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ON SOLVING THE DIRICHLET BOUNDARY PROBLEM FOR THE POISSON EQUATION BY THE METHOD OF CONFORMAL MAPPING

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Abstract. In the present paper, the question of solution of the Dirichlet boundary problem for the Poisson equation by the method of conformal mapping (MCM) is considered. It is shown that application of the method is especially effective in the case where a particular solution to the Poisson equation cannot be written explicitly. The cases of both finite and infinite domains are considered. Illustrative examples are given.

რეზიუმე. წარმოდგენილ ნაშრომში განხილულია პუასონის განტოლებისათვის დირიხლეს სასაზღვრო ამოცანის ამოხსნის საკითხი კონფორმულ ასახვათა მეთოდით. ნაჩვენებია, რომ ამ მეთოდის გამოყენება განსაკუთრებით ეფექტურია იმ შემთხვევაში, როდესაც პუასონის განტოლების კერძო ამონახსნი არ იწერება ცხადი სახით. განხილულია როგორც სასრული ასევე უსასრულო არეთა შემთხვევები. მოცემულია საილუსტრაციო მაგალითები.

1. The Principle of Solution of Boundary Problems by the MCM

Let a domain D in the plane $z = x+iy \equiv (x, y)$ be bounded by a piecewise smooth contour S without multiple points (i.e., S is a simple contour). We assume that a plane boundary problem consists in finding a function $u(x, y)$ in the domain D under boundary conditions on S.

As it often happens, the problem can be solved in a relatively simple way under more complicated boundary conditions for canonical domains such as a disk, a circular ring, a square and so on. Hence, there are attempts to transfer the boundary problem posed for the initial (basic) domain D to the canonical domain G with boundary γ. Obviously in this case generally the following is being changed: 1) The given differential equation; 2) the domain in which the unknown function is sought; 3) the boundary conditions.

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As early as the middle of the 19-th century conformal mappings were widely used for transference of a number of plane problems of mathematical physics to canonical domains. To implement the transference an analytic function $z = \omega(\zeta)$ conformally mapping the domain G in the plane $\zeta = \xi + i\eta$ onto the domain D is applied (see, e.g., [1]–[4]).

The range of problems solvable by the MCM is very wide. In particular, the method has been applied successfully in problems of hydro and aerodynamics, elasticity, filtration etc.

Thus many boundary problems can be reduced to a problem of finding the function $z = \omega(\zeta)$. Note that the solution of boundary problems can easily be constructed when the function $z = \omega(\zeta)$ is either a rational or polynomial. The mentioned circumstance was organically connected with the development of methods for constructing conformally mapping functions.

Since conformally mapping functions $z = \omega(\zeta)$ can be written in explicit form only for a rather narrow family of domains, one has usually to resort to approximate methods. Hence quite a number of approximate methods of constructing mapping functions appeared (see, e.g., [1], [4]–[10]).

It should be noted that in solving boundary problems by the MCM the following circumstance takes place [11]. The function $z = \omega(\zeta)$ which is constructed approximately, maps conformally the canonical domain G onto the domain $\widetilde{D}$ which is close to D, and thus, practically, the problem stated for the domain D with boundary S is solved for the domain $\widetilde{D}$ with the simple boundary $\widetilde{S}$. Here we mean that the conditions which ensure the existence and uniqueness of a solution to a mathematical problem are fulfilled for the domain $\widetilde{D}$.

The possibility of such approach is due to the following facts. When passing from a practical problem to a mathematical model, the idealization of both the physical properties of the medium and the contour S takes place. Since the real boundary does not coincide with the ideal boundary S, the contour S has a tolerance field in which it can vary almost arbitrarily (without change of type). Physically this means that a small change of data induces a small change of effect, and mathematically this means that a solution depends continuously on the data. Therefore, in solving correct problems by the MCM, we have to find a function $z = \omega(\zeta)$ such that the deviation of a simple contour $\widetilde{S}$ from the given boundary S be within admissible limits. It is evident, that if $\widetilde{S} \to S$, then $\widetilde{u}(x, y) \to u(x, y)$, where $u(x, y)$ is a solution of the initial problem, and $\widetilde{u}(x, y)$ is a solution of the problem for the domain $\widetilde{D}$ with boundary $\widetilde{S}$.

