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ON THE ABSOLUTE CONVERGENCE OF TRIGONOMETRIC FOURIER SERIES

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Abstract. The problem of convergence of the series

$$\sum_{n=1}^{\infty} \gamma_n \, \rho_n^r(f), \quad 0 < r < 2$$

is considered, where $\rho_n(f) = \big(a_n^2(f) + b_n^2(f)\big)^{\frac{1}{2}}$, $a_n(f)$, $b_n(f)$ are the coefficients of the Fourier trigonometric series of the function f, and $\{\gamma_n\}$ is the sequence of positive numbers satisfying definite conditions.

From the obtained results follow the theorems of Bernstein, Szasz, Stechkin, Chelidze and Konyushkov.

რეზიუმე. შესწავლილია

$$\sum_{n=1}^{\infty} \gamma_n \, \rho_n^r(f), \quad 0 < r < 2$$

მწკრივის კრებადობის საკითხი, სადაც $\rho_n(f) = \big(a_n^2(f) + b_n^2(f)\big)^{\frac{1}{2}}$, $a_n(f)$ და $b_n(f)$ არიან f ფუნქციის ფურიეს ტრიგონომეტრიული მწკრივის კოეფიციენტები, ხოლო $\{\gamma_n\}$ დადებით რიცხვთა მიმდევრობაა, რომელიც აკმაყოფილებს გარკვეულ პირობებს.

მიღებული შედეგებიდან გამომდინარეობს ბერშტეინის, სასის, სტეჩკინის, ჭელიძის, კონიუშკოვის თეორემები.

Let $f \in L(T)$, $T = [-\pi, \pi]$, $\rho_k(f) = \big(a_k^2(f) + b_k^2(f)\big)^{\frac{1}{2}}$, where $a_k(f)$ and $b_k(f)$ are the Fourier coefficients of the function f.

We say [1] that the sequence $\{\gamma_k\}$ of nonnegative numbers belongs to the class A_α, $\alpha \geq 1$, if there exists $\varkappa_\alpha > 0$, such that for any natural n the inequality

$$\Bigg(\sum_{k=2^n+1}^{2^{n+1}} \gamma_k^\alpha \Bigg)^{\frac{1}{\alpha}} \leq \varkappa_\alpha \, 2^{n \frac{1-\alpha}{\alpha}} \sum_{k=2^{n-1}+1}^{2^n} \gamma_k$$

is fulfilled.

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It is not difficult to show that $A_{\alpha_1} \subset A_{\alpha_2}$ if $\alpha_1 > \alpha_2$. Note also that $\overline{A} \subset A_\alpha$, $\alpha \geq 1$, where $\overline{A}$ is the class of numbers introduced by P. Ul'yanov [2] as follows: $\{\gamma_k\} \in \overline{A}$, if there exists $\varkappa \geq 1$, such that

$$\max_{2^n < k \leq 2^{n+1}} \gamma_k \leq \varkappa \min_{2^{n-1} < k \leq 2^n} \gamma_k.$$

Let $f \in L_p(T)$, $p \geq 1$. We introduce the notation

$$C_{2^k}(f;p) = \left(2^k \int\limits_T \left|f\left(x + \frac{\pi}{2^{k+1}}\right) - f\left(x - \frac{\pi}{2^{k+1}}\right)\right|^p dx\right)^{\frac{1}{p}}.$$

and prove the following

Theorem 1. *Let* $f \in L_p(T)$, $p \in (1,2]$, $r \in \left(0, \frac{p}{p-1}\right)$, *and* $\{\gamma_k\} \in A_{\frac{p}{p-rp+r}}$, *then*

$$\sum_{n=3}^{\infty} \gamma_k \, \rho_n^r(f) \leq \varkappa_{p,r} \sum_{n=1}^{\infty} C_{2^{n+1}}^r(f;p) \sum_{k=2^{n-1}+1}^{2^n} \gamma_k. \tag{1}$$

