Proceedings of A. Razmadze
Mathematical Institute
Vol. **141** (2006), 15–27

ON THE CAUCHY PROBLEM FOR ONE QUASI-LINEAR HYPERBOLIC EQUATION WITH ORDER AND TYPE DEGENERACY

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Abstract. On a plane, we consider the Cauchy problem for one quasi-linear second order hyperbolic equation whose order and type degenerate on some set of points. The conditions for which the Cauchy problem has no solution, are established. In the case if these conditions are not fulfilled, the solution of the problem is constructed explicitly. The structure of the domain of definition of the solution is described.

რეზიუმე. განხილულია კოშის ამოცანა სიბრტყეზე მეორე რიგის კვაზიწრფივი ჰიპერბოლური განტოლებისათვის, რომელიც წერტილთა გარკვეულ სიმრავლეზე განიცდის ტიპის და რიგის გადაგვარებას. დადგენილია პირობები, რომლისთვისაც ამოცანას არ აქვს ამონახსნი. ასევე ამ პირობების შეუსრულობის შემთხვევაში ცხადი სახით აგებულია ამოცანის ამონახსნი და აღწერილია მისი განსაზღვრის არის სტრუქტურა.

In the present paper, on a plane of variables x, y we consider the initial Cauchy problem for a quasi-linear nonstrictly hyperbolic equation of second order

$$y\big\{(q^2-q)u_{xx}-(2pq+q-p-1)u_{xy}+(p^2+p)u_{yy}\big\}=F(x,y,u,p,q), \quad (1)$$

where $p\equiv u_x$ and $q\equiv u_y$ are the Monge designations, and F is the given for all finite values of its arguments, twice continuously differentiable function.

The roots

$$\lambda_1=-\frac{p+1}{q} \quad \text{and} \quad \lambda_2=\frac{p}{1-q} \quad (2)$$

of the characteristic equation, corresponding to (1), under the condition

$$p-q+1=0 \quad (3)$$

may coincide. This indicates that the type of equation (1) depends on its solutions. If the condition (3) is fulfilled everywhere, then it determines a

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2000 *Mathematics Subject Classification.* 35l70, 35l80, 35L75.

Key words and phrases. Hyperbolic, nonlinear, order degeneracy, type degeneracy, general integral, domain of definition.

class of parabolic solutions of equation (1). Otherwise, a solution is hyperbolic. If the condition (3) is violated everywhere, except possibly at isolated points or curves, then we deal with the mixed hyperbolo-parabolic solutions. Therefore equation (1) can be referred to a class of non-strictly hyperbolic equations with an acceptable parabolic degeneracy.

Moreover, on the axis $y = 0$ the order of equation (1) is likewise degenerates. Therefore we can consider equation (1) as a nonlinear variant of the Euler-Darboux equation [1–3].

It is well-known that for the right-hand side

$$F = -p(p+1)(p-q+1)$$

equation (1), taking the form

$$y\big\{(q^2-q)u_{xx}-(2pq+q-p-1)u_{xy}+(p^2+p)u_{yy}\big\} = -p(p+1)(p-q+1) \tag{4}$$

has the integral which is expressed by means of two arbitrary functions $f, g \in C^2(\mathbb{R}^1)$ as follows [4–5]:

$$f(u+x) + g(u-y) - y^2 = 0. \tag{5}$$

Below, the Cauchy problem will be considered for equation (4).

Using general integrals, the Cauchy problem, as well as some other problems including characteristic ones, have been investigated for quasi-linear, non-strictly hyperbolic equations of different types in [4–11].

As the basis in considering the Cauchy problem for equation (4) is assumed to be the representation of its general integral (5). The problem under consideration is somewhat modified, but equivalent to the Cauchy problem, since on the data support we assign a derivative of the first order in an arbitrary tangential direction and not the derivative with respect to the normal [12].

Let γ be the given unclosed smooth arc of the Jordan curve uniquely projected on the coordinate axes. Suppose that it is represented parametrically in terms of strictly monotone functions $\lambda, \mu \in C^2[0,l]$ of the arc length s counting from one of its points. Let $\varphi \in C^2[0,l]$, $\psi \in C^1[0,l]$ be the given functions, too.

The problem is formulated as follows: Find a regular solution $u(x,y)$ of equation (4) together with its domain of definition, satisfying the conditions

$$u\big|_{\gamma} = \varphi(s), \tag{6}$$

$$u_y\big|_{\gamma} = \psi(s). \tag{7}$$

Since equation (4) is nonlinear, the solvability of the above-posed problem depends on the functions φ and ψ. Therefore before we proceed to the investigation of our problem it is necessary to show to what extent the initial perturbations φ and ψ are consistent with equation (4).

First of all, we determine values of the characteristic roots (2) of equation (4) on the support γ of the initial data (6), (7). These roots are expressed in terms of the values of the first order derivatives of an unknown solution. The derivative $u_y[\lambda(s), \mu(s)]$ on the data support γ is defined by means of the condition (7). Another derivative u_x can be determined by differentiating the condition (6) with respect to the argument s,

$$u_x\big[\lambda(s), \mu(s)\big] \cdot \lambda'(s) + u_y\big[\lambda(s), \mu(s)\big] \cdot \mu'(s) = \varphi'(s)$$

and also by substituting the derivative $u_y = \psi(s)$ into the obtained equality:

$$u_x\big[\lambda(s), \mu(s)\big] = \frac{\varphi' - \psi\mu'}{\lambda'} \equiv p(s). \tag{8}$$

From our assumptions regarding the arc γ, it follows that this derivative on the entire data support (6), (7) is bounded.

Substituting the values u_x and u_y into (2), we express λ_1 and λ_2 trough the initial data

$$\lambda_1 = -\frac{\varphi' - \psi\mu' + \lambda'}{\psi\,\lambda'}\,, \quad \lambda_2 = \frac{\varphi' - \psi\mu'}{\lambda'(1-\psi)}\,, \tag{9}$$

which allow one to determine directions of characteristics at every point of the curve γ. Surely, it is not excepted that the characteristic directions at some points may coincide. This occurs, for example, when $\lambda_1|_\gamma = \lambda_2|_\gamma$, what is equivalent to the equality

$$\varphi' + \lambda' - \psi(\lambda' + \mu') = 0. \tag{10}$$

In this case equation (4) on the data support γ degenerates parabolically. As is seen, this degeneracy depends not only on the curve γ, but also on the derivatives $u_x = p(s)$ and $u_y = \psi(s)$ on that curve. This is, actually, an effect of nonlinearity of equation (4).

For the sake of simplicity, we introduce the notation

$$H(s) \equiv \frac{\psi^2 - \psi}{\lambda'}\,p' - \frac{p^2 + p}{\mu'}\,\psi',$$

where $p(s) = \frac{\varphi'(s) - \psi(s)\mu'(s)}{\lambda'(s)}$.

Theorem. *If everywhere on the interval* $[0, l]$ *is fulfilled only one of the conditions*

$$\varphi' = -\lambda', \tag{11}$$

or

$$\varphi' = \mu', \tag{12}$$

then the problem (4), (6), (7) *for* $H \neq \frac{F}{y}\big|_\gamma = -\frac{p(p+1)(p-q+1)}{y}\big|_\gamma$ *has no solution at all, but for*

$$H = \frac{F}{y}\,\Big|_\gamma \tag{13}$$

it has infinitely many solutions.

Proof. Assume first that $\varphi' = -\lambda'$. Then on the basis of the above-obtained values (9) of the roots λ_1 and λ_2 on γ, we can conclude that the characteristic direction defined at all points γ by the root λ_1 coincides with the tangent direction of the curve γ:

$$\lambda_1 = -\frac{\varphi' - \psi\,\mu' + \lambda'}{\psi\,\lambda'} = \frac{\mu'}{\lambda'}\,.$$

As is known [13], the curve whose tangent has at every point the characteristic direction is itself the singular characteristic. Therefore the data support γ turns out to be the characteristic of equation (4) of the family which corresponds to the root λ_1.

Had we restricted ourselves to the analytic solutions $u(x,y)$ and, following Cauchy [7], endeavored to represent a solution of equation (4) in the neighborhood of the arc γ in the form of a Taylor series, we would have to determine values of all its derivatives of arbitrary order at every point of that arc. The first order derivatives are already known and represented by formulas (7) and (8). Hence we have to determine values of the second order derivatives. Differentiating formally (7) and (8), we obtain for them two relations

$$\lambda'(s)\,u_{xx}\big[\lambda(s),\mu(s)\big] + \mu'(s)\,u_{xy}\big[\lambda(s),\mu(s)\big] = p'(s)$$

and

$$\lambda'(s)\,u_{xy}\big[\lambda(s),\mu(s)\big] + \mu'(s)\,u_{yy}\big[\lambda(s),\mu(s)\big] = \psi'(s)$$

which we supplement with equation (4), taken along the arc γ:

$$A(s)\,u_{xx}\big[\lambda(s),\mu(s)\big] + B(s)\,u_{xy}\big[\lambda(s),\mu(s)\big] + C(s)\,u_{yy}\big[\lambda(s),\mu(s)\big] = \overline{F}(s),$$

where we introduced the notation

$$\begin{aligned}
&A(s) = \psi^2(s) - \psi(s),\\
&B(s) = -2p(s)\,\psi(s) - \psi(s) + p(s) + 1,\\
&C(s) = p^2(s) + p(s),\\
&\overline{F}(s) = \frac{F}{y}\Big|_{\gamma} = -\frac{1}{\mu(s)}\,p(s)\big[p(s)+1\big]\,\big[p(s) - \psi(s) + 1\big].
\end{aligned}$$

The above three relations we consider as a system of linear algebraic equations with respect to the second order derivatives.

By the condition (11), the determinant of the system,

$$\psi(\psi-1){\mu'}^2 + p^2(p+1){\lambda'}^2 + (2p\psi + \psi - p - 1)\lambda'\mu' = (\varphi' + \lambda')(\varphi' - \mu'),$$

vanishes everywhere on γ.

Consequently, for the three-component vectors $\overrightarrow{m_1} = (\lambda'(s), \mu'(s), 0)$, $\overrightarrow{m_2} = (0, \lambda'(s), \mu'(s))$, $\overrightarrow{m_3} = (A(s), B(s), C(s))$ there exist the functions $\alpha(s)$, $\beta(s)$ such that the equality

$$\alpha \, \overrightarrow{m_1} + \beta \, \overrightarrow{m_2} = \overrightarrow{m_3}$$

holds. The latter allows us to determine

$$\alpha = \frac{A}{\lambda'} \quad \text{and} \quad \beta = -\frac{C}{\mu'}\,.$$

If between the right-hand sides of the system there exists the dependence

$$\alpha p' + \beta \psi' = \overline{F},$$

or substituting the values α and β,

$$\frac{\psi^2 - \psi}{\lambda'} - \frac{p^2 + p}{\mu'}\,\psi' = \overline{F}(s),$$

then the problem will have infinitely many solutions. Otherwise, the problem is unsolvable, which was to be demonstrated. The case $\varphi' = \mu'$ is treated analogously.

The case when the conditions (11) and (12) are fulfilled simultaneously, is of interest. Under such an assumption, $\lambda' + \mu' = 0$. This implies that a slope of the curve γ is everywhere equal to -1. Therefore the arc γ is a segment of the straight line $y + x = c$ with an arbitrary constant c.

On the other hand, taking into account the conditions (11) and (12) in (9), we can conclude that

$$\lambda_1(s) = \lambda_2(s) = -1.$$

Thus we arrive at the conclusion that in the case under consideration the arc γ, being the segment of the straight line $x + y = c$, belongs to both characteristic families. In other words, equation (4) on the given segment has characteristic parabolic degeneracy [14].

It should be noted that this degeneracy is caused by the functions φ, λ, μ and by their derivatives, and hence does not depend on the values of the derivative $u_y = \psi$ on γ.

As for the existence of a solution when the conditions (11) and (12) are satisfied simultaneously, equality (13) in this case is fulfilled identically everywhere on the segment γ, and the problem (4), (6), (7), according to the theorem, has infinitely many solutions.

Consider now the case in which none of the conditions (11), (12) is fulfilled. Then the following theorem is valid.

Theorem. *If* $\varphi' \neq -\lambda'$, $\varphi' \neq \mu'$, *then the problem* (4), (6), (7) *is solvable, and the integral is defined in the domain bounded by the curves*

$$\mu^2(a)+\int\limits_a^{h(\varphi(a)+\lambda(a)-(x+y))} \frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\lambda'+\mu')}\,ds=y^2,\ a=0,\ a=l,$$

$$\mu^2(a)+\int\limits_a^{k(\varphi(a)-\mu(a)+x+y)} \frac{2\mu(\varphi'-\mu'\psi)(\varphi'+\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds=y^2,\quad a=0,\quad a=l.$$

To prove the theorem, we assign the general integral (5) to the problem (4), (6), (7) and extend the D'Alembert method to our case ([15], [16]).

Let the general integral (5) be subjected to the initial condition (6). Taking into account that all three values

$$x=\lambda(s),\quad y=\mu(s),\quad u=\varphi(s) \tag{14}$$

along the data support are known, for determination of arbitrary functions f and g we obtain the relation

$$f\big[\varphi(s)+\lambda(s)\big]+g\big[\varphi(s)-\mu(s)\big]=\mu^2(s). \tag{15}$$

Further, differentiated with respect to y the general integral

$$f'(u+x)u_y+g'(u-y)(u_y-1)=2y$$

with regard for (7) and (14) yields

$$f'\big[\varphi(s)+\lambda(s)\big]\psi(s)+g'\big[\varphi(s)-\mu(s)\big](\psi(s)-1)=2\mu(s). \tag{16}$$

Thus the initial conditions (6) and (7) together with the general integral (5) result in the relations (15) and (16) which will allow one to determine an exact type of arbitrary functions f and g.

Towards this end, we differentiate (15) with respect to s and the obtained result

$$f'\big[\varphi(s)+\lambda(s)\big](\varphi'(s)+\lambda'(s))+g'\big[\varphi(s)-\mu(s)\big](\varphi'(s)-\mu'(s))=2\mu(s)\mu'(s) \tag{17}$$

we consider together with (16) as the linear algebraic system with respect to the derivatives f' and g'. Let the initial functions and the data support be chosen in such a way that equation (4) does not degenerate parabolically along the support, i.e., according to (10), the determinant of the system (16), (17) is different from zero. In this case, the derivatives f' and g' of arbitrary functions are defined uniquely:

$$f'\big(\varphi(s)+\lambda(s)\big)=\frac{2\mu(\varphi'-\mu'\psi)}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,, \tag{18}$$

$$g'\big(\varphi(s)-\mu(s)\big)=\frac{2\mu(\mu'\psi-\varphi'-\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,. \tag{19}$$

The arbitrary functions f and g themselves are defined by integrating (18) and (19) in the limits (s_0, s), where s_0 is an arbitrary value of the argument s from the interval $[0, l]$:

$$f\big(\varphi(s)+\lambda(s)\big)=\int\limits_{s_0}^{s}\frac{2\mu(\varphi'-\mu'\psi)(\varphi'+\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds+f\big[\varphi(s_0)+\lambda(s_0)\big], \tag{20}$$

$$g\big(\varphi(s)-\mu(s)\big)=\int\limits_{s_0}^{s}\frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds+g\big[\varphi(s_0)-\mu(s_0)\big]. \tag{21}$$

To determine once and for all the functions f and g, we have to express the value s from the functional equalities

$$\varphi(s)+\lambda(s)=z, \tag{22}$$

$$\varphi(s)-\mu(s)=\zeta \tag{23}$$

in terms of the functions z and ζ, respectively. By our assumptions, the derivatives of the combinations (22) and (23) with respect to s are different from zero everywhere on the support γ. If at least one of them vanishes, i.e. if (11) or (12) is fulfilled, the question on the solvability of the problem has already been considered above.

Assume now that there exist unique solutions of equations (22) and (23):

$$s=k(z),\quad z\in\big[\varphi(0)+\lambda(0),\varphi(l)+\lambda(l)\big],$$
$$s=h(\zeta),\quad \zeta\in\big[\varphi(0)+\lambda(0),\varphi(l)+\lambda(l)\big],$$

which satisfy respectively the normalization conditions

$$k\big[\varphi(0)+\lambda(0)\big]=0,\quad h\big[\varphi(0)-\mu(0)\big]=0.$$

Our assumptions on the unique solvability of the functional equations (22), (23) does not contradict the conditions of the theorem concerning an implicit function.

Thus we have

$$f(z)=\int\limits_{s_0}^{k(z)}\frac{2\mu(\varphi'-\mu'\psi)(\varphi'+\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds+f\big[\varphi(s_0)+\lambda(s_0)\big], \tag{24}$$
$$z\in\big[\varphi(0)+\lambda(0),\varphi(l)+\lambda(l)\big],$$

$$g(\zeta)=\int\limits_{s_0}^{h(\zeta)}\frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds+g\big[\varphi(s_0)-\mu(s_0)\big], \tag{25}$$
$$z\in\big[\varphi(0)-\mu(0),\varphi(l)-\mu(l)\big].$$

Next, we substitute the obtained functions f and g expressed by formulas (24) and (25) in the general integral (5) and construct the integral of the problem (4), (6), (7):

$$\int_{s_0}^{k(u+x)} \frac{2\mu(\varphi'-\mu'\psi)(\varphi'+\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds + \int_{s_0}^{h(u-y)} \frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds + \\ + f\big[\varphi(s_0)+\lambda(s_0)\big] + g\big[\varphi(s_0)-\mu(s_0)\big] = y^2. \tag{26}$$

But taking into account that the condition (6) is fulfilled everywhere on γ, we have

$$f\big[u(\lambda(s'),\mu(s'))+\lambda(s')\big] + g\big[u(\lambda(s'),\mu(s'))-\mu(s')\big] = \mu^2(s'),$$

no matter how the value s' from the interval $[0,l]$ is, including for $s'=s_0$. Thus from (26) we obtain the final form of the integral of the problem under consideration:

$$\int_{s_0}^{k(u+x)} \frac{2\mu(\varphi'-\mu'\psi)(\varphi'+\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds + \int_{s_0}^{h(u-y)} \frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds + \\ + \mu^2(s_0) = y^2. \tag{27}$$

The occurrence of an arbitrary parameter s_0 in the integral (27) of the problem (4), (6), (7) shows as if there exist infinitely many such integrals. But such a dependence of the integral (27) on the constant parameter s_0 is formal.

Indeed, if we divide one of the integral summands, say the second one, into two integrals as it follows

$$\int_{s_0}^{h(u-y)} \frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds =$$

$$= \int_{s_0}^{k(u+x)} \frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds + \int_{k(u+x)}^{h(u-y)} \frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds$$

and then combine two integral terms with common limits $[s_0, k(u+x)]$ of integration, we will find that

$$\int_{k(u+x)}^{h(u-y)} \frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds +$$

$$+\int\limits_{s_0}^{k(u+x)}\left[\frac{2\mu(\varphi'-\mu'\psi)(\varphi'+\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}+\frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\right]ds.$$

Simple calculations show that the second integral summand in the latter relation is equal to $-\mu^2(s_0)+\mu^2[k(u+x)]$. Consequently, the integral (27) can finally be rewritten as

$$\mu^2[k(u+x)]+\int\limits_{k(u+x)}^{h(u-y)}\frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds=y^2. \qquad (28)$$

Thus we have proved that the integral (27) of the problem under consideration is equivalent to the integral (28) and does not depend on an arbitrary parameter s_0. Hence the problem (4), (6), (7) has the unique integral which is constructed explicitly by formula (28).

The integral (28) can now be rewritten equivalently in the form

$$\mu^2[h(u-y)]+\int\limits_{k(u-y)}^{h(u+x)}\frac{2\mu(\varphi'-\mu'\psi)(\varphi'+\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds=y^2. \qquad (29)$$

if in (27) we divide the first integral summand and not the second one. Summing the representations (28) and (29), we can get the integral of the problem in a symmetric form:

$$\mu^2[k(u+x)]+\mu^2[h(u-y)]+$$
$$+\int\limits_{k(u+x)}^{h(u-y)}2\mu\frac{2\varphi'\mu'\psi-2\varphi'^2-2\varphi'\lambda'+\mu[\varphi'+\lambda'-\psi(\lambda'-\mu')]}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds=2y^2.$$

In such a representation the characteristic invariants $u+x$ and $u-y$ gain no advantage, they are equivalent.

In what follows, the use will be made of the representation of the integral of the problem (4), (6), (7) which is more convenient in the given special case.

To find a domain of definition of a solution we have to describe the structure of both characteristic families. Relying on the integral (28) or (29), we can get an explicit representation of characteristics coming out of the points $(\lambda(s),\mu(s))$, $s\in[0,l]$ of the support of initial data γ.

We fix a value a of the parameter s from the interval $[0,l]$. This value makes it possible to find a special point $(\lambda(a),\mu(a))\in\gamma$ through which go the characteristics of both families. Along the characteristic corresponding to the root λ_1 we estimate the value of the invariant $u+x$. This value along the entire characteristic is equal to $\varphi(a)+\lambda(a)$.

Moreover, on the same characteristic we can find the value of the invariant $u-y$. Indeed,

$$u(x,y)-y=u(x,y)+x-(x+y)=\varphi(a)+\lambda(a)-(x+y).$$

Taking into account the values of both invariants along the unknown characteristic in the representation (28) of the general integral, we obtain

$$\mu^2(a)+\int\limits_a^{h(\varphi(a)+\lambda(a)-(x+y))}\frac{2\mu(\mu'\psi-\varphi'-\lambda')(\varphi'-\mu')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds=y^2. \tag{30}$$

Thus we have managed to construct the equation which describes the family characteristic of the root λ_1 coming out of the point of the support γ for the parameter a.

The value of the parameter $s=a$ is taken arbitrarily from the interval $[0,l]$. Assigning all values of the interval to the above-mentioned parameter, by formula (30) we can find all family characteristics of the root λ_1. As it turns out, this family of curves is one-parametric.

The characteristic family corresponding to the root λ_2

$$\mu^2(a)+\int\limits_a^{k(\varphi(a)-\mu(a)+x+y)}\frac{2\mu(\varphi'-\mu'\psi)(\varphi'+\lambda')}{\varphi'+\lambda'-\psi(\mu'+\lambda')}\,ds=y^2 \tag{31}$$

is constructed analogously on the basis of the representation (29). The only difference is that along these characteristics the invariants are equal to

$$u(x,y)-y=\varphi(a)-\mu(a),$$
$$u(x,y)+x=\varphi(a)-\mu(a)+x+y.$$

Naturally, (31) is likewise one-parametric family of curves with values of the parameter from the same interval $[0,l]$.

In order for the domain of definition of a solution of the problem (4), (6), (7) to have no singularities, we have to find some conditions for both families (30) and (31). In particular, the curves belonging to different families, but coming out of the common point for $s=a$, should not intersect at any other point, i.e., the system of equations (30), (31) must possess a unique solution $x=\lambda(a)$, $y=\mu(a)$.

Moreover, the curves of one and the same family must have no singular points. To express this condition by a formula, say for the family (30), we consider the expression

$$\frac{dx}{dy}=\frac{(\mu-y)(\varphi'+\lambda'-\psi\mu')-y\psi\lambda'}{\mu(\varphi'+\lambda'-\psi\mu')}\,, \tag{32}$$

which is obtained from (30) by means of differentiation and simple transformation. As is known, the family (30) has no singular points if both the numerator and denominator in (32) do not vanish simultaneously [17]. Consequently, assuming that there exists $\varepsilon > 0$ for which

$$\frac{(\mu - y)(\varphi' + \lambda' - \psi\mu') - y\psi\lambda'}{\mu(\varphi' + \lambda' - \psi\mu')} > \varepsilon, \tag{33}$$

the existence of singular points of the family (30) is excluded.

To get explicit representations of the characteristic families (30) and (31), we have to determine the variable x by y or vice versa. However, on the plane of new variables

$$\xi = x + y, \quad \eta = y$$

these representations will be explicit, and on the basis of these representations we can show all structural properties of the above-mentioned families.

Thus we consider two one-parametric families with the general parameter $a \in [0, l]$.

$$\eta^2 - \mu^2(a) = \int\limits_a^{h(\varphi(a)+\lambda(a)-\xi)} \frac{2\mu(\mu'\psi - \varphi' - \lambda')(\varphi' - \mu')}{\varphi' + \lambda' - \psi(\mu' + \lambda')}\, ds, \tag{34}$$

$$\eta^2 - \mu^2(a) = \int\limits_a^{k(\varphi(a)-\mu(a)+\xi)} \frac{2\mu(\varphi' - \mu'\psi)(\varphi' + \lambda')}{\varphi' + \lambda' - \psi(\mu' + \lambda')}\, ds. \tag{35}$$

The support of the initial data γ will again be represented parametrically: $\xi = \xi(s) = \lambda(s) + \mu(s)$, $\eta = \eta(s) = \mu(s)$. For any value of the parameter $s = a^*$, through every point of the support $\xi^* = \xi(a^*)$, $\eta^* = \eta(a^*)$ passes one and only one curve of the family (34). If we draw through that point a straight line $\xi = a^*$, parallel to the axis η, this line will intersect the curves of the family (34) coming out of another points of the support. Obviously, a set of these points of intersection contains also a point ξ^*, η^*. Moreover, this set is continuous. This conclusion follows from (34) for $\xi = \xi^*$. The set of points of intersection is defined as follows:

$$\eta^2 = \mu^2(a) + \int\limits_a^{h(\varphi(a)+\lambda(a)-\lambda(a^*)-\mu(a^*))} \frac{2\mu(\mu'\psi - \varphi' - \lambda')(\varphi' - \mu')}{\varphi' + \lambda' - \psi(\mu' + \lambda')}\, ds.$$

As is seen, owing to the continuous dependence of values a, a^*, η on the straight line $\xi = \xi^*$, the set is likewise continuous. Since the value a^* is taken arbitrarily, it becomes obvious that the family (34) covers continuously a part of the plane ξ, η. This part lies between two characteristics (34) taken for $a = 0$ and $a = l$.

As for another asymptotes or singular (discriminant) points, they are excluded by our assumptions (33) regarding the initial perturbations.

Analogous reasoning allows us to conclude that the family (35) covers continuously the curvilinear strip bounded by the curves of that family which come out of the end points of the support.

Our reasoning regarding the above-mentioned strips remains valid on the plane of the initial variables x and y, and the intersection of these two strips is, in fact, the domain of definition of the solution of the problem (4), (6), (7), which was to be demonstrated.

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(Received 11.01.2006)

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