2. The Dirichlet Boundary Problem for Poisson's Equation and a General Principle of Its Solution

Let a domain D in the plane z be bounded by a simple contour S. Consider the Dirichlet boundary problem for the Poisson equation

$$\Delta u(z) = \varphi(z), \qquad z \in D, \tag{2.1}$$

$$u(z) = g(z), \qquad z \in S, \tag{2.2}$$

where $u(z)$ is an unknown function, $u(z) \equiv u(x, y) \in C^2(D) \bigcap C(\overline{D})$, $\varphi(z) \in C^1(D) \bigcap C(\overline{D})$ and $g(z) \in C(S)$ are given functions, Δ is the Laplace operator.

It is known [12] that the problem (2.1), (2.2) is correct, i.e., the solution exists, is unique and depends continuously on the data. If the domain D is infinite, then for the uniqueness of the solution of the problem (2.1), (2.2) we require in addition that

$$u(\infty) = \lim u(z) = c, \quad \text{for} \quad z \to \infty,$$

where c is a real constant and $|c| < \infty$. The constant c cannot be fixed in advance, it should be found while solving the problem (2.1), (2.2).

We note that many problems of the theory of heat conductivity, electrostatics, the torsion of homogeneous and isotropic beams and rods reduce to the problem (2.1), (2.2) (see [13], [14]). In the case of a finite domain D a number of interesting works are devoted to the numerical solution of the problem (2.1), (2.2) (see, e.g., [15]–[24]).

A general principle of solution of the problem (2.1), (2.2) consists in seeking its solution in the form

$$u(z) = u_0(z) + v(z), \tag{2.3}$$

where $u_0(z)$ is a particular solution of the equation (2.1) ($u_0(z) \in C(\overline{D})$), and $v(z)$ is a solution of the following Dirichlet boundary problem for the Laplace equation:

$$\Delta v(z) = 0, \quad z \in D, \tag{2.4}$$

$$v(z) = g(z) - u_0(z), \quad z \in S, \tag{2.5}$$

which can be solved e.g., by the method of fundamental solutions [22], [23], [25], [26].

Katsurada and Okamoto obtained extensive results concerning error bounds and convergence of the method in such domains as a disk, its exterior, or an annulus [27], [28], and then generalized the results to Jordan domains [29]–[31].

Thus while solving the problem (2.1), (2.2) we find first an analytic form of the function $u_0(z)$. It is evident that the particular solution $u_0(z)$ is defined up to a harmonic function in D. The difficulty in construction of $u_0(z)$ in explicit form, evidently, depends on the form of $\varphi(z)$. As an illustration we consider some cases with a finite domain D.

1. $\varphi(z)$ is a polynomial,

$$\varphi(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}x^iy^j. \tag{2.6}$$

A particular solution $u_o(z)$ has the form

$$u_0(z)\equiv u_0(x,y)=\frac{1}{a+b}\Bigg[a\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}\frac{(-1)^{[j/2]}}{(i+2)(i+1)C_{i+j+2}^{j}}P_{i+j+2,j}(x,y)+$$
$$+b\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}\frac{(-1)^{[i/2]}}{(j+2)(j+1)C_{i+j+2}^{i}}\overline{P}_{i+j+2,i}(x,y)\Bigg], \tag{2.7}$$

where

$$P_{i+j+2,j}(x,y)=\sum_{k=0}^{[j/2]}(-1)^kC_{i+j+2}^{2k+\{j/2\}}x^{i+j+2-2k-\{j/2\}}y^{2k+\{j/2\}},$$
$$\overline{P}_{i+j+2,i}(x,y)=\sum_{k=0}^{[i/2]}(-1)^kC_{i+j+2}^{2k+\{i/2\}}x^{2k+\{i/2\}}y^{i+j+2-2k-\{i/2\}}.$$

Here, $\{i/2\}$ denotes the remainder of the ratio $i/2$, i.e., 1 when i is odd, and 0 when i is even; $[k]$ denotes the whole part of a number k; a and b are arbitrary finite numbers and $a^2+b^2\neq 0$; C_m^k is the number of combinations of k objects from m ones (we put $C_m^0=1$); $C_m^k=C_m^{m-k}$.

