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ON SOME DIFFERENTIAL PROPERTIES OF AN INDEFINITE INTEGRAL WITH A PARAMETER

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Abstract. In the present paper we investigate an indefinite integral with a parameter from the point of view of the existence of strong and mixed partial derivatives, as well as of Bettazzi derivative.

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The goal of the present paper is to investigate functions represented by the integral with a variable upper bound and a parameter.

1. Strong Partial and Mixed Derivatives

Below, the use will be made of the notion of strong and angular partial derivatives ([1]).

Let the function f of two variables be finite and summable on the rectangle $Q = \{(x, y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}$. Consider the corresponding integral with a variable upper bound and a parameter ([1], p. 119–123)

$$p(x, y) = \int\limits_a^x f(t, y)\, dt, \tag{1}$$

some of which properties will be established below.

Theorem 1. I. *Let the following conditions be fulfilled:*

1) *the function f is continuous on the rectangle $r = [a_1, b_1] \times [c_1, d_1]$, where $a \le a_1 < b_1 \le b$, $c \le c_1 < d_1 \le d$, and let $f(t, c)$ be continuous on the segment $[a_1, b_1]$;*

2) *for every $x \in [a, b]$, the function f is absolutely continuous with respect to y on the segment $[c, d]$;*

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3) *the partial derivative* f'_y *with respect to* y *is summable on* Q, *and at every point* $x_0 \in (a,b)$ *is partial continuous with respect to* x *uniformly with respect to* y *on* $[c,d]$ ([1], *p.* 40).

Then at every point $M=(x,y)\in r^0$ *the equality*

$$p'_{[x]}(M) = f(t,y) \tag{2}$$

holds.

II. *If the function* f'_y *is continuous on the rectangle* $r_2=[a,b]\times[c_1,d_1]$, *then at every point* $M=(x,y)\in r_2^0$ *the equality*

$$p'_{[y]}(M) = \int\limits_a^x f'_y(t,y)\,dt \tag{3}$$

is fulfilled.

Proof. I. We have the relations

$$\frac{p(x+h,y+k)-p(x,y+k)}{h} = \frac{1}{h}\int\limits_x^{x+h} f(t,y+k)\,dt =$$

$$= \frac{1}{h}\int\limits_x^{x+h}\big[f(t,y+k)-f(t,y)\big]\,dt + \frac{1}{h}\int\limits_x^{x+h} f(t,y)\,dt. \tag{4}$$

The function f is uniformly continuous on the rectangle r, therefore for every number $\varepsilon>0$ there exists a positive number $\delta=\delta(\varepsilon)$, such that

$$\Bigg|\int\limits_x^{x+h}\big[f(t,y+k)-f(t,y)\big]\,dt\Bigg| < \varepsilon|h| \tag{5}$$

when $(x,y)\in r^0$ and $\max\{|h|,|k|\}<\delta$. Hence the limit of the first summand in the right-hand side of (4) is equal to zero as $(h,k)\to(0,0)$ and $(x,y)\in r^0$.

Consider now the second summand in the right-hand side of (4). The integral with a variable upper bound (1), containing the parameter y, is finite at all points $(x,y)\in Q$. On the basis of the conditions 2) and the first part of 3), the equality

$$\int\limits_a^x f(t,y)\,dt = \int\limits_a^x\int\limits_c^y f'_\tau(t,\tau)\,dt\,d\tau + \int\limits_a^x f(t,c)\,dt \tag{6}$$

is valid for every point $(x,y)\in Q^0$.

Moreover, from the second part of the conditions 3), we have the equalities

$$\frac{\partial}{\partial x}\int\limits_a^x\int\limits_c^y f'_\tau(t,\tau)\,dt\,d\tau = \int\limits_c^y f'_\tau(x,\tau)\,d\tau = f(x,y)-f(x,c) \tag{7}$$

at every point $(x, y) \in r^0$, since the continuity of the function

$$\varphi(t) = \int_c^y f'_\tau(t, \tau)\, d\tau$$

at every point $t \in (a, b)$ follows from the equality

$$\varphi(t + h) - \varphi(t) = \int_c^y \left[f'_\tau(t + h, \tau) - f'_\tau(t, \tau)\right] d\tau.$$

In addition, for every $x \in (a_1, b_1)$ the equality

$$\Big(\int_a^x f(t, c)\, dt\Big)'(x) = f(x, c) \tag{8}$$

is valid since the function $f(t, c)$ is continuous on $[a_1, b_1]$.

From the equalities (6)–(8) it follows that

$$\Big(\int_a^x f(t, y)\, dt\Big)'_x(x, y) = f(x, y) \tag{9}$$

at every point $(x, y) \in r^0$.

