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THE INFLUENCE OF WALL PERMEABILITY ON THE STABILITY OF FLOWS BETWEEN TWO ROTATING CYLINDERS WITH A PRESSURE GRADIENT ACTING ROUND THE CYLINDERS

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Abstract. The conditions of stability and instability of flow between rotating permeable cylinders with the fluid pumping round the annular space are studied. Considering the both axisymmetric and oscillatory three-dimensional perturbation, the calculation of neutral curves is given.

რეზიუმე. შეისწავლება ორ ფოროვან ცილინდრს შორის მოთავსებული სითხის დინების მდგრადობისა და არამდგრადობის პირობები, როდესაც ხდება სითხის დატუმბვა წრიული მიმართულებით. როგორც ღერძსიმეტრიული, ისე რხევითი სამგანზომილებიანი შეშფოთებებისათვის აგებულია მდგრადობის ნეიტრალური მრუდები.

1. The instability of the rotating fluid applied, in particular, to the problem of stability of motion between rotating cylinders has been investigated by Rayleigh [1]. Neglecting viscosity, it was found that the motion of the rotating fluid is stable or instable depending on whether the circulation square increases monotonically from the axis of rotation, or not. For the cylinders rotating in one direction by this criterion it was stated that the necessary and sufficient condition for instability of a nonviscous flow with respect to axisymmetric disturbance is $\Omega_2 R_2^2 < \Omega_1 R_1^2$, where R_1 and R_2 are the radii, and Ω_1, Ω_2 are, respectively, angular velocity of the inner and outer cylinders.

For the flows between rigid cylinders with the fluid pumping round the annular space, using approximation for a narrow gap [1,2] and for a finite gap [3] the Rayleigh's criterion for an velocity distribution, corresponding to the viscous flow states that there exist both stable and instable layers of fluid in the basic flow.

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Our aim is to determine by the Rayleigh's criterium the influence of wall permeability on the conditions of instability of the flow between permeable cylinders with the fluid pumping round the annular space.

Let a viscous incompressible flow fill up the space between two rotating coaxial cylinders. The external mass forces are assumed to be absent, the flux of velocity through the cross-section of cylinders space is zero and the fluid inflow s through one cylinder is equal to the fluid outflow through the other. It is also assumed that the constant pressure gradient acts on the flow in the azimuthal direction $\left(\frac{\partial P}{\partial \theta}\right)_0$ due to the fluid pumping round the annular space. The pumping may be in the direction of the rotating cylinders or opposed to them.

We use Navier-Stokes equations in the cylindrical coordinates (r, θ, z) with the axis directed along the axis of the cylinders

$$\frac{d\vec{v}'}{dt} = -\frac{1}{\rho}\nabla p' - \nu \operatorname{rot}\operatorname{rot}\vec{v}', \quad \operatorname{div}\vec{v}' = 0, \quad \int_0^{2\pi}\int_{R_1}^{R_2} v_z' r dr d\theta = 0 \tag{1.1}$$

and the boundary conditions

$$R_1 v_r'|_{r=R_1} = R_2 v_r'|_{r=R_2} = s, \quad v_\theta|_{r=R_i} = \Omega_i R_i, \quad v_z'|_{r=R_i} = 0 \ \ (i=1,2), \tag{1.2}$$

where $v'(t, v_r', v_\theta', v_z')$ is the velocity vectors, ν is the kinematic viscosity, ρ' is the density,

$$\nabla = \left\{\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \theta}, \frac{\partial}{\partial z}\right\},$$
$$\frac{d\vec{v}'}{dt} = \frac{\partial \vec{v}'}{\partial t} + (\vec{v}', \nabla)\vec{v}' + \left\{-\frac{1}{r^2}(v_\theta')^2, \frac{1}{r}v_r' v_\theta', 0\right\}.$$