2. $\varphi(x,y)$ has the form

$$\varphi(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}\sin(b_ix+c_i)\sin(d_jy+e_j).$$

Then

$$u_0(x,y)=-\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}(b_i^2+d_j^2)^{-1}\sin(b_ix+c_i)\sin(d_jy+e_j).$$

3. $\varphi(x,y)$ has the form

$$\varphi(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}e^{b_ix+c_jy}.$$

Then

$$u_0(x,y)=\sum_{i=0}^{n}\sum_{j=0}^{n}a_{ij}(b_i^2+c_j^2)^{-1}e^{b_ix+c_jy},$$

where a_{ij},b_i,c_j,d_j,e_j are given numbers.

4. If functions $\varphi_1(x,y), \varphi_2(x,y)$ are harmonic in the domain D, then the functions [15, 23]

$$u_1(x,y) = xH_1(x,y)/2, \quad u_2(x,y) = yH_2(x,y)/2,$$

are particular solutions of the equation (2.1). Here $H_1(x,y)$ and $H_2(x,y)$ are harmonic in D and

$$\frac{\partial H_1(x,y)}{\partial x} = \varphi_1(x,y), \quad \frac{\partial H_2(x,y)}{\partial y} = \varphi_2(x,y), \quad (x,y) \in D,$$

$$H_1(x,y) = \int\limits_{x_0}^{x} \varphi_1(t,y)\,dt + h_1(y), \quad h_1(y) = -\int\limits_{y_0}^{y} (y-t)\frac{\partial \varphi_1(x_0,t)}{\partial x}\,dt,$$

$$H_2(x,y) = \int\limits_{y_0}^{y} \varphi_2(x,t)\,dt + h_2(x), \quad h_2(x) = -\int\limits_{x_0}^{x} (x-t)\frac{\partial \varphi_2(t,y_0)}{\partial y}\,dt,$$

(x_0, y_0) is an arbitrarily fixed point in D.

In [20], [21], [23] an approximate expression for a particular solution is constructed on the basis of an approximation of $\varphi(x,y)$. Obviously, in this case the accuracy of the solution of the problem (2.1), (2.2) depends on the accuracy of both the solution of the problem (2.4), (2.5) and approximation of $\varphi(x,y)$.

In general, if the particular solution $u_0(x,y)$ cannot be written explicitly, then as a rule, a logarithmic potential is used as $u_0(z)$:

$$u_0(z) = u_0(x,y) = \frac{1}{2\pi}\int\limits_{D} \varphi(t) \ln|z-t| dt_1\,dt_2, \tag{2.8}$$

$$t = t_1 + it_2, \quad z \in D, \quad t \in \overline{D}.$$

It is well known ([12], [13]) that if $\varphi(z) \in C^1(D) \bigcap C(\overline{D})$, the improper integral (2.8) is continuous on $\overline{D}$, twice continuously differentiable and satisfies the equation (2.1) in D.

Since the integral (2.8) cannot be expressed by elementary functions, the numerical integration of the integral (2.8) strongly increases the computation time of the boundary problem (2.1), (2.2). For example, in [16] the integral (2.8) is computed by means of numerical quadrature, and in [17]–[19] quasi-Monte-Carlo methods are applied to approximate integrals of type (2.8).

In [24], an improved method for calculating integrals of type (2.8) is given for finite domains. It gives especially high accuracy if a domain D is a disk.

If D is infinite, direct calculation of integrals of type (2.8) by means of the above methods is impossible or represents a very difficult problem. If, besides, the function $u_0(z)$ cannot be written explicitly, it is reasonable to solve the problem (2.1), (2.2) by the method of conformal mapping.

It is evident that the method of conformal mapping can be applied to the case, where the particular solution $u_0(z)$ can be written explicitly.

3. The Transference of the Boundary Problem onto a Disk by the MCM

Suppose that a domain D is finite or infinite(i.e., z-plane with a hole) and G is the unit disk($|\zeta| < 1$). Let the function $z = \omega(\zeta)$ conformally map the disk G onto a domain D, and the function $\zeta = f(z)$ be inverse of $z = \omega(\zeta)$, i.e., $\zeta = \omega^{-1}(z)$. It is clear from above mentioned that the systems

$$\begin{cases} x = x(\xi, \eta) \\ y = y(\xi, \eta) \end{cases} \qquad \text{and} \qquad \begin{cases} \xi = \xi(x, y) \\ \eta = \eta(x, y) \end{cases} \tag{3.1}$$

are mutually inverse.