Proof. As is known ([7], p. 609),

$$f\left(x + \frac{\pi}{2^{n+2}}\right) - f\left(x - \frac{\pi}{2^{n+2}}\right) \sim 2 \sum_{k=1}^{\infty} (b_k \cos kx - a_k \sin kx) \sin \frac{k\pi}{2^{n+2}}.$$

Therefore according to the Hausdorff-Young theorem ([7], p. 211), if $q = \frac{p}{p-1}$, we have

$$\begin{aligned}
&\sum_{k=1}^{\infty} \rho_k^q(f) \left|\sin \frac{k\pi}{2^{n+2}}\right|^q \leq \\
&\quad \leq \varkappa_p \left(\int\limits_T \left|f\left(x + \frac{\pi}{2^{n+2}}\right) - f\left(x - \frac{\pi}{2^{n+2}}\right)\right|^p dx\right)^{\frac{1}{p-1}} = \\
&\quad = \varkappa_p \, 2^{-\frac{n+1}{p-1}} \, C_{2^{n+1}}^{\frac{p}{p-1}}(f;p).
\end{aligned}$$

If we take into account that

$$\left|\sin \frac{k\pi}{2^{n+2}}\right|^q \geq 2^{-q}, \quad 2^n < k \leq 2^{n+1},$$

the above inequality yields

$$\begin{aligned}
\sum_{k=2^n+1}^{2^{n+1}} \rho_k^q(f) &\leq \varkappa_q \sum_{k=2^n+1}^{2^{n+1}} \rho_k^q(f) \left|\sin \frac{k\pi}{2^{n+2}}\right|^q \leq \\
&\leq \varkappa_{p,q} \, 2^{-\frac{n+1}{p-1}} \, C_{2^{n+1}}^{\frac{p}{p-1}}(f;p).
\end{aligned} \tag{2}$$

Using Hölder's inequality and the fact that $\{\gamma_k\} \in A_{\frac{p}{p-rp+r}}$, we obtain

$$\sum_{k=2^n+1}^{2^{n+1}} \gamma_k\, \rho_k^r(f) \le \left(\sum_{k=2^n+1}^{2^{n+1}} \gamma_k^{\frac{p}{p-rp+r}} \right)^{\frac{p-rp+r}{p}} \left(\sum_{k=2^n+1}^{2^{n+1}} \rho_k^q(f) \right)^{\frac{r}{q}} \le$$

$$\le \varkappa_{p,r}\, 2^{\frac{nr(1-p)}{p}} \left(\sum_{k=2^n+1}^{2^{n+1}} \rho_k^q(f) \right)^{\frac{r}{q}} \left(\sum_{k=2^{n-1}+1}^{2^n} \gamma_k \right).$$

From the latter inequality, using (2), we find that the relation

$$\sum_{k=2^n+1}^{2^{n+1}} \gamma_k\, \rho_k^r(f) \le \varkappa_{p,r}\, 2^{\frac{nr(1-p)}{p}} \cdot 2^{-\frac{(n+1)r}{(p-1)q}}\, C_{2^{n+1}}^{\frac{pr}{(p-1)q}}(f;p) \left(\sum_{k=2^{n-1}+1}^{2^n} \gamma_k \right) \le$$

$$\le \varkappa_{p,r}\, 2^{-nr}\, C_{2^{n+1}}^r(f;p) \left(\sum_{k=2^{n-1}+1}^{2^n} \gamma_k \right) \tag{3}$$

is valid.

Summarizing inequalities (3), we will obtain inequality (1) which is being proved. □

Corollary 1.

$$\sum_{k=1}^{\infty} \gamma_k\, \rho_k^r(f) \le \varkappa_{p,r} \sum_{k=1}^{\infty} \gamma_k\, k^{-\frac{r(p-1)}{p}}\, \omega^r\left(\frac{1}{k}; f\right)_{L_p},$$

where $\omega\left(\frac{1}{k}; f\right)_{L_p}$ *is the modulus of continuity of the function* f *in the norm* $L_p(T)$.