We choose the scales $R_1, \Omega_1 R_1$, $1/\Omega_1$, $\nu\rho'\Omega_1$, respectively, for length, velocity, time and pressure. Under these assumptions, the system (1.1)–(1.2) admits the following exact solution with velocity vector $\vec{v}_0 = \{v_{0r}, v_{0\theta}, v_{0z}\}$ and pressure p_0:

$$v_{0r} = \frac{\varkappa_0}{r}, \quad v_{0z} = 0,$$
$$v_{0\theta} = \begin{cases} \frac{1}{2\varkappa}\left(\frac{\partial p_0}{\partial \theta}\right)_0\left(-r + A_1 r^{\varkappa+1} + \frac{B_1}{r}\right) + A r^{\varkappa+1} + \frac{B}{r}, & \varkappa \neq -2, \\ \frac{1}{4}\left(\frac{\partial p_0}{\partial \theta}\right)_0\left(r - \frac{A_1' \ln r + 1}{r}\right) + \frac{A' \ln r + 1}{r}, & \varkappa = -2, \end{cases} \tag{1.3}$$
$$\frac{1}{\mathrm{Re}}\frac{\partial p_0}{\partial r} = \frac{v_{0\theta}^2}{r} + \frac{\varkappa_0^2}{r^3},$$

where

$$A_1 = \frac{R^2-1}{R^{\varkappa+2}-1}, \quad B_1 = \frac{R^2(R^\varkappa-1)}{R^{\varkappa+2}-1}, \quad A = \frac{\Omega R^2-1}{R^{\varkappa+2}-1},$$
$$B = \frac{R^2(R^\varkappa-\Omega)}{R^{\varkappa+2}-1}, \quad A_1' = \frac{R^2-1}{\ln R}, \quad A' = \frac{\Omega R^2-1}{\ln R}, \tag{1.4}$$
$$R=\frac{R_2}{R_1}, \quad \Omega=\frac{\Omega_2}{\Omega_1}, \quad \varkappa_0=\frac{s}{\Omega_1 R_1^2}, \quad s=R_i v_{0r}|_{r=R_i} \quad (i=1,2), \quad \mathrm{Re}=\frac{\Omega_1 R_1^2}{\nu},$$

$\varkappa = s/\nu$ is the radial Reynolds number.

According to the Rayleigh criterion, neglecting the viscosity, the necessary and sufficient condition of the flow instability will be

$$\Phi(r) = \frac{d}{dr}(r^2\omega_{0\theta})^2 < 0, \quad \omega_{0\theta} = \frac{v_{0\theta}}{r},$$

while for $\Phi(r) > 0$ the flow is stable.

For the flow (1.3)–(1.4) we have

$$\Phi(r) = 2r^3\omega_{0\theta}g(r), \tag{1.5}$$

where

$$g(r) = \frac{v_{0\theta}}{r} + \frac{dv_{0\theta}}{dr} =$$
$$= \begin{cases} \dfrac{1}{2\varkappa}[-2+(\varkappa+2)A_1r^\varkappa] + (\varkappa+2)A, & \varkappa \neq -2 \\ \dfrac{1}{4}\Big(2-\dfrac{A_1'}{r^2}\Big) + \dfrac{A'}{r^2}, & \varkappa = -2. \end{cases} \tag{1.6}$$

Suppose the cylinder rotates in the same direction $\Omega > 0$ and pumping flow is opposed to them,

$$P_\theta = \Big(\frac{\partial p_0}{\partial\theta}\Big)_0 < 0. \tag{1.7}$$

Consider the case $\varkappa > 0$, i.e. there takes place fluid inflow through the inner cylinder. In this case, taking into account values of the coefficients (1.4), it is not difficult to see that

$$\omega_{0\theta} < 0, \quad 1 < r < R \tag{1.8}$$

and likewise

$$g(r) > 0 \quad \text{for} \quad r > \Big[\frac{2(R^{\varkappa+2}-1)}{(\varkappa+2)(R^2-1)}\Big]^{\frac{1}{\varkappa}}, \quad 0 < \Omega R^2 < 1, \tag{1.9}$$

$$g(r) < 0 \quad \text{for} \quad r < \Big[\frac{2(R^{\varkappa+2}-1)}{(\varkappa+2)(R^2-1)}\Big]^{\frac{1}{\varkappa}}, \quad \Omega R^2 > 1. \tag{1.10}$$

As is known, the condition $\Omega R^2 > 1$ is necessary and sufficient for the stability of a nonviscous and sufficient for a viscous Couette flows [4] in the whole flow region between the cylinders, while for the flow (1.3)–(1.4) with pumping the condition of stability (1.10) is sufficient only in some part of its region. When $\varkappa$ increases, the region stability likewise increases. Passing to the limit as $\varkappa \to \infty$, since

$$\lim_{\varkappa\to\infty}\Big[\frac{2(R^{\varkappa+2}-1)}{(\varkappa+2)(R^2-1)}\Big]^{\frac{1}{\varkappa}}=2,$$

we can conclude that the flow is stable in the whole flow region $1 < r < R$, if the conditions $\Omega R^2 > 1$ is fulfilled.