It should be noted that

$$\lim_{h\to 0} \frac{1}{h} \int_x^{x+h} f(t, y)\, dt = \Big(\int_a^x f(t, y)\, dt\Big)'_x(x, y). \tag{10}$$

By the definition of the strong partial derivative ([1], p. 79), from equality (4), by means of (9), (10) and (5), we obtain

$$p'_{[x]}(M) = f(x, y), \quad M = (x, y) \in r^0.$$

Thus we have established equality (2).

II. By the continuity of the partial derivative f'_y on r_2, we obtain equality (3) by using Theorem 6.2.2 from [1] (p. 126) regarding f'_y. Thus the theorem is complete. □

Corollary 1. *Under the assumptions* $1) - 3)$ *of Theorem* 1*, we have the equality*

$$\lim_{\substack{k\to 0\\ |h|\le c|k|}} \frac{1}{k} \int_y^{y+k} \int_a^x f'_\tau(t, \tau)\, dt\, d\tau = \int_a^x f'_y(t, y)\, dt, \quad (x, y) \in r^0 \tag{11}$$

for every constant $c > 0$.

Proof. We have the following relations:

$$\frac{p(x+h,y+k)-p(x+h,y)}{k}=\frac{1}{k}\int\limits_a^{x+h}\big[f(t,y+k)-f(t,y)\big]\,dt=$$

$$=\frac{1}{k}\int\limits_a^{x}\big[f(t,y+k)-f(t,y)\big]\,dt+\frac{1}{k}\int\limits_x^{x+h}\big[f(t,y+k)-f(t,y)\big]\,dt. \qquad (12)$$

We rewrite the second summand in the right-hand side of (12) in the form

$$\lim_{k\to 0}\frac{h}{k}\cdot\frac{1}{h}\int\limits_x^{x+h}\big[f(t,y+k)-f(t,y)\big]\,dt$$

and make use of the condition $|h|\le c|k|$ with regard for the continuity of f. Then we obtain the equality

$$\lim_{\substack{k\to 0\\ |h|\le c|k|}}\frac{1}{k}\int\limits_x^{x+k}\big[f(t,y+k)-f(t,y)\big]\,dt=0. \qquad (13)$$

Notice now that the existence of the strong partial derivative implies that of the angular partial derivative, and their equality. Therefore equality (11) follows from equalities (3),(12) and (13) with regard for the absolute continuity of f with respect to y. Thus our Corollary is proved. $\square$

Corollary 2. *Let for the function f the assumptions $1)-3)$ of Theorem 1 be fulfilled, and let the angular partial derivative $f'_{\widehat{y}}$ be finite in the neighborhood $U(M_0)\subset r^0$ of the point $M_0\in r^0$. Then the equality*

$$p''_{[x],\widehat{y}}(M)=f'_{\widehat{y}}(M), \quad M\in U(M_0) \qquad (14)$$

is valid.

Proof. By equality (2), we have the relations

$$p''_{[x],\widehat{y}}(M)=\big(p'_{[x]}\big)'_{\widehat{y}}(M)=\big(f(x,y)\big)'_{\widehat{y}}(M)=f'_{\widehat{y}}(M), \quad M\in U(M_0). \quad \square$$

Corollary 3. *If the partial derivative f'_y likewise satisfies the conditions $1)-3)$ of Theorem 1 mentioned in* I, *then the equality*

$$p''_{[y],[x]}(M)=f'_{[y]}(M), \quad M\in U(M_0) \qquad (15)$$

holds.

Proof. By equality (3), we have the relations

$$p''_{[y],[x]}(M)=\big(p'_{[y]}\big)'_{[x]}(M)=\Bigg(\int\limits_a^{x}f'_y(t,y)\,dt\Bigg)_{[x]}(M). \qquad (16)$$

If in equality (1) instead of f we take f'_y, then by equality (2), the last term in (16) is equal to $f'_y(M)$, $M \in U(M_0)$. But owing to the continuity of f'_y in $U(M_0)$, we can replace $f'_y(M)$ by $f'_{[y]}(M)$ ([1], pp. 74–75). $\square$

Corollary 4. *The fulfilment of the conditions* 1)$-$3) *of Theorem* 1 *implies that* strgrad $p(x,y)$ *is finite, in particular, that the differential* $dp(x,y)$ *exists and the equality*

$$dp(x,y) = f(x,y)\,dx + \left(\int\limits_a^x f'_y(t,y)\,dt\right)dy, \quad (x,y) \in r^0 \tag{17}$$

is fulfilled.

2. Unilateral Bettazzi Derivatives

Below, the use will be made of the notion of strong ${}^{\pm}$limits (see [2]), ${}^{\pm}$Bettazzi derivatives ([3]) and the sets

$$\begin{aligned} A_1^+ &= \big\{(x,y) \in Q : x > x_0\big\}, \qquad & A_1^- &= \big\{(x,y) \in Q : x < x_0\big\}, \\ A_2^+ &= \big\{(x_0,y) \in Q : y > y_0\big\}, \qquad & A_2^- &= \big\{(x_0,y) \in Q : y < y_0\big\}, \\ A_{12}^+ &= A_1^+ \cup A_2^+, \qquad & A_{12}^- &= A_1^- \cup A_2^- . \end{aligned}$$

Obviously, $A_{12}^+ \cap A_{12}^- = \varnothing$ and $A_{12}^+ \cup A_{12}^- = Q \backslash \{M_0\}$.
We have the following

Theorem 2. *Let the function f be summable with respect to x on $[a,b]$ for every fixed $y \in [c,d]$. Let for the finite on Q function $p(x,y)$ from equality* (1) *the following conditions be fulfilled:*

1) *At every point $(x,y) \in Q$ there exists the finite partial derivative f'_y which is assumed to be summable on Q;*

2) *f'_y has a finite strong ${}^+$limit at the point $M_0 = (x_0,y_0) \in Q$.*

Then the function $p(x,y)$ has at the point M_0 a finite ${}^+$Bettazzi derivative ${p'_s}^+(M_0)$, and the equality

$${p'_s}^+(M_0) = {f'_y}^+(M_0) \tag{18}$$

is fulfilled.

Proof. We have the equality

$$\Delta_{[M_0]} p(x,y) = \int\limits_{x_0}^x \big[f(t,y) - f(t,y_0)\big]\,dt.$$

From the condition $f'_y \in L(Q)$ it follows by Fubini's theorem that for almost all $t \in [a,b]$ the function $f_y(t,\tau)$ is summable on $[c,d]$. Moreover, $f'_y(t,\tau)$ is everywhere finite. Therefore, by the well-known theorem ([4], p. 251), the equality

$$f(t,y)-f(t,y_0)=\int\limits_{y_0}^{y} f'_\tau(t,\tau)\,d\tau$$

is valid. Hence

$$\Delta_{[M_0]}p(x,y)=\int\limits_{x_0}^{x}\bigg(\int\limits_{y_0}^{y} f'_\tau(t,\tau)d\tau\bigg)dt$$

from which, owing to $f'_\tau\in L(Q)$, we have the representation in the form of the double integral

$$\Delta_{[M_0]}p(x,y)=\int\limits_{x_0}^{x}\int\limits_{y_0}^{y} f'_\tau(t,\tau)\,dt\,d\tau.$$

Thus we have

$$p'_s{}^{+}(M_0)=\lim_{\substack{(x,y)\to M_0\\(x,y)\in A^+_{12}}}\frac{1}{(x-x_0)(y-y_0)}\int\limits_{x_0}^{x}\int\limits_{y_0}^{y} f'_\tau(t,\tau)\,dt\,d\tau. \tag{19}$$

Now, by virtue of the definition of $^+$Bettazzi derivative ([3]) and by Theorem 1 from [3] we have the equality

$$p'_s{}^{+}(M_0)=\lim_{\substack{M\to M_0\\M\in A^+_{12}}} f'_s(M).$$

The right-hand side of the last equality is finite by the assumption 2). Therefore by the property of the derivative, it is equal to $f'_y{}^{+}(M_0)$. □

Analogously, we prove the following

Theorem 3. *Let the function f satisfy the condition* 1) *of Theorem* 2 *and, moreover, let the following condition*
2′) *f'_y has a finite strong $^-$limit at M_0*
be fulfilled.
Then $p(x,y)$ at M_0 has a finite Bettazzi $^-$derivative $p'_s{}^{-}(M_0)$, and

$$p'_s{}^{-}(M_0)=f'_y{}^{-}(M_0). \tag{20}$$

From Theorems 2 and 3, taking into account Propositions I.1.1 of [2] and 2 of [3], we have

Theorem 4. *Let the function f satisfy the condition* 1) *of Theorem* 2 *and, moreover, let the condition*
2″) *f'_y has a finite strong limit at the point M_0*
be fulfilled.

Then the function $p(x,y)$ has at the point M_0 the Bettazzi derivative $p_s'(M_0)$, and the equality

$$p_s'(M_0) = f_y'(M_0) \tag{21}$$

holds.

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