Proof. It is easily seen that

$$C_{2^n}(f;p) \le \varkappa\, 2^{\frac{n}{p}}\, \omega\left(\frac{1}{2^{n+1}}; f\right)_{L_p}.$$

Therefore it follows from inequality (3) that

$$\sum_{k=2^n+1}^{2^{n+1}} \gamma_k\, \rho_k^r(f) \le \varkappa_{p,r} 2^{-nr} 2^{\frac{(n+1)r}{p}}\, \omega^r\left(\frac{1}{2^n}; f\right)_{L_p} \sum_{2^{n-1}+1}^{2^n} \gamma_k \le$$

$$\le \varkappa_{p,r} 2^{-\frac{nr}{q}}\, \omega^r\left(\frac{1}{2^n}; f\right)_{L_p} \sum_{2^{n-1}+1}^{2^n} \gamma_k \le$$

$$\le \varkappa_{p,r} \sum_{2^{n-1}+1}^{2^n} \gamma_k\, \omega^r\left(\frac{1}{k}; f\right)_{L_p} k^{-\frac{r}{q}}.$$

Summarizing the last inequalities with respect to k, we have

$$\sum_{k=3}^{\infty} \gamma_k\, \rho_k^r(f) \le \varkappa_{p,r} \sum_{k=2}^{\infty} \gamma_k\, \omega^r\Big(\frac{1}{k}; f\Big)_{L_p} k^{-\frac{r(p-1)}{p}}.$$

Moreover, if we take into account the fact that

$$\rho_k(f) = \bigg|\frac{1}{4\pi}\int\limits_T \Big(f(x) - f\Big(x+\frac{\pi}{k}\Big)\Big)e^{-ikx}dx\bigg| \le \varkappa_q\, \omega\Big(\frac{1}{k}; f\Big)_{L_p},$$

we will obtain the inequality which is being proved. □

Corollary 1 for $r = 1$, $\gamma_n = 1$, $p = 2$, leads to the O. Szasz theorem on the absolute convergence of Fourier series [3].

Corollary 2. *Let $f \in L_2(T)$ and the series*

$$\sum_{k=1}^{\infty} C_{2^k}(f;2)$$

converge. Then $f \in A$, i.e.,

$$\sum_{k=1}^{\infty} \rho_k(f) < +\infty.$$

Corollary 2, obtained earlier by V. Chelidze [5], follows from Theorem 1, when $r = 1$, $\gamma_n = 1$, $p = 2$.

Corollary 3. *Let $f \in C(T) \cap V_s(T)$,* [1] *$s \ge 1$, $\{\gamma_n\} \in A_{\frac{2}{2-r}}$, $r \in (0,2)$, then*

$$\sum_{k=2}^{\infty} \gamma_k\, \rho_k^r(f) \le \varkappa_{r,s} \sum_{k=2}^{\infty} \gamma_k\, k^{-r} \omega^{\frac{(2-s)r}{2}}\Big(\frac{1}{k}; f\Big).$$

Proof. It can be easily verified that for any integers k the equality

$$\int\limits_T \Big|f\Big(x+\frac{k\pi}{2^n}\Big) - f\Big(x+\frac{(k-1)\pi}{2^n}\Big)\Big|^p dx = \int\limits_T \Big|f\Big(x+\frac{\pi}{2^{n+1}}\Big) - f\Big(x-\frac{\pi}{2^{n+1}}\Big)\Big|^p dx$$

is fulfilled. The latter results in

$$C_{2^n}(f;p) = \bigg(\int\limits_T \sum_{k=1}^{2^n} \Big|f\Big(x+\frac{k\pi}{2^n}\Big) - f\Big(x+\frac{(k-1)\pi}{2^n}\Big)\Big|^p dx\bigg)^{\frac{1}{p}},$$

whence

$$C_{2^n}(f;2) \le \varkappa \sqrt{\omega^{2-s}\Big(\frac{\pi}{2^n}; f\Big)\, V_s(f)}\,,$$

