{\beta_i} \rangle$, g_i can be obtained from a finite number of summands

$$g_i = \pi_i\varphi d_1 + \pi_i\varphi d_2 + \cdots + \pi_i\varphi d_n, \quad n \leq i. \tag{6}$$

If $n < i$, then $O(d_n) < O(g_i)$. Therefore height of the element $\pi_i\varphi d_n \in \langle C_{\beta_i} \rangle$ is bigger than that of g_i. Consequently, height of the right-hand side of (6) must bigger than that of g_i, which is impossible.

Let $n = i$. Then we have three possible cases.

1. $h(\pi_i\varphi d_i) > h(g_i)$;
2. $h(\pi_i\varphi d_i) < h(g_i)$;
3. $h(\pi_i\varphi d_i) = h(g_i)$.

In the first case the above argument implies

$$h(\pi_i\varphi d_1 + \pi_i\varphi d_2 + \cdots \pi_i\varphi d_i) > h(g_i).$$

Therefore

$$g_i \neq \pi_i\varphi d_1 + \pi_i\varphi d_2 + \cdots + \pi_i\varphi d_i$$

In the second case height of every summand of the right-hand side of (6) except the last one, is bigger than that of g_i; height of the last summand is less than $h(g_i)$. Therefore $h(g_i) = h(\pi_i\varphi d_i) < h(g_i)$, but this again leads to the contradiction. Hence only the third case

$$h(\pi_i\varphi d_i) = h(g_i), \quad i = 1, 2, \ldots$$

can take place. Hence

$$O(\pi_i\varphi d_i) = O(g_i), \quad e(g_i) = e(\pi\varphi d_i) = e(d_i). \tag{7}$$

Images of $\pi\varphi d_i$, $i = 1, 2, \ldots$ are elements with finite carrier from $\overline{C}$. Therefore there exist elements $d_{i_1}, d_{i_2}, \ldots$ from B', such that carriers of the elements of $\pi\varphi d_{i_j}$ do not intersect, $j = 1, 2, \ldots$.

By definition (3) of the group B', in the group A there exists an element of finite order p^n with an infinite carrier which can, without loss of generality, be taken in the form

$$t = p^{e(d_{i1})-n} d_{i_1} + p^{e(d_{i2})-n} d_{i_2} + \cdots .$$

Obviously, $t \in \overline{B}'$, $O(t) = p^n$, $t \in A$, $t \in T$.

On the other hand,

$$\pi\varphi(p^{e(d_{ij})-n}d_{ij}) = p^{e(d_{ij})-n}\pi\varphi d_{ij} \neq 0,$$

since $e(\pi\varphi d_{i_j}) = e(d_{i_j})$, $j = 1, 2, \ldots$ by virtue of (7).

We have

$$\pi\varphi t = p^{e(d_{i_1})-n}\pi\varphi d_{i_1} + p^{e(d_{i_2})-n}\pi\varphi d_{i_2} + \cdots \tag{8}$$

The right-hand side of (8) is an element of the group $\overline{C}$ with infinite carrier. Therefore $\pi\varphi t \notin C$, hence $\varphi t \notin T$, i. e. φ does not induce an endomorphism on T. The theorem is proved.

It follows from the theorem that in general for the description of the lattice of fully invariant subgroups of the group $T^{\bullet}$, when T is a direct sum of a separable p-group and of a direct sum of cyclic p-groups, it is impossible to use the indicator function H, because of the condition 4 of A. Mader's theorem. Consequently, we have to seek for another function possessing the needed properties.

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Author's address:

I. Tsereteli Kutaisi State University
Department of Mathematics
59, Tamar Mephe Str.
Georgia