Consequently, for the rotating cylinders in the same direction with the fluid inflow through the inner cylinder if the pumpimg flow is opposed to them and the condition $\Omega R^2 > 1$ is fulfilled, then for the increasing radial Reynolds number, the flow (1.3)–(1.4) becomes stable in the whole flow region.

Consider the case $\varkappa < 0$ for the fluid inflow through the outer cylinder. In this case (1.8)remains valid, and for $\varkappa \neq -2$

$$g(r) < 0, \quad \text{for} \quad r^{\varkappa} > \frac{2(R^{\varkappa+2}-1)}{(\varkappa+2)(R^2-1)}, \quad \mu R^2 > 1, \tag{1.11}$$

$$g(r) > 0, \quad \text{for} \quad r^{\varkappa} < \frac{2(R^{\varkappa+2}-1)}{(\varkappa+2)(R^2-1)}, \quad \mu R^2 < 1. \tag{1.12}$$

If $\varkappa = -2$, we have

$$g(r) > 0, \quad \text{for} \quad r > \Big[\frac{R^2-1}{2\ln R}\Big]^{1/2}, \quad \Omega R^2 > 1, \tag{1.13}$$

$$g(r) < 0, \quad \text{for} \quad r < \Big[\frac{R^2-1}{2\ln R}\Big]^{1/2}, \quad \Omega R^2 < 1. \tag{1.14}$$

Thus in this case, taking into account (1.7), (1.8), (1.12) and (1.14) we obtain, that $\phi(r) < 0$ and flow (1.3)–(1.4) is unstable. Passing to the limit as $\varkappa \to -\infty$ we have:

$$\lim_{\varkappa\to-\infty}\Big[\frac{2(R^{\varkappa+2}-1)}{(\varkappa+2)(R^2-1)}\Big]^{\frac{1}{\varkappa}}=1.$$

Consequently, for the rotating cylinders in the same direction with the fluid inflow through the outer cylinder if the pumping flow is opposed to them and the condition $\Omega R^2 < 1$ is fulfilled, by the increasing in the absolute value radial Reynolds number, the flow (1.3)–(1.4) becomes instable in the whole flow region.

2. In [4] in the framework of the linear theory of stability the first loss of stability of fluid between rigid rotating cylinders in the presence of a transverse pressure gradient has been investigated.Unlike the earlier works cited in [4], we consider both the stationary axisymmetric and oscillatory three-dimensional perturbations for a wide gap between the cylinders. It was stated that when the stationary flow first losses its stability depending on P_θ there may take place the stationary axisymmetric as well as oscillatory flows in the azimuthal direction.

In the present work we are interested in the effect of the radial Reynolds number $\varkappa$ on the instability of the main flow (1.3)–(1.4).

Consider an infinitesimal perturbation of the basic flow (1.3)–(1.4) and describe the perturbed flow by

$$v' = \vec{v}_0 + \vec{v}, \quad p' = p_0 + p.$$

The linearized equations of motion then follow from Navier-Stokes equations and the continuity equation. For axisymmetric disturbances that are periodic in the axial direction one can look for separated solutions of the form

$$\vec{v}(v_r, v_\theta, v_z) = \{u_0(r), v_0(r), w_0(r)\}e^{i\alpha z}, \quad p = p_{10}(r)e^{i\alpha z},$$

where α is the wave number of the disturbance in the axial direction.

