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THE TWO-WEIGHTED INEQUALITIES FOR GENERALIZED MARCINKIEWICZ INTEGRALS

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Abstract. In the present paper we establish the two-weighted inequalities for generalized Marcinkiewicz integrals. We consider the integrals with multiple kernels as well.

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Let P be a closed set of the space R^n and $\delta(y)$ be a distance from the point y to the set P. J. Marcinkiewicz was the first who studied the following integral transforms [1]:

$$Jf(x) = \int\limits_P \frac{(\delta(y))^\lambda}{|x-y|^{n+\lambda}} f(y)\,dy, \quad \lambda > 0 \tag{1.1}$$

and

$$Jf(x) = \int\limits_{\{y:\delta(y)\le\delta_0<1\}} \frac{(\log\frac{1}{\delta(y)})^{-1}}{|x-y|^n} f(y)\,dy. \tag{1.2}$$

These integrals are of importance in the theory of Fourier series. Modification of the above integrals have been considered by L. Carleson [2] and A. Zygmund [3]. For example, A. Zygmund studied the following integral transformations:

$$(J^*f)(x) = \int\limits_{R^n} \frac{[\delta(y)]^\lambda}{(|x-y|+\delta(y))^{n+\lambda}} f(y)\,dy, \tag{1.3}$$

$$(J^*f)(x) = \int\limits_{\{\delta(y)\le\delta_0<1\}} \frac{[\lg\frac{1}{\delta(y)}]^{-1}}{(|x-y|+\delta(y))^n} f(y)\,dy. \tag{1.4}$$

It is evident that if $x \in P$, then $(J^*f)(x)$ and $(J^*f)(x) = (Jf)(x)$. In the theory of singular and hypersingular integrals the most important turned out to be the integrals written in the form (1.3) and (1.4). In this section

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we shall prove the two-weight inequality for generalized Marcinkiewicz integrals. This generalization has been considered by Calderon [4] who proved the one-weighted inequality for the Muckenhoupt A_p classes.

Let on the set $(0,\infty)\times[0,\infty)$ be defined the nonnegative function $\varphi(\rho,t)$ satisfying the following conditions:

(1) for every fixed t, $t\in[0,\infty)$ the function $(\rho+t)^{-n}\varphi(\rho,t)$ is nonincreasing with respect to ρ, and

$$\lim_{\rho\to\infty}(\rho+t)^{-n}\varphi(\rho,t)=0;$$

(2) there exists the positive constant c such that

$$\int\limits_0^\infty \rho^{n-1}(\rho+t)^{-n}\varphi(\rho,t)\,d\rho\le c,$$

for every nonnegative t.

Let $\psi(y)\ge 0$ be a measurable function such that the function $\varphi(|x-y|,\psi(y))$ is measurable. Consider the integral

$$Kf(x)=\int\limits_{R^n}\frac{\varphi(|x-y|,\psi(y))}{(|x-y|+\psi(y))^n}\varphi\big(|x-y|,\psi(y)\big)f(y)\,dy. \tag{1.5}$$

Obviously, when

$$\varphi(\rho,t)=\frac{t^2}{(\rho+t)^2},\quad \psi(y)=\delta(y),$$

we obtain $Kf(x)=J^*f(x)$.

Let $w(x)$ be a nonnegative locally integrable function. We define the space L^p_w as follows:

$$L^p_w=\big\{f:\|f\|_{L^p_w}<\infty\big\},\quad\text{where}\quad \big\|f\big\|_{L^p_w}=\bigg(\int\limits_K\big|f(x)\big|^p w(x)\,dx\bigg)^{1/p}.$$

Definition 1.1. Let v and w be nonnegative increasing functions, and $v(\pi-)<\infty$, $w(\pi-)<\infty$. $(v,w)\in a_p$, if the condition

$$\sup_{0<x<\pi}\int\limits_x^\infty\frac{v(t)}{t^p}\,dt\bigg(\int\limits_0^x w^{1-p'}(t)\,dt\bigg)^{p-1}<\infty$$

is fulfilled.

Definition 1.2. Let v and w be nonnegative decreasing functions, and $v(\pi-)<\infty$, $w(\pi-)<\infty$. $(v,w)\in b_p$ if

$$\sup_{0<x<\pi}\int\limits_0^x v(t)\,dt\bigg(\int\limits_x^\infty w^{1-p'}(t)t^{-p'}\,dt\bigg)^{p-1}<\infty$$

is fulfilled.

In the sequel, we shall consider the radial weights $\sigma(x) = w(|x|)$, $\rho(x) = v(|x|)$, $x \in R^n$. To prove the two-weighted inequalities for the generalized Marcinkiewicz integral we shall use the following criteria of the two-weighted inequalities (in the case of monotone weights) for maximal Hardy–Littlewood functions

$$Mf(x) = \sup_{r>0} \frac{1}{B(x,r)} \int\limits_{B(x,r)} \big|f(y)\big|\,dy,$$

where $B(x,r)$ is a ball in an n-dimensional Euclidean space.

Theorem A. *If $1 < p < \infty$, v and w are increasing even functions, then for the boundedness of the operator $H : L^p_\sigma \to L^p_\rho$ it is necessary and sufficient that $(v,w) \in a_p$*

Theorem B. *If $1 < p < \infty$, v and w are decreasing functions, then for the boundedness of the operator $H : L^p_\sigma \to L^p_\rho$ it is necessary and sufficient that the condition*

$$\sup_{r>0} \frac{1}{r} \int\limits_0^r v(t)\,dt \bigg(\frac{1}{r} \int\limits_0^r w^{1-p'}(t)\,dt\bigg)^{p-1} < \infty.$$

be fulfilled.

The above theorems can be proved analogously to the case of singular Calderon–Zygmund integrals treated in their joint work by D. Edmunds and V. Kokilashvili [5].

Theorem 1.1. *Let $1 < p < \infty$ and, moreover, let the condition $(v,w) \in b_p$ be fulfilled. Let $\rho(x) = v(|x|)$ and $\sigma(x) = w(|x|)$. Then there exists the positive constant c such that for any function $f \in L^p_\sigma(R^n)$ the inequality*

$$\int\limits_{R^n} \big|(Kf)(x)\big|^p \rho(x)\,dx \le c \int\limits_{R^n} \big|f(x)\big|^p \sigma(x)\,dx \tag{1.6}$$

holds.

To prove the theorem we shall need one lemma which establishes aconnection between the generalized Marcinkiewicz integral and the maximal Hardy–Littlewood function.

Lemma. *Let f and g be some positive measurable functions defined on an n-dimensional space. Then the inequality*

$$\int\limits_{R^n} Kf(x)g(x)\,dx \le c \int\limits_{R^n} f(x)Mg(x)\,dx \tag{1.7}$$

holds, where the constant c does not depend on the functions f and g.

Proof. We have

$$\int\limits_{R^n}(Kf)(x)g(x)\,dx=$$

$$=\int\limits_{R^n}f(y)\int\limits_{R^n}\frac{1}{[|x-y|+\psi(y)]^n}\varphi\big(|x-y|,\psi(y)\big)g(x)\,dx\le$$

$$\le\int\limits_{R^n}f(y)\sup_t\int\limits_{R^n}\frac{1}{[|x-y|+t]^n}\varphi\big(|x-y|,t\big)g(x)\,dx. \tag{1.8}$$

Let

$$G(y,\rho)=\int\limits_{|x-y|\le\rho}g(x)\,dx,$$

then

$$G(y,\rho)\le c\rho^n(Mg)(y)$$

and

$$\int\limits_{R^n}\frac{1}{[|x-y|+t]^n}\varphi\big(|x-y|,t\big)g(x)g(x)\,dx=\int\limits_{\rho=0}^{\infty}\frac{\varphi(\rho,t)}{(\rho+t)^n}\,dG(y,\rho)=$$

$$=-\int\limits_{\rho=0}^{\infty}G(y,\rho)d\frac{\varphi(\rho,t)}{(\rho+t)^n}\le-c(Mg)(y)\int\limits_{\rho=0}^{\infty}\rho^n d\frac{\varphi(\rho,t)}{(\rho+t)^n}=$$

$$=nc(Mg)(y)\int\limits_{0}^{\infty}\frac{\rho^{n-1}}{(\rho+t)^n}\varphi(\rho,t)\,d\rho\le nc(Mg)(y).$$

Taking into account the signs of the functions $G(y,\rho)$ and $(\rho+t)^{-n}\varphi(\rho,t)$ and the fact that these functions tend to zero as $\rho\to 0$ and $\rho\to\infty$, respectively, then after integration by parts and substitution into (1.8) we obtain the desired result. □

Proof of Theorem 1. Obviously,

$$\big\|Kf\big\|_{L^p_\rho}=\sup_g\bigg|\int\limits_{R^n}(Kf)(x)g(x)\,dx\bigg|, \tag{1.9}$$

where the least upper bound is taken for all functions g such that

$$\int\limits_{R^n}\big|g(x)\big|^{p'}\rho^{1-p'}(x)\,dx\le 1$$

or

$$\big\|g\big\|_{L^{p'}_{\rho^{1-p'}}}\le 1. \tag{1.10}$$

On the other hand, using Hölder's inequality, we obtain

$$\int\limits_{R^n} \big|f(x)\big|Mg(x) \leq \big\|f\big\|_{L^\rho_\sigma} \big\|Mg\big\|_{L^{p'}_{\sigma^{1-\rho'}}} \tag{1.11}$$

Applying now Theorem A and the condition (1.6), we find that

$$\int\limits_{R^n} \big|f(x)\big|Mg(x)\,dx \leq c\big\|f\big\|_{L^\rho_\sigma} \big\|g\big\|_{L^{p'}_{\sigma^{1-\rho'}}} \leq c\big\|f\big\|_{L^\rho_\sigma}. \tag{1.12}$$

Making use of inequalities (1.9), (1.10) and (1.11), we complete the proof of the theorem. □

Theorem 1.2. *Let $1 < \rho < \infty$, v and w be nonnegative even and increasing functions and let the condition*

$$\sup_{r>0} \frac{1}{r} \int\limits_0^r w^{1-\rho}(t)\,dt \bigg(\frac{1}{r}\int\limits_0^r v(t)\,dt\bigg)^{p'-1} < \infty$$

be fulfilled. If $\rho(x) = v(|x|)$ and $\sigma(x) = w(|x|)$, then there exists the positive constant c such that for any function $f \in L^p_\sigma(R^n)$ the inequality

$$\int\limits_{R^n} \big|(Mf)(x)\big|^p \rho(x)\,dx \leq c \int\limits_{R^n} \big|f(x)\big|^p \sigma(x)\,dx$$

holds.

Proof. Obviously,

$$\big\|Kf\big\|_{L^p_\rho} = \sup_g \bigg|\int\limits_{R^n} (Kf)(x)g(x)\,dx,\bigg| \tag{1.13}$$

where the least upper bound is taken for all functions g such that

$$\big\|g\big\|_{L^{p'}_{\rho^{1-p'}}} \leq 1.$$

On the other hand, by virtue of Hölder's inequality we obtain

$$\int\limits_{R^n} \big|f(x)\big|Mg(x)\,dx \leq \big\|f\big\|_{L^p_\sigma} \big\|Mg\big\|_{L^{p'}_{\sigma^{1-p'}}}. \tag{1.14}$$

From Theorem B and the condition (1.6) it follows that

$$\int\limits_{R^n} \big|f(x)\big|Mg(x)\,dx \leq c\big\|f\big|_{L^p_\sigma} \big\|g\big\|_{L^{p'}_{\rho^{1-p'}}} \leq \big\|f\big\|_{L^p_\sigma}. \tag{1.15}$$

Using inequalities (1.13), (1.14) and (1.15), we complete the proof of the theorem. □

2. **The Multiple Generalized Marcinkiewicz Integrals.** Here we have established the two-weighted inequalities for multiple generalized Marcinkiewicz integrals.

Consider the integral transformation

$$Kf(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{\varphi_1(|x-t|,x_1(t))}{|x-t|+x_1(t)}\,\frac{\varphi_2(|y-s|,x_2(s))}{|y-s|+x_2(s)}\,f(t,s)dtds, \qquad (2.1)$$

where the functions φ_i and $x_i (i=1,2)$ satisfy the same conditions as those of the above functions φ and x.

Using twice inequality (1.7), we can conclude that for every pair of non-negative functions given on R^2, the inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}Kf(x,y)g(x,y)\,dxdy\le c\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x,y)M_1(M_2f)(x,y)\,dxdy \qquad (2.2)$$

is valid, where $M_1(M_2f)(x,y)$ is the iterated maximal function,

$$M_1(M_2f)(x,y)=\sup_h\frac{1}{h}\int_{x-h}^{x+h}\left(\sup_k\frac{1}{k}\int_{y-k}^{y+k}\big|f(t,s)\big|\,ds\right)dt.$$

The following statements are true.

Theorem 2.1 *Let $1<p<\infty$, v and w be the weighted functions such that they are even and increasing with respect to every variable. Assume also that $w(x,y)=w_1(x)w_2(y)$. If the condition*

$$\sup_{a,b>0}\left(\frac{1}{ab}\int_0^a\int_0^b w^{1-p'}(x,y)dxdy\right)\left(\frac{1}{ab}\int_0^a\int_0^b v(x,y)dxdy\right)^{p'-1}<\infty \qquad (2.3)$$

is fulfilled, then the operator K is bounded from $L^p_w(R^2)$ to the space $L^p_v(R^2)$.

Theorem 2.2. *Let $1<p<\infty$. Suppose that the weighted functions of two variables are even, decreasing with respect to every variable, and $w(x,y)=w_1(x)w_2(y)$. If, moreover, the condition*

$$\sup_{a,b>0}\left(\int_a^{\infty}\int_b^{\infty}\frac{w^{1-p'}(x,y)}{(xy)^{p'}}dxdy\right)\left(\int_0^a\int_0^b v(x,y)dxdy\right)^{p'-1}<\infty \qquad (2.4)$$

is fulfilled, then the operator K is bounded from $L^p(R^2)$ to the space $L^p_v(R^2)$.

The above stated theorems can be proved by the same way as Theorems 1.1 and 1.2 but here we have only to use inequality (2.2) and the criteria of the boundedness of iterated maximal functions from $L^p_w(R^2)$ to $L^p_v(R^2)$ established in [6].

Now we will proceed to investigating the one-weighted estimate for the operator K for which we prove the analogue of the theorem due to Calderon.

Introduce the following

Definition. We say that the weight $w : R^2 \to R^1$ belongs to the class $A_p(\mathbb{J})$ if

$$\sup \frac{1}{|\mathbf{J}|} \int\limits_{\mathbf{J}} w(x,y)dxdy \left(\frac{1}{|\mathbf{J}|} \int\limits_{\mathbf{J}} w^{1-p'}(x,y)dxdy \right)^{p-1} < \infty, \tag{2.5}$$

where the supremum is taken over all rectangles with sides parallel to the coordinate axes.

These classes have been introduced in [7]. It was proved that they characterize the classes of those weighted functions for which the strong is the maximal function and multiple Hilbert transforms are bounded in $L^p_w(R^2)$ (see, for e.g., [8]).

It has also been proved in [7] that the condition $w \in A_p(\mathbb{J})$ is equivalent to the simultaneous fulfilment of the following two conditions:

$$\sup_y \sup_I \frac{1}{|I|} \int\limits_{R^1} w(x,y)\, dx \left(\frac{1}{|I|} \int\limits_{R^1} w^{1-p'}(x,y)dxdy \right)^{p-1} < \infty \tag{2.6}$$

and

$$\sup_x \sup_I \frac{1}{|I|} \int\limits_{R^1} w(x,y)\, dy \left(\frac{1}{|I|} \int\limits_{R^1} w^{1-p'}(x,y)dydy \right)^{p-1} < \infty. \tag{2.7}$$

The classes $A_p(\mathbb{J})$ are narrower than the classes A_p introduced by B. Mukenhoupt, when the exact upper bound is taken over all squares. For example, for the power functions $w(x) = |x|^\alpha$ we have

$$w \in A_p \Leftrightarrow -2 < \alpha < 2(p-1),$$

while

$$w \in A_p(\mathbb{J}) \Leftrightarrow -1 < \alpha < p-1.$$

Theorem. *Let $1 < p < \infty$ and $w \in A_p(\mathbb{J})$. Then the operator K is bounded in $L^p_w(R^2)$.*

The proof of this theorem is similar to that of Theorem 1.1, but we have only to apply inequality (2.2) and the fact that the condition $w \in A_p(\mathbb{J})$ is equivalent to $w^{1-p'} \in A_{p'}(\mathbb{J})$, $p' \dfrac{p}{p-1}$. The latter is obvious by the definition of the class $A_p(\mathbb{J})$.

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