It is well-known ([1], [2]) that for functions $\omega(\zeta)$ and $f(z)$ the following conditions take place:

$$\begin{aligned} &f'(z) \neq 0 \quad \text{for} \quad z \in D, \quad \omega'(\zeta) \neq 0 \quad \text{for} \quad \zeta \in G, \\ &f'(z) = \frac{1}{\omega'(\zeta)} \quad \text{for} \quad z \in D, \quad \zeta \in G \ \ (z \leftrightarrow \zeta). \end{aligned} \tag{3.2}$$

Taking into account that the function $u(x, y)$ was unknown in the domain D, after transference of the problem (2.1), (2.2) onto disk G the unknown function will be

$$u^*(\xi, \eta) = u(x(\xi, \eta), y(\xi, \eta)) \equiv u(x, y). \tag{3.3}$$

Let us note that in order to find a value of the function $u(x, y)$ at any point of the initial domain D it is necessary to know the function $\zeta = f(z)$. Indeed, since the functions $\zeta = f(z)$ and $z = \omega(\zeta)$ are mutually inverse, at any point $z = (x, y)$ of D, $u(x, y) = u^*(\xi, \eta)$, where $\xi + i\eta = \zeta = f(z)$.

Due to the analyticity of the function $\zeta = f(z) \equiv \xi(x, y) + i\eta(x, y)$, using (3.1), (3.3) we easily obtain that after the transformation the Laplace operator $\Delta u(x, y)$ takes the form (see [1], [4])

$$\Delta u(x, y) = |f'(z)|^2 \Delta u^*(\xi, \eta). \tag{3.4}$$

From (3.4) we get the known result saying that a function, harmonic before transformation, remains harmonic, since $|f'(z)| \neq 0$ for $z \in D$. According to (3.2) and (3.4), after transference to the domain G the problem (2.1), (2.2) takes the form

$$\Delta u^*(\xi, \eta) = |\omega'(\zeta)|^2 \varphi^*(\zeta) \equiv \psi^*(\zeta), \quad \zeta = (\xi, \eta) \in G, \tag{3.5}$$

$$u^*(\tau) = g^*(\tau), \quad \tau = (\xi, \eta) \in \gamma, \tag{3.6}$$

where

$$\varphi^*(\zeta) \equiv \varphi^*(\xi, \eta) = \varphi(x(\xi, \eta), y(\xi, \eta)), \quad g^*(\tau) \equiv g^*(\xi, \eta) = g(x(\xi, \eta), y(\xi, \eta)).$$

If the domain D is infinite, then in order to have $\psi^*(\zeta) \in C^1(G) \bigcap C(\overline{G})$ we should require in addition that

$$\varphi(z) = O\big(\frac{1}{|z|^{4+\alpha}}\big) \quad \text{for} \quad |z| \to \infty, \quad \alpha \geq 0. \tag{3.7}$$

Indeed, if for the mapping $z = \omega(\zeta)$ we assume for definiteness that the point $z = \infty$ transforms into the point $\zeta = 0$, then at the point $\zeta = 0$ the function $\omega(\zeta)$ will have a simple pole, i.e., the function $\omega(\zeta)$ will have the form

$$\omega(\zeta) = \frac{c}{\zeta} + \omega_1(\zeta), \tag{3.8}$$

where $c = c_1 + ic_2$ is a constant, and a function $\omega_1(\zeta)$ is analytic in the disk G. It is evident that the function $\omega(\zeta)$ cannot have other singularities in G, since in this case the mapping will not be one-to-one. From (3.8) we have

$$\begin{aligned} \omega'(\zeta) &= -\frac{c}{\zeta^2} + \omega_1'(\zeta), \\ |\omega'(\zeta)|^2 &= \frac{|\zeta^2\omega_1'(\zeta) - c|^2}{|\zeta|^4}. \end{aligned} \tag{3.9}$$