Thus the neutral curves, corresponding to the axisymmetric perturbations, is sought by solving a spectral problem for a system of ordinary differential equations in dimensionless form:

$$\begin{gathered}
(L^* - \alpha^2)u_0 = \frac{dp_{10}}{dr} - 2\,\mathrm{Re}\,\omega_{0\theta}v_0 \\
(L^{**} - \alpha^2)v_0 = -\,\mathrm{Re}\,g(r)u_0 \\
(L^* + \frac{1-\varkappa}{r^2} - \alpha^2)w_0 = \alpha p_{10} \\
\frac{du_0}{dr} + \frac{u_0}{r} - \alpha w_0 = 0, \\
u_0 = v_0 = w_0 = 0 \quad (r = 1, R),
\end{gathered} \tag{2.1}$$

where

$$L^* = \frac{d^2}{dr^2} + \frac{1-\varkappa}{r}\frac{d}{dr} - \frac{1-\varkappa}{r^2}, \quad L^{**} = \frac{d^2}{dr^2} + \frac{1-\varkappa}{r}\frac{d}{dr} - \frac{1+\varkappa}{r^2},$$

$\mathrm{Re} = \frac{\Omega_1 R_1^2}{\nu}$ is the Reynolds number, $\omega_{0\theta} = \dfrac{v_{0\theta}}{r}$ and $g(r)$ are given by (1.3) and (1.6).

When considering the three-dimensional oscillatory perturbations, we seek for a solution of linearized disturbance equation in the form

$$\begin{gathered}
\vec{v}(tv_r, v_\theta, v_z) = e^{ict}\{u_1(r), v_1(r), w_1(r)\}e^{-i(m\theta+\alpha z)}, \\
P = p_1(r)e^{-i(m\theta+\alpha z)}e^{ict},
\end{gathered}$$

where c is an unknown cyclic frequency (phase velocity of azimuthal waves), and m is an azimuthal wave number. Having divided the variables, we obtain the spectral problem for the system of differential equations

$$\begin{gathered}
\Big[L^* - \alpha^2 - \frac{m^2}{r^2} - i\,\mathrm{Re}(c - m\omega_{0\theta})\Big]u_1 = \frac{dp_1}{dr} - 2\,\mathrm{Re}\,\omega v_1 - \frac{2im}{r^2}v_1 \\
\Big[L^{**} - \alpha^2 - \frac{m^2}{r^2} - i\,\mathrm{Re}(c_0 - m\omega_{0\theta})\Big]v_1 = \frac{im}{r}p_1 - \mathrm{Re}\,g(r)u_1 + \frac{2im}{r^2}u_1 \\
\Big[L^* + \frac{1-\varkappa}{r^2} - \alpha^2 - \frac{m^2}{r^2} - i\,\mathrm{Re}(c - \omega_{0\theta})\Big]w_1 = -i\alpha p_1 \qquad (2.2) \\
\frac{du_1}{dr} + \frac{u_1}{r} - \frac{im}{r}v - i\alpha w = 0 \\
u_1 = v_1 = \omega_1 = 0 \quad (r = 1, R).
\end{gathered}$$

The solution of that spectral problem allowes one to find neutral curves which corresponds to the initiation of azimuthal waves for three-dimensional oscillatory perturbations.

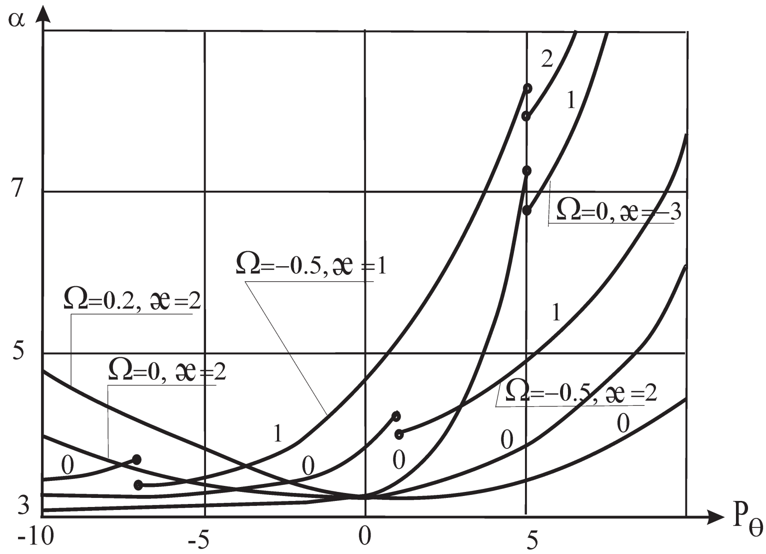

Fig. 1.