[1] $V_s(T)$ is the class of 2π periodic functions with bounded S-variation

for $p = 2$. Therefore, using Theorem 1, we conclude that

$$\sum_{n=2}^{\infty} \gamma_n \rho_n^r(f) \leq \varkappa_{r,s} \sum_{n=1}^{\infty} 2^{-nr} C_{2^{n+1}}^r(f;2) \sum_{k=2^{n-1}+1}^{2^n} \gamma_n \leq$$

$$\leq \varkappa_{r,s} \sum_{n=1}^{\infty} 2^{-nr} \omega^{\frac{(2-s)r}{2}} \Big(\frac{1}{2^n}; f\Big) \sum_{k=2^{n-1}+1}^{2^n} \gamma_k \leq$$

$$\leq \varkappa_{r,s} \sum_{n=1}^{\infty} \sum_{k=2^{n-1}+1}^{2^n} k^{-r} \omega^{\frac{(2-s)r}{2}} \Big(\frac{1}{n}; f\Big) \gamma_k =$$

$$= \varkappa_{r,s} \sum_{n=2}^{\infty} \gamma_n \, n^{-r} \, \omega^{\frac{(2-s)r}{2}} \Big(\frac{1}{n}; f\Big).$$

Thus Corollary 3 is proved. □

Corollary 3 has been obtained by A. Zygmund for $\gamma_k = 1$, $r = 1$, $s = 1$ (see [6]).

Corollary 4. *If* $f \in C(T)$, $r \in (0,2)$, $\{\gamma_n\} \in A_{\frac{2}{2-r}}$, *then*

$$\sum_{k=1}^{\infty} \gamma_k \rho_k^r(f) \leq \varkappa_r \sum_{k=1}^{\infty} \gamma_k \, k^{-\frac{r}{2}} \, \omega^r \Big(\frac{1}{k}; f\Big). \tag{4}$$

Proof. Corollary 4 is obtained from Corollary 1 for $p = 2$ by means of the inequality

$$\omega\Big(\frac{1}{n}; f\Big)_{L_2} \leq \varkappa \, \omega\Big(\frac{1}{n}; f\Big).$$

From Corollary 4 we arrive at O. Szasz theorem which is formulated as follows: If $f \in \operatorname{Lip}\beta$, $0 < \beta \leq 1$, then for any number $r > \frac{2}{1+2\beta}$ the series

$$\sum_{n=1}^{\infty} \rho_n^r(f)$$

converges.

Indeed, if $f \in \operatorname{Lip}\beta$, then $\omega\big(\frac{1}{n}; f\big) \leq \varkappa \, n^{-\beta}$; by the condition, $\gamma_n = 1$ and $r > \frac{2}{1+2\beta}$, i.e. $\frac{r}{2} + r\beta > 1$, and hence (4) implies

$$\sum_{n=1}^{\infty} \rho_n^r(f) \leq \varkappa_r \sum_{n=1}^{\infty} n^{-\frac{r}{2}} \, n^{-\beta r} = \varkappa_r \sum_{n=1}^{\infty} n^{-\left(\frac{r}{2}+\beta r\right)} < +\infty. \qquad \square$$

Theorem 2. *Let* $f \in L_p(T)$, $p \in (1,2]$, $r \in \big(0, \frac{p}{p-1}\big)$, $\{\gamma_k\} \in A_{\frac{p}{p-rp+r}}$, *then*

$$\sum_{n=1}^{\infty} \gamma_n \rho_n^r(f) \leq \varkappa_{p,r} \sum_{n=1}^{\infty} \gamma_n \, n^{-\frac{r(p-1)}{p}} \, E_n^r(f)_{L_p},$$

where $E_n(f)_{L_p}$ is the best approximation of the function f by trigonometric polynomials whose order is not more than n in the norm $L_p(T)$.