Using (3.7), (3.8) and (3.9) we have

$$|\varphi^*(\zeta)| < \frac{A|\zeta|^{4+\alpha}}{|c + \zeta\omega_1(\zeta)|^{4+\alpha}} \quad \text{for} \quad \zeta \in \overline{G},$$

$$|\psi^*(\zeta)| < \frac{A|\zeta|^{\alpha}|\zeta^2\omega_1'(\zeta) - c|^2}{|\zeta\omega_1(\zeta) + c|^{4+\alpha}} < B \quad \text{for} \quad \zeta \in \overline{G},$$

where A and B are real numbers. If we choose the origin of coordinates of plane z outside of D, then $\zeta\omega_1(\zeta) + c \neq 0$ for $\zeta \in G$ and consequently $\psi^*(\zeta) \in C^1(G) \bigcap C(\overline{G})$, Q.E.D.

Thus we obtain again the Dirichlet boundary problem for the Poisson equation with changed right hand sides, however for the disk G.

Evidently, for approximate solution of the problem (3.5), (3.6) we can use the general method described in Section 2, however for the same purpose we can apply any other variant. In particular, a solution to the problem (3.5), (3.6) is sought in the form of the sum (see [1], pp. 595–597)

$$u^*(\zeta) = v(\zeta) + w(\zeta), \tag{3.10}$$

where $v(\zeta)$ is a solution to the Dirichlet boundary problem

$$\Delta v(\zeta) = 0, \quad \zeta \in G, \tag{3.11}$$

$$v(\tau) = g^*(\tau), \quad \tau \in \gamma, \tag{3.12}$$

and the function $w(\zeta)$ represents a solution to the boundary problem

$$\Delta w(\zeta) = \psi^*(\zeta), \quad \zeta \in G, \tag{3.13}$$

$$w(\tau) = 0, \quad \tau \in \gamma. \tag{3.14}$$

The problem (3.11), (3.12) can be approximately solved by the method of fundamental solutions or the Poisson integral ([1], [2], [13])

$$v(\zeta) = v(r,\vartheta) = \frac{1}{2\pi}\int\limits_0^{2\pi} g^*(\tau)\frac{1-r^2}{1-2r\cos(\theta-\vartheta)+r^2}\,d\theta, \tag{3.15}$$

where $\zeta = (r\cos\vartheta, r\sin\vartheta)$, $\tau = (\cos\theta, \sin\theta)$, $0 \le \vartheta \le 2\pi$, $0 \le \theta \le 2\pi$, $0 \le r < 1$, can be applied. Expression (3.15) is meaningless when r=1, however it is known [2], that

$$\lim_{\zeta\to\tau} v(\zeta) \equiv \lim_{\substack{r\to 1\\ \vartheta\to\theta}} v(r,\vartheta) = g^*(\theta) \equiv g^*(\tau), \quad \zeta \in G, \ \ \tau \in \gamma.$$

Concerning the problem (3.13), (3.14), its solution has the form [1]

$$\begin{aligned} w(\zeta) = w(r,\vartheta) &= \frac{1}{2\pi}\iint\limits_G \psi^*(t)\ln\frac{|\zeta-t|}{|1-\zeta\overline{t}|}\,dt_1\,dt_2 = \\ &= \frac{1}{2\pi}\iint\limits_G \psi^*(t)\ln|\zeta-t|\,dt_1\,dt_2 - \\ &\quad - \frac{1}{2\pi}\iint\limits_G \psi^*(t)\ln|1-\zeta\overline{t}|dt_1\,dt_2 = I_1 + I_2\,, \end{aligned} \tag{3.16}$$

where $t = (t_1,t_2) \equiv t_1 + it_2 = \rho e^{i\theta}$, $\zeta = (\xi,\eta) = re^{\vartheta}$, $\zeta \in G$, $t \in \overline{G}$.

Since in the problem (3.13), (3.14) $\psi^*(\zeta) \in C^1(G)\bigcap C^(\overline{G})$, logarithmic potential I_1 satisfies the Poisson equation (3.13) in the domain G and is continuous in $\overline{G}$ (is harmonic outside G). The function I_2 is harmonic in G, continuous in $\overline{G}$ and

$$\lim_{\zeta\to\gamma}(I_1+I_2) = 0, \quad \zeta \in G.$$

For calculation of the integral (3.16) we can apply the method described in [24].

4. Examples of Application of Conformal Mapping

In the examples below (for control of the accuracy of the solution of the problem (2.1), (2.2) by the method of conformal mapping) the role of $\varphi(z)$ was played by a function for which a particular solution to equation (2.1) can be written explicitly. Namely, in the case of a finite simply connected domain we take the function

$$\varphi(z) = \varphi(x,y) = x^2y^3. \tag{4.1}$$

In the case of function (4.1) in formula (2.6) we have: $n = 3$; $a_{11} = a_{12} = a_{21} = a_{32} = a_{33} = 0$; $a_{23} = 1$. By (2.7) we get that the particular solution of equation (2.1) for the function (4.1) has the form

$$u_0(z) = \frac{1}{a+b}\bigg(-a\frac{x^6y}{60} + a\frac{x^4y^3}{12} - b\frac{y^7}{420} + b\frac{x^2y^5}{20}\bigg), \tag{4.2}$$

where a and b are arbitrary finite numbers which do not vanish simultaneously.

In order to obtain from (4.2) a simpler particular solution we consider two cases: 1) $a \neq 0,\ b = 0$; 2) $a = 0,\ b \neq 0$. Consequently, we come to the following particular solutions of the equation (2.1)

$$u_0^1(z) = -\frac{x^6y}{60} + \frac{x^4y^3}{12},$$
$$u_0^2(z) = -\frac{y^7}{420} + \frac{x^2y^5}{20}.$$

In Example 4.1 the function

$$g(z) = -\frac{x^6y}{60} + \frac{x^4y^3}{12}, \quad z \in S, \tag{4.3}$$

is taken as a boundary function. It is clear that for functions (4.1) and (4.3) the exact solution of the problem (2.1), (2.2) will be

$$u(z) = -\frac{x^6y}{60} + \frac{x^4y^3}{12}, \quad z \in D.$$

In the case of an infinite domain D as $\varphi(z)$ we take

$$\varphi(z) = \frac{4}{|z|^4}, \tag{4.4}$$

where $|z| = \sqrt{x^2+y^2}$ and it is assumed that the origin lies outside of the domain D (for example in the "center" of a hole). It is easy to see that for function (4.4) the particular solution of equation (2.1) has the form

$$u_0(z) = \frac{1}{|z|^2}, \quad z \in D.$$

If as a boundary function we take

$$g(z) = \frac{1}{|z|^2}, \quad z \in S, \tag{4.5}$$

then the exact solution to the problem (2.1), (2.2) for the functions (4.4) and (4.5) will have the form

$$u(z) = u(x,y) = \frac{1}{|z|^2}, \quad z \in D.$$

Evidently, in the above case $u(z) \in C^2(D)\bigcap C(\overline{D})$, $\varphi(z) \in C^1(D)\bigcap C(\overline{D})$ and

$$u(z) = O\bigg(\frac{1}{|z|^2}\bigg), \quad \varphi(z) = O\bigg(\frac{1}{|z|^4}\bigg) \quad \text{for} \quad z \to \infty.$$

Example 4.1. Let the domain D be the interior of the Pascal limacon with the equation
$$S : z = K(\tau + a\tau^2),$$
where $0 \le a \le 0.5, \quad K > 0, \quad \tau = e^{i\theta}, \quad 0 \le \theta \le 2\pi.$

It is known [3] that a function $z = \omega(\zeta)$ which conformally maps the disk G onto the interior of the indicated limacon has the form $\omega(\zeta) = K(\zeta + a\zeta^2), \quad \zeta = \xi + i\eta \in G$. We have
$$|\omega'(\zeta)|^2 = K^2\big[(1+2a\xi)^2 + 4a^2\eta^2\big],$$
$$z = x + iy = K(\zeta + a\zeta^2), \ \ x = K\big[\xi + a(\xi^2 - \eta^2)\big], \ \ y = K\eta(1+2a\xi). \tag{4.6}$$
Consequently, for the functions (4.1) and (4.3) the problem (2.1), (2.2) after transference onto the disk G will have the form (see (3.5), (3.6))
$$\Delta u^*(\xi,\eta) = \psi^*(\xi,\eta), \quad (\xi,\eta) \in G, \tag{4.7}$$
$$u^*(\xi,\eta) = g^*(\xi,\eta), \quad (\xi,\eta) \in \gamma, \tag{4.8}$$
where
$$\psi^*(\xi,\eta) = K^7\big[(1+2a\xi)^2 + 4a^2\eta^2\big]\big[\xi + a(\xi^2-\eta^2)\big]^2(\eta + 2a\xi\eta)^3,$$
$$g^*(\xi,\eta) = \frac{K^7\big[\xi + a(\xi^2-\eta^2)\big]^4(\eta+2a\xi\eta)^3}{12} - \\ - \frac{K^7\big[\xi + a(\xi^2-\eta^2)\big]^6(\eta+2a\xi\eta)}{60}.$$
It is evident that, $\psi^*(\xi,\eta) \in C^1(G)\bigcap C(\overline{G})$, $g^*(\xi,\eta) \in C(\gamma)$.

Example 4.2. Let D be the exterior of the ellipse $S: \ x = a\cos t, \ y = b\sin t, \ 0 \le t \le 2\pi$. As it is known [3] the function which conformally maps the disk G onto the exterior of the above mentioned ellipse has the form
$$z = \omega(\zeta) = c\left(\frac{1}{\zeta} + d\zeta\right), \quad \zeta \in G, \tag{4.9}$$
where
$$c = \frac{a+b}{2}, \quad d = \frac{a-b}{a+b}, \quad a \ge b.$$
From (4.9) we have
$$\begin{aligned} |\omega'(\zeta)|^2 &= c^2\big[d^2 + \frac{1 - 2d(\xi^2-\eta^2)}{(\xi^2+\eta^2)^2}\big], \\ x = c\xi\big(d + \frac{1}{\xi^2+\eta^2}\big), &\quad y = c\eta\big(d - \frac{1}{\xi^2+\eta^2}\big). \end{aligned} \tag{4.10}$$

Thus, if in the problem (2.1), (2.2) as the functions $\varphi(z)$ and $g(z)$ we will take (4.4) and (4.5) respectively, then after transference onto the disk G it will have the form
$$\Delta u^*(\xi,\eta) = \psi^*(\xi,\eta), \quad (\xi,\eta) \in G, \tag{4.11}$$

$$u^*(\xi,\eta) = g^*(\xi,\eta), \quad (\xi,\eta) \in \gamma, \tag{4.12}$$

where

$$\psi^*(\xi,\eta) = \frac{4[d^2(\xi^2+\eta^2)^2 - 2d(\xi^2-\eta^2)+1]}{c^2[d^2(\xi^2+\eta^2)^2 + 2d(\xi^2-\eta^2)+1]^2}, \quad (\xi,\eta) \in G,$$

$$g^*(\xi,\eta) = \frac{1}{c^2[d^2+2d(\xi^2-\eta^2)+1]}, \quad (\xi,\eta) \in \gamma.$$

Since

$$|1+d\zeta^2|^2 = d^2(\xi^2+\eta^2)^2 + 2d(\xi^2-\eta^2)+1 \neq 0, \quad (\xi,\eta) \in \overline{G} \ \ (0 \le d < 1),$$

therefore $\psi^*(\xi,\eta) \in C^1(G)\bigcap C(\overline{G})$, $g^*(\xi,\eta) \in C(\gamma)$.

Finally we note that for concrete numbers K, a, c and d the solution to the problems (4.7), (4.8) and (4.11), (4.12) must be sought in the form (3.10).

Let us assume that we have found the value of an approximate solution of the problem (4.7), (4.8) (or (4.11), (4.12)) at a point $(\xi,\eta) \in G$. Using the principle of the conformal mapping method we conclude that actually we have found an approximate value of the solution $u(x,y)$ to the problem (2.1), (2.2) for functions (4.1) and (4.3) (or (4.4) and (4.5)) at a point $(x,y) \in D$, where x and y are defined by the relation $z = \omega(\zeta)$, i.e., from (4.6) (or from (4.10)).

5. Concluding Remarks

We have presented an algorithm of approximate solution of the Dirichlet boundary problem for the Poisson equation. This algorithm is a synthesis of the MCM and an improved method [24] for calculating integrals of logarithmic potential type. An application of the described algorithm is especially effective in the case when a particular solution to the Poisson equation cannot be written explicitly. In the case of an infinite domain for an application of the presented algorithm an additional condition is established for the right-hand side of the Poisson equation.

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