The problems on eigenvalues (2.1) and (2.2) have been solved numerically by the shooting method for fixed R, α, m, P_θ, $\varkappa$, Ω. The use was made of the calculating algorithm realized by V. Kolesov on computers for Couette flow [5]. The problems were reduced to the Cauchy boundary value problems for six first order differential equations with real and complex variables, respectively.

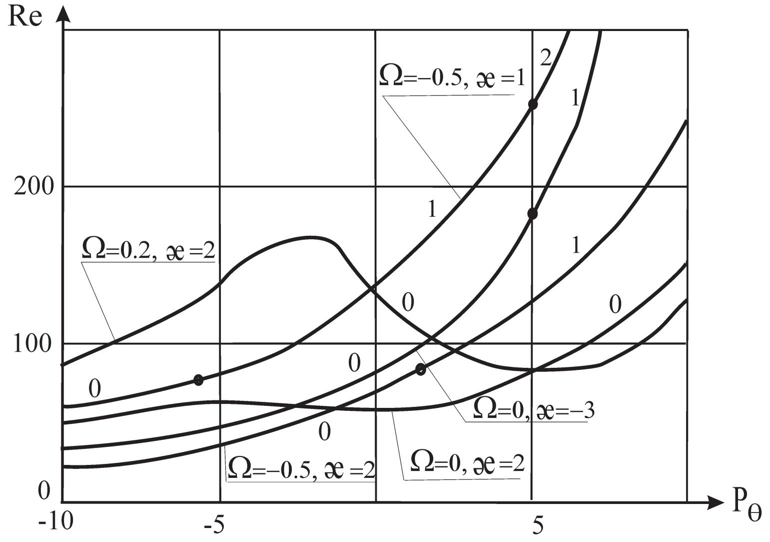

Fig. 2.

For numerical integration of these problems the standard Runge-Couette method was used. Calculations were performed by the numerical minimization of Re with respect to the wave numbers m and α in the case $R = 2$ (radius of the outer cylinder is twice as large as that of the inner cylinder) and $-10 \le P_\theta \le 10$. Figs. 1 and 2 show the dependence of the minimized critical Reynolds number and axial wave number α on P_θ under different values of the radial Reynolds number $\varkappa$ and Ω. The segments of the curves on which the azimuthal wave number m is constant, are denoted by the 0, 1, 2. These numbers correspond to the axisymmetric ($m = 0$) and oscillatory three-dimensional perturbations with periods $2\pi(m = 1)$ and $\pi(m = 2)$. On the neutral curves $\mathrm{Re} = \mathrm{Re}(P_\theta)$ there are the points at which perturbations with different wave numbers are equally dangerous, and the curves $\alpha = \alpha(P_\theta)$ at these points are discontinuous.

As calculations show, for $\Omega = 0$ and $\Omega = 0.2$, i.e. when the outer cylinder is at rest, or when both cylinders rotate in the same direction, for the flowing fluid through the inner cylinder $\varkappa > 0$, in the range of variation P_θ for the first loss of stability there take place only stationary axisymmetric flows, unlike rigid cylinders [4]: instability leads to a new steady secondary axisymmetric flow only for $P_\theta < 0$ (pumping flow is opposed to the inner cylinder), while for $P_\theta > 0$, i.e. the pumping flow has the same direction, as the inner cylinder there take place both the stationary axissymmetic and auto-oscillatory regimes with period 2π in the azimuthal direction. An analogous picture can be seen for the flow (1.3)–(1.4), but only for the flowing fluid through the outer cylinder.

When the cylinders rotate in opposite directions, then for the flowing fluid through the inner cylinder $\varkappa > 0$ and for $P_\theta > 0$ (the pumping flow and the inner cylinder rotate in the same direction), after the loss of stability of main stationary flow there take place stationary axisymmetric flows, while for $P_\theta < 0$ we have oscillatory three-dimensional flows with period 2π in the azimuthal direction. For $\varkappa < 0$ we can see the same picture as for the rigid cylinders, i.e., depending on P_θ there take place both the stationary and the auto-oscillatory motions with periods 2π and π.

Thus under the flowing fluid through the inner cylinder when cylinders rotate in the same direction, or the outer cylinder is at rest then the principle of exchange of stabilities holds in the considered range of variation of the constant pressure gradient in the azimuthal direction.

In the rest cases, depending on $\varkappa$ and P_θ, there take place either axisymmetric or auto-oscillatory regimes.

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