Proof. Applying the Hausdorff-Young inequality to the function $f(x)-S_{2^{n-1}}(f;x)$, we obtain

$$\sum_{k=2^n+1}^{\infty}\rho_k^q(f)\le \varkappa_p\biggl(\int_T \bigl|f(x)-S_{2^n}(f,x)\bigr|^p dx\biggr)^{\frac{1}{p-1}},$$

where $q=\frac{p}{p-1}$.

By the M. Riesz theorem ([8], p. 424), we have

$$\bigl\|f-S_{2^n}(f)\bigr\|_{L_p}\le \varkappa\, E_{2^n}(f)_{L_p},$$

whence

$$\sum_{k=2^n+1}^{2^{n+1}}\rho_k^q(f)\le \sum_{k=2^n+1}^{\infty}\rho_n^q(f)\le \varkappa_p\bigl(E_{2^n}(f)\bigr)_{L_p}^{\frac{p}{p-1}}.$$

Using Hölder's inequality and the fact that $\{\gamma_k\}\in A_{\frac{p}{p-rq+r}}$, we obtain

$$\sum_{k=2^n+1}^{2^{n+1}}\gamma_k\,\rho_k^r(f)\le\biggl(\sum_{k=2^n+1}^{2^{n+1}}\rho_k^q(f)\biggr)^{\frac{r}{q}}\biggl(\sum_{k=2^n+1}^{2^{n+1}}\gamma_k^{\frac{p}{p-rp+r}}\biggr)^{\frac{p-rp+r}{p}}\le$$

$$\le \varkappa_{p,r}\,E_{2^n}^r(f)_{L_p}\cdot 2^{-\frac{nr(p-1)}{p}}\sum_{k=2^{n-1}+1}^{2^n}\gamma_k\le$$

$$\le \varkappa_{p,r}\sum_{k=2^{n-1}+1}^{2^n}\gamma_k\,E_k^r(f)_{L_p}\,k^{-\frac{r(p-1)}{p}}.$$

Summarizing the last inequality with respect to n, we find that

$$\sum_{n=3}^{\infty}\gamma_n\,\rho_n^r(f)\le\sum_{n=2}^{\infty}\gamma_k\,n^{-\frac{r(p-1)}{p}}\,E_n^r(f)_{L_p}.$$

Since

$$\rho_k(f)\le\varkappa_p\,E_k(f)_{L_p},$$

we can conclude that Theorem 2 is valid. □

For $\gamma_n=n^{\beta}$, from Theorem 2 follows A. Konyushkov theorem ([7], p. 647).

Corollary 5. *Let $f\in C(T)$, $\{\gamma_k\}\in A_{\frac{2}{2-r}}$, then for any $r\in(0,2)$,*

$$\sum_{n=1}^{\infty}\gamma_n\,\rho_n^r(f)\le\varkappa_{p,r}\sum_{n=1}^{\infty}\gamma_n\,n^{-\frac{r}{2}}\,E_n^r(f). \tag{5}$$

Corollary 5 follows from Theorem 2 for $p=2$ by using the inequality

$$E_n(f)_{L_2} \leq \sqrt{2n}\, E_n(f).$$

For $\gamma_n = 1$ and $r = 1$, Corollary 5 leads to S. Stechkin's theorem [4].

Note that O. Szasz [3] and S. Stechkin's theorems are equivalent. It should be noted that O. Szasz and S. Stechkin's theorems follow from V.Chelidze's theorem (see Corollary 2). All this is shown in V. Chelidze's work [5].

There naturally arises the question whether V. Chelidze's theorem is stronger than O. Szasz and S. Stechkin's theorems, or are they equivalent?

To prove their equivalence, it is sufficient to show that if

$$\sum_{1}^{\infty} C_{2^n}(f;2) < +\infty,$$

then

$$\sum_{n=1}^{\infty} E_{2^n}(f) 2^{\frac{n}{2}} < +\infty. \tag{6}$$

We have the following inequality

$$\big(E^2_{2^{n-1}}(f) - E^2_{2^n}(f)\big)^{\frac{1}{2}} \geq E_{2^{n-1}}(f) - E_{2^n}(f).$$

Therefore for $p=2$, inequality (2) results in

$$E_{2^{n-1}}(f) - E_{2^n}(f) \leq \big(E_{2^{n-1}}(f) - E_{2^n}(f)\big)^{\frac{1}{2}} = \\ = \bigg(\sum_{k=2^{n-1}}^{2^n} \rho_k^2(f)\bigg)^{\frac{1}{2}} \leq 2^{-\frac{n}{2}}\, C_{2^n}(f;2).$$

Consequently,

$$\sum_{n=1}^{\infty} 2^{\frac{n}{2}} \big(E_{2^{n-1}}(f) - E_{2^n}(f)\big) < +\infty.$$

Moreover,

$$\sum_{n=k}^{\infty} 2^{\frac{n}{2}} \big(E_{2^{n-1}}(f) - E_{2^n}(f)\big) \geq 2^{\frac{k}{2}}\, E_{2^{k-1}}(f),$$

whence

$$\lim_{n\to\infty} 2^{\frac{n}{2}}\, E_{2^n}(f) = 0. \tag{7}$$

Using the Abel transformation and taking into account (7), we obtain

$$\sum_{n=1}^{\infty} 2^{\frac{n-1}{2}}\, E_{2^{n-1}}(f) \leq \sum_{n=1}^{\infty} 2^{\frac{n-1}{2}} \big(E_{2^{n-1}}(f) - E_{2^n}(f)\big) < +\infty,$$

i.e. Stechkin's condition (6) is valid.

Consider now the question that the obtained results are final. Towards this end, we formulate in terms of the lemma a particular case of S.N. Bernstein's result ([7], pp. 618–623).

Lemma (S. N. Bernstein). *For any natural k there exist the trigonometric polynomial*

$$T_{2^k}(x) = \sum_{n=2^{k-1}+1}^{2^k} \cos(kx + \alpha_n) \tag{8}$$

and the absolute constant $\varkappa$, such that

$$\left\|T_{2^k}(x)\right\|_C \leq \varkappa\, 2^{\frac{k}{2}}. \tag{9}$$

Introduce the notation

$$\Gamma_k = \sum_{n=2^{k-1}+1}^{2^k} \gamma_n .$$

The following theorem is valid.

Theorem 3. *Let $\{\gamma_k\} \in A_{\frac{2}{2-r}}$, $0 < r \leq 1$ and for some $\beta > r$ the sequence $2^{-\frac{\beta k}{2}}\Gamma_k$ does not decrease and $\{e_n\}$ does not increase, $\lim\limits_{n\to\infty} e_n = 0$ and*

$$\sum_{n=1}^{\infty} \gamma_n\, n^{-\frac{r}{2}}\, e_n^r = +\infty. \tag{10}$$

Then there exists $f \in C(T)$, such that

$$E_n(f) \leq e_n$$

and

$$\sum_{n=1}^{\infty} \gamma_n\, \rho_n^r(f) = +\infty. \tag{11}$$

Proof. Consider the polynomials satisfying the conditions (8) and (9). Let

$$u_k = \varkappa^{-1}\, 2^{-\frac{k}{2}}\, (e_{2^k} - e_{2^{k+1}}) \tag{12}$$

and

$$f(x) = \sum_{n=1}^{\infty} u_k\, T_{2^k}(x). \tag{13}$$

From the condition (9) we have

$$\left|u_k\, T_{2^k}(x)\right| \leq \varkappa^{-1} 2^{-\frac{k}{2}} (e_{2^k} - e_{2^{k+1}}) \varkappa 2^{\frac{k}{2}} = e_{2^k} - e_{2^{k+1}}.$$

Consequently,

$$\sum_{k=1}^{\infty} u_k\, T_{2^k}(x)$$

converges uniformly, and $f \in C(T)$.

It is not difficult to show that for particular sums of the series of f the equality

$$S_{2^k}(f;x) = \sum_{\nu=1}^{k} u_\nu \, T_{2^\nu}(x)$$

is valid.

Thus we obtain

$$E_{2^k}(f) \le \max \left| f(x) - S_{2^k}(f) \right| \le \sum_{\nu=k+1}^{\infty} u_\nu \left| T_{2^\nu}(x) \right| \le$$

$$\le \sum_{\nu=k+1}^{\infty} (e_{2^\nu} - e_{2^{\nu+1}}) = e_{2^{k+1}}. \tag{14}$$

For the given n we choose k, such that $2^k \le n < 2^{k+1}$. Since the sequences $\{e_n\}$ and $\{E_n(f)\}$ are nonincreasing, from (14) it follows that

$$E_n(f) \le E_{2^k}(f) \le e_{2^{k+1}} \le e_n.$$

Let us show that (12) is valid.

It can be easily verified that if $2^{k-1} + 1 \le n < 2^k$,

$$\rho_n(f) = u_k = \varkappa^{-1} \, 2^{-\frac{k}{2}} (e_{2^k} - e_{2^{k+1}}).$$

Consequently,

$$\sum_{n=2}^{\infty} \gamma_n \, \rho_n^r(f) = \varkappa^{-r} \sum_{k=1}^{\infty} 2^{-\frac{kr}{2}} \big(e_{2^k} - e_{2^{k+1}}\big)^r \sum_{n=2^{k-1}+1}^{2^k} \gamma_n =$$

$$= \varkappa^{-r} \sum_{k=1}^{\infty} 2^{-\frac{kr}{2}} \, \Gamma_k \big(e_{2^k} - e_{2^{k+1}}\big)^r. \tag{15}$$

Using the inequality

$$(a+b)^r \le a^r + b^r \quad (a, b > 0, \quad 0 < r \le 1),$$

(15) yields

$$\sum_{n=2}^{\infty} \gamma_n \, \rho_n^r(f) \ge \varkappa^{-r} \sum_{k=1}^{\infty} 2^{-\frac{kr}{2}} \, \Gamma_k \big(e_{2^k}^r - e_{2^{k+1}}^r\big).$$

For the divergence of the series $\sum\limits_{n=1}^{\infty} \gamma_n \rho_n^r(f)$ it is sufficient to prove that

$$\sum_{k=1}^{\infty} \Gamma_k \, 2^{-\frac{kr}{2}} \big(e_{2^k}^r - e_{2^{k+1}}^r\big) = +\infty.$$

Assume the contrary. Let

$$\sum_{k=1}^{\infty}\Gamma_k\,2^{-\frac{kr}{2}}\big(e^r_{2^k}-e^r_{2^{k+1}}\big)<+\infty. \tag{16}$$

By the condition of the theorem, the sequence $\Gamma_k e^{-\frac{k\beta}{2}}$ does not decrease for some $\beta>r$, hence $\big\{\Gamma_k\,e^{-\frac{kr}{2}}\big\}$ does not likewise decrease.

Estimate the sum

$$\sum_{k=n}^{\infty}\Gamma_k\,2^{-\frac{kr}{2}}\big(e^r_{2^k}-e^r_{2^{k+1}}\big)\ge\Gamma_n\,2^{-\frac{nr}{2}}\sum_{k=n}^{\infty}\big(e^r_{2^k}-e^r_{2^{k+1}}\big)=$$
$$=\Gamma_n\,2^{-\frac{nr}{2}}\,e^r_{2^n}.$$

Taking into account (16), from the last inequality it follows that

$$\lim_{n\to\infty}\Gamma_n\,2^{-\frac{nr}{2}}\,e^r_{2^n}=0. \tag{17}$$

Using the Abelian transformation, we obtain

$$\sum_{k=1}^{N}2^{-\frac{kr}{2}}\,\Gamma_k\,e^r_{2^k}=\sum_{k=1}^{N-1}\big(e^r_{2^k}-e^r_{2^{k+1}}\big)\sum_{i=1}^{k}\Gamma_i\,e^{-\frac{ir}{2}}+$$
$$+\sum_{i=1}^{N}\Gamma_i\,2^{-\frac{ir}{2}}\,e^r_{2^N}. \tag{18}$$

$\big\{\Gamma_k\,e^{-\frac{k\beta}{2}}\big\}$ does not decrease for $\beta>r$, therefore

$$\sum_{i=1}^{k}\Gamma_i\,2^{-\frac{ir}{2}}=\sum_{i=1}^{k}\Gamma_i\,2^{-\frac{ir}{2}}\cdot e^{-\frac{i\beta}{2}}e^{\frac{i\beta}{2}}\le\Gamma_k\,2^{-\frac{k\beta}{2}}\sum_{i=1}^{k}2^{\frac{i(\beta-r)}{2}}\le$$
$$\le\varkappa_{r,\beta}\,\Gamma_k\,e^{-\frac{k\beta}{2}}\cdot2^{\frac{k(\beta-r)}{2}}=\varkappa_{r,\beta}\,\Gamma_k\,e^{-\frac{kr}{2}}.$$

Using now the last inequality, from (18) we obtain

$$\sum_{k=1}^{N}2^{-\frac{kr}{2}}\,\Gamma_k\,e^r_{2^k}\le\varkappa_{r,\beta}\sum_{k=1}^{N-1}\big(e^r_{2^k}-e^r_{2^{k+1}}\big)\Gamma_k\,2^{-\frac{kr}{2}}+$$
$$+\varkappa_{N,r}\,\Gamma_N\,2^{-\frac{Nr}{2}}\,e^r_{2^N}.$$

Passing in the latter inequality to the limit, as $N\to\infty$, and using the conditions (16) and (17), we obtain

$$\sum_{k=1}^{\infty}2^{-\frac{kr}{2}}\,\Gamma_k\,e^r_{2^k}<+\infty. \tag{19}$$

By the condition, $\{\gamma_n\} \in A_{\frac{2}{2-r}}$, $0 < r \leq 1$. it is clear that $\frac{2}{2-r} \geq 1$, hence $A_{\frac{2}{2-r}} \subseteq A_1$ and $\{\gamma_n\} \in A_1$, i.e.

$$\sum_{n=2^{k-1}+1}^{2^k} \gamma_n \leq \sum_{n=2^{k-2}+1}^{2^{k-1}} \gamma_n.$$

Consequently, $\Gamma_k \leq \varkappa \Gamma_{k-1}$.

It can be easily verified that

$$\sum_{n=2}^{\infty} \gamma_n \, n^{-\frac{r}{2}} \, e_n^r = \sum_{k=1}^{\infty} \sum_{n=2^{k-1}+1}^{2^k} \gamma_n \, n^{-\frac{r}{2}} \, e_n^r \leq$$
$$\leq \sum_{k=1}^{\infty} 2^{\frac{(k-1)r}{2}} \, e_{2^{k-1}}^r \, \Gamma_k \leq \varkappa \sum_{k=1}^{\infty} 2^{\frac{(k-1)r}{2}} \, e_{2^{k-1}}^r \, \Gamma_{k-1}.$$

whence, by the condition (19),

$$\sum_{n=1}^{\infty} \gamma_n \, n^{-\frac{r}{2}} \, e_n^r < +\infty,$$

which contradicts the condition (10). Hence

$$\sum_{n=1}^{\infty} \gamma_n \, \rho_n^r(f) = +\infty.$$

It follows from Theorem 3 that Corollary 5 is unimprovable.

We can easily see that when $\gamma_n = 1$, $n = 1, 2, \dots$, $\{\gamma_n\} \in A_{\frac{2}{2-r}}$ and $\{\Gamma_k 2^{-\frac{\beta k}{2}}\}$, $\beta = r + \varepsilon$ does not decrease for any $\varepsilon \in (0, 2-r)$, $0 < r \leq 1$.

Therefore when $r = 1$, from Theorem 3 follows S. N. Bernstein's theorem ([7], p. 618). □

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