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CONTACT PROBLEM FOR ORTHOTROPIC PLATE WITH AN ELASTIC SEMI-INFINITE INCLUSION

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Abstract. A contact problem of the theory of elasticity of infinite orthotropic plates with an elastic semi-infinite inclusion of variable rigidity is considered. The problem is reduced to the system of integral differential equations with variable coefficient of singular operator. If such coefficient varies with power law we can manage to investigate the obtained equations, to get exact solutions and to establish behavior of unknown contact stresses at the ends of elastic inclusion.

რეზიუმე. განხილულია დრეკადობის თეორიის საკონტაქტო ამოცