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APPLICATION OF THE BOUNDARY ELEMENT METHOD TO THE SOLUTION OF THE PROBLEM OF DISTRIBUTION OF STRESSES IN AN ELASTIC BODY WITH A CIRCULAR HOLE WHOSE INTERIOR SURFACE CONTAINS RADIAL CRACKS

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Abstract. Using the boundary element method, i.e., the combined method of fictitious loads and discontinuous displacements, we obtained numerical solutions of two-dimensional (plane deformation) boundary value problems on the elastic equilibrium of infinite and finite homogeneous isotropic bodies having circular holes with radial cracks and cuts of finite length.

რეზიუმე. სასაზღვრო ელემენტთა მეთოდით, კერძოდ ფიქტიურ დატვირთათა და წყვეტილ გადაადგილებათა კომბინირებული მეთოდით, მიღებულია წრიულ ხვრელიანი უსასრულო და სასრული ერთგვაროვანი იზოტროპული სხეულების დრეკად წონასწორობაზე ორგანზომილებიანი (ბრტყელი დეფორმაცია) სასაზღვრო ამოცანების რიცხვითი ამონახსნები. აღნიშნული სხეულები შეიცავენ რადიალურ ბზარებს ან სასრული სიგრძის ჭრილებს.

In the projects of underground constructions, in particular, of tunnels, one have to take into account how a number of cracks on tunnel walls and their length affect the distribution of stresses. The mathematical model of that practical problem are the boundary value problems which we consider in the present paper.

In the first part of the work, using the boundary element method [1], we solve the two-dimensional (plane deformation) problem on the elastic equilibrium of an infinite homogeneous isotropic body with a circular hole having radial cracks. Cracks of the same length L pass along $\alpha = 0$, $2\pi/k, \dots, (k-1)2\pi/k$, where $k \geq 1$ is an integer defining their number. The interior surface of the body is free from stresses, and all-round tension is given at infinity. Numerical results are obtained and the graphs of one ($k = 1$), two ($k = 2$) and four ($k = 4$) cracks are presented.

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The second part of the work is devoted to finding in the polar coordinates r, α an elastic equilibrium of a finite body occupying the domain $\Omega = \{r_1 < r < r_2,\ 0 < \alpha < \pi\}$. Note that nonzero stresses are given for $r = r_1$, and zero stresses are given for $r = r_2$; the conditions of symmetry and anti-symmetry [2] are assigned for $\alpha = 0$ and $\alpha = \pi$, respectively. It is not difficult to understand that this problem coincides with the boundary value problem on the elastic equilibrium of a circular ring with a radial cut on which contours the symmetry conditions are fulfilled.

I. In an infinite domain $\Omega = \{r_1 < r < \infty,\ 0 < \alpha < 2\pi\}$ with a circular hole $r = r_1$ containing radial cracks of length L, find a solution of a system of equations of equilibrium with respect to the unknowns D, K, u, v,

$$\begin{aligned} &\frac{\partial D}{\partial r} - \frac{1}{r}\frac{\partial K}{\partial \alpha} = 0, \quad \frac{1}{r}\frac{\partial (ru)}{\partial r} + \frac{1}{r}\frac{\partial v}{\partial \alpha} = \frac{D}{\lambda + 2\mu}, \\ &\frac{1}{r}\frac{\partial D}{\partial \alpha} + \frac{\partial K}{\partial r} = 0, \quad \frac{1}{r}\frac{\partial (rv)}{\partial r} - \frac{1}{r}\frac{\partial u}{\partial \alpha} = \frac{K}{\mu}, \end{aligned} \tag{1}$$

with the following boundary conditions:

$$r = r_1: \quad \sigma_{rr} = 0, \quad \sigma_{r\alpha} = 0, \tag{2}$$

$$r = \infty: \quad \sigma_{rr} = p = const, \quad \sigma_{r\alpha} = 0, \tag{3}$$

$$\alpha = 0, \quad 2\pi: \quad \sigma_{\alpha\alpha} = 0, \quad \sigma_{r\alpha} = 0 \quad (r_1 < r < r_1 + L), \tag{4}$$

where u and v are the components of the displacement vector, $\sigma_{\alpha\alpha}, \sigma_{rr}, \sigma_{r\alpha}$ are the components of the stress vector in the polar coordinates, $\frac{D}{\lambda+2\mu}$ is the divergence of the displacement vector, $\frac{K}{\mu}$ is the rotor of the displacement vector; λ and μ are the known constants.

For the numerical solution of the problem we use both the combined method of fictitious loads [1] and the method of discontinuous displacements [1], [3].

To solve the exterior problem with the given nonzero stresses at infinity, we have to formulate the boundary conditions in additional stresses. The boundary conditions (2),(3) and (4) in additional stresses can be written as follows:

$$r = r_1: \ \sigma_{rr} = -p, \quad \sigma_{r\alpha} = 0, \tag{5}$$

$$\alpha = 0, \quad 2\pi: \ \sigma_{\alpha\alpha} = -p, \quad \sigma_{r\alpha} = 0 \quad (r_1 < r < r_1 + L). \tag{6}$$

If the boundary is divided into N segments (elements) of small length, then we can assume that constant normal $\sigma_{rr}^i = -p$ (or $\sigma_{\alpha\alpha}^i = -p$) and tangential $\sigma_{r\alpha}^i = 0$ stresses act on every i-th element over all its length. Then the boundary conditions (5),(6) take the form

$$r = r_1: \ \sigma_{rr}^i = -p, \quad \sigma_{r\alpha}^i = 0, \tag{7}$$

$$\alpha = 0, \quad 2\pi: \ \sigma_{\alpha\alpha}^i = -p, \quad \sigma_{r\alpha}^i = 0 \quad (r_1 < r < r_1 + L). \tag{8}$$

For every boundary element we select uniformly distributed concentrated forces. For example, for the j-th element we assume that the tangential P_s^j and normal P_n^j stresses are distributed continuously. Thus for the j-th element we have applied fictitious stresses P_s^j and P_n^j and also real stresses σ_s^j and σ_n^j which are caused by the stresses applied to all boundary elements.

Using the solution of Kelvin's problem for the plane deformation [4] and formulas for transformation of coordinates [5] (with regard for the segments orientation), we can calculate real stresses σ_s^j and σ_n^j at the midpoint of each segment, $i = 1, \ldots, N_1$. Thus we obtain the following formulas:

$$\begin{aligned} \sigma_s^i \equiv \sigma_{r\alpha}^i &= \sum_{j=1}^{N_1} \left(A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j \right), \\ \sigma_n^i \equiv \sigma_{rr}^i &= \sum_{j=1}^{N_1} \left(A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j \right), \end{aligned} \tag{9}$$

where $A_{ss}^{ij}, A_{sn}^{ij}, A_{ns}^{ij}, A_{nn}^{ij}$ are the boundary coefficients of influence of stresses for the problem under consideration. For example, coefficient A_{ns}^{ij} supplies with the real normal stress at the center of the i-th segment, caused by the constant unit tangential load ($P_s^j = 1$)), applied to the j-th segment.

Consider now that part of the domain which contains a crack having two contours. The method of fictitious loads does not fit for the solution of crack problems, since the influence of elements on one contour is indistinguishable from that on the other contour. For the solution of problems of such a type the use can be made of the boundary element method which is also called the method of discontinuous displacements [1]. This method is based on the analytic solution of the problem on an infinite plane whose displacements have constant in size discontinuity in the limits of an infinite segment. Analytic solution of that problem has been obtained by S.L. Crouch [3].

If the crack is divided into N_2 segments (elements) of small length, then we can assume that the displacement discontinuity is constant in the limits of every element length. On the basis of the analytic solution obtained by S.L. Crouch [3], one can find influence of a separate elementary displacement discontinuity on the displacements and stresses at an arbitrary point of an infinite rigid body. For example, tangential and normal stresses at the i-th element center can be expressed in terms of the components of the j-th element displacement discontinuity.

If the elementary displacement discontinuity is placed on each of the segments along the crack, then we obtain

$$\left.\begin{aligned} \sigma_s^i \equiv \sigma_{r\alpha}^i &= \sum_{j=N_1+1}^{N} \left(C_{ss}^{ij} D_s^j + C_{sn}^{ij} D_n^j\right), \\ \sigma_n^i \equiv \sigma_{\alpha\alpha}^i &= \sum_{j=N_1+1}^{N} \left(C_{ns}^{ij} D_s^j + C_{nn}^{ij} D_n^j\right), \end{aligned}\right\} \; i = N_1+1, \dots, N = N_1+N_2, \quad (10)$$

where $C_{ss}^{ij}, C_{sn}^{ij}, C_{ns}^{ij}, C_{nn}^{ij}$ are the boundary coefficients of influence for stresses. For example, coefficient C_{ns}^{ij} supplies with the normal stress (σ_n^i) at the center of the i-th segment, which is caused by the constant unit displacement discontinuity directed tangentially along the j-element $(D_s^j = 1)$.

For the boundary conditions to be satisfied on $r = r_1$, we use formulas (9) which are obtained by the method of fictitious loads, while for the cracks we use formulas (10), obtained by the method of discontinuous displacements. Thus we obtain the system of $2N$ linear equations with $2N$ unknowns $(N = N_1 + N_2)$

$$\left\{\begin{aligned} &\sum_{j=1}^{N_1} \left(A_{ss}^{ij} P_s^j + A_{sn}^{ij} P_n^j\right) + \\ &\qquad + \sum_{j=N_1+1}^{N} \left(C_{ss}^{ij} D_s^j + C_{sn}^{ij} D_n^j\right) = 0, \\ &\sum_{j=1}^{N_1} \left(A_{ns}^{ij} P_s^j + A_{nn}^{ij} P_n^j\right) + \\ &\qquad + \sum_{j=N_1+1}^{N} \left(C_{ns}^{ij} D_s^j + C_{nn}^{ij} D_n^j\right) = -p, \end{aligned}\right. \qquad i = 1, \dots, N. \qquad (11)$$

The stresses P_s^j and P_n^j in the above equations are fictitious. They were introduced as auxiliary unknowns and does not make physical sense. Along with the above-said, linear combinations of fictitious loads (9) provide us with real tangential and normal stresses which are aimed to satisfy the boundary conditions, while the unknowns D_s^j and D_n^j represent discontinuous displacements.

After the system (11) is solved by any numerical method (here by the Gaussian method), we can express displacements and stresses at an arbitrary point.

On PC in the MATLAB system we have obtained numerical results and constructed graphs for the boundary problem (1),(2),(3),(4) (or (1),(7),(8)) for $\nu = 0,3$, $E = 710^4 (H/m^2)$, $N_1 = 360$, $N_2 = 20$, $p = 10 (H/m^2)$,

$r_1 = 1(m)$. In particular, we obtained numerical results for three values of crack length $L = 0,1;\ 0,5;\ 0,8\,(m)$.

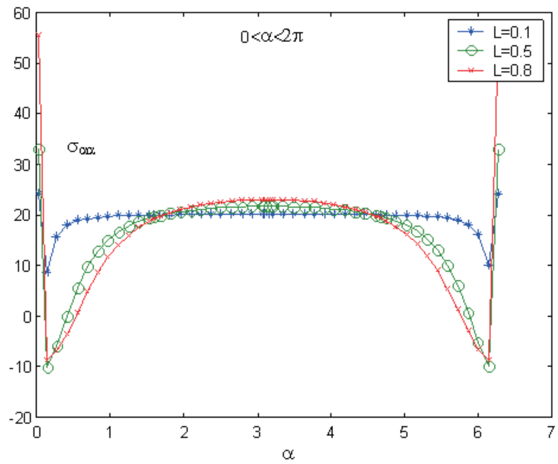

Fig.1.

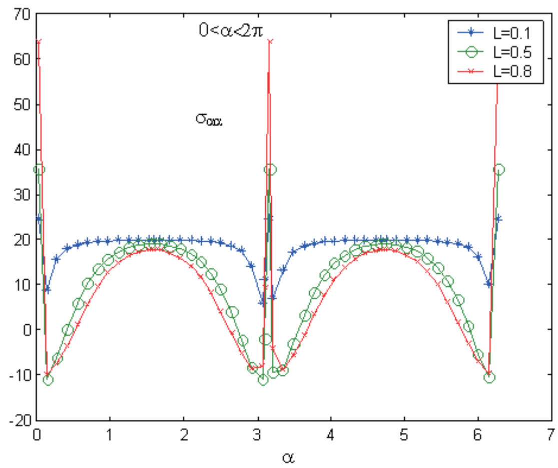

Fig.2.

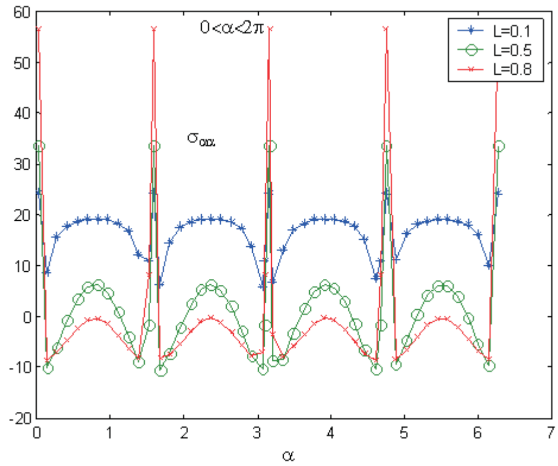

Fig.3.

In Fig. 1 we can see the values of $\sigma_{\alpha\alpha}$ for $r = r_1$ $(0 < \alpha < 2\pi)$ and of one radial crack along $\alpha = 0$ of length $L = 0,1$, $L = 0,5$ and $L = 0,8$. In Fig. 2 are given the values of $\sigma_{\alpha\alpha}$ for $r = r_1$ and of two radial cracks along $\alpha = 0$ and $\sigma = \pi$ of length $L = 0,1$, $L = 0,5$ and $L = 0,8$. Fig. 3 displays the values of $\sigma_{\alpha\alpha}$ for $r = r_1$ and of four radial cracks along $\alpha = 0$, $\alpha = \frac{\pi}{2}$, $\alpha = \pi$ and $\alpha = \frac{3}{2}\pi$ of length $L = 0,1$, $L = 0,5$ and $L = 0,8$.

Relaying on the above-given graphs, we can draw the following conclusions.

As a number of cracks and their length increase, concentration of stresses $\sigma_{\alpha\alpha}$, strange as it may seem, decreases. Knowing this fact, sometimes engineers make themselves the so-called technical cracks to reinforce tunnel constructions.

We will now proceed to solving the boundary value problem on the elastic equilibrium of a circular ring with a cut along the radius. The problem is formulated as follows.

II. In the domain $\Omega_1 = \{r_1 < r < r_2,\ 0 < \alpha < \pi\}$ we seek for a solution of the system of equilibrium equations (1) with the following boundary conditions:

$$\begin{aligned} &a)\ \ r = r_1:\ \ \sigma_{rr} = p\cos\frac{\alpha}{2},\ \ \sigma_{r\alpha} = 0,\\ &b)\ \ r = r_2:\ \ \sigma_{rr} = 0,\quad \sigma_{r\alpha} = 0,\\ &c)\ \ \alpha = 0:\ \ v = 0,\quad \sigma_{r\alpha} = 0,\\ &d)\ \ \alpha = \pi:\ \ u = 0,\quad \sigma_{\alpha\alpha} = 0. \end{aligned} \tag{12}$$

The above-formulated problem is solved by the boundary element method. At the characteristic points of the domain we obtain stresses for $\nu = 0,3$, $E = 710^4(H/m^2)$, $r_1 = 1(m)$, $r_2 = 40(m)$, $p = 10(H/m^2)$. The semicircles $r = r_1$ and $r = r_2$ are divided into 180 equal arcs, whereas the linear parts of the boundary $\alpha = 0$ and $\alpha = \pi$ are divided into 40 equal segments.

It should be noted that the problem considered above can be solved analytically. This solution will be given below.

III. The method of separation of variables yields an exact solution of the boundary value problem (1),(2). The solution is constructed by means of the general representation of a solution through two harmonic functions φ_1 and φ_2, in addition,

$$\sigma_{rr} = -\frac{\mu}{r^2}\left(\frac{\partial^2\varphi}{\partial\alpha^2} + r\frac{\partial\varphi}{\partial r}\right),$$

$$\sigma_{\alpha\alpha} = -\mu\frac{\partial^2\varphi}{\partial r^2},$$

$$\sigma_{r\alpha} = \sigma_{\alpha r} = \mu\left(\frac{1}{r}\frac{\partial^2\varphi}{\partial r\partial\alpha} - \frac{1}{r^2}\frac{\partial\varphi}{\partial\alpha}\right),$$

where $\varphi = 2\varphi_2 + \frac{\lambda+\mu}{\lambda+2\mu}r^2\frac{\partial\varphi_1}{\partial r}$.

The boundary conditions (12c,d) are satisfied if

$$\varphi_1 = \left(Ar^{-1/2} + Br^{1/2}\right)\cos\frac{\alpha}{2},$$

$$\varphi_2 = \left(Cr^{-1/2} + Dr^{1/2}\right)\cos\frac{\alpha}{2},$$

The constants A, B, C, D are defined if are satisfied the boundary conditions (12a,b).

It is important to indicate that the solution found in the domain Ω_1 may be continuously extended through the boundary $\alpha = \pi$. As a result, we obtain the domain $\Omega_2 = \{r_1 < r < r_2,\, 0 < \alpha < 2\pi\}$ with a cut. On the contours of the cut $\alpha = 0$ and $\alpha = 2\pi$ the conditions $v = 0$, $\sigma_{r\alpha} = 0$ are fulfilled.

Thus we have obtained exact and approximate solutions in the annular domain with the cut on the contours of which the symmetry conditions are fulfilled.

Comparison of results obtained by the boundary element method and the exact solution shows that the results are in a good agreement (some graphs are given). This circumstance allows us to conclude that the application of the method of boundary elements proved to be correct for the solution of boundary value problems considered in the present paper.

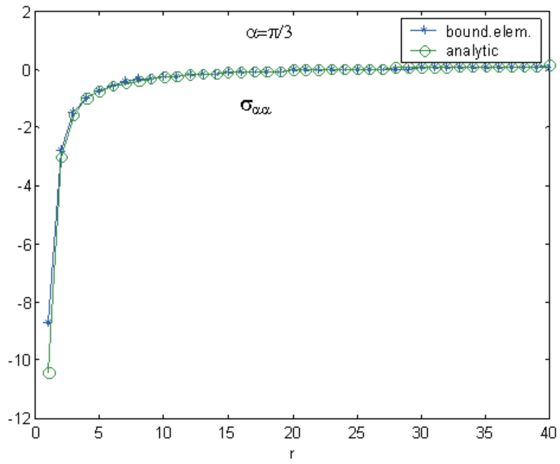

Fig.4.

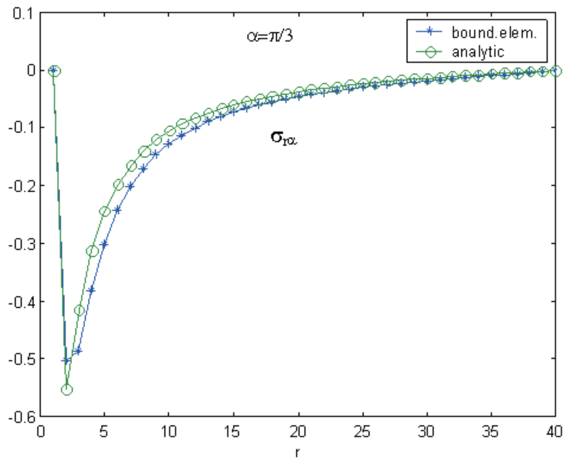

Fig.5.

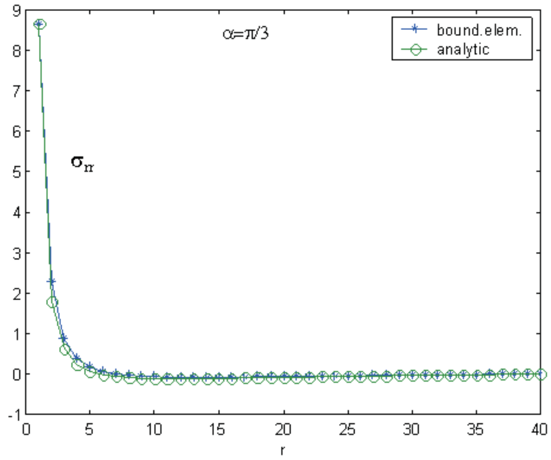

Fig.6.

In Fig. 4 we can see the graphs of exact and approximate solutions of the problem for the stress $\alpha_{\alpha\alpha}$ for $\alpha = \frac{\pi}{3}(r_1 < r < r_2))$; Fig. 5 shows the graphs of exact and approximate solutions of the problem for the stress $\sigma_{r\alpha}$ for $\alpha = \frac{\pi}{3}(r_1 < r < r_2)$; Fig. 6 gives the graphs of exact and approximate solutions of the problem for the stress σ_{rr} for $\alpha = \frac{\pi}{3}(r_1 < r < r_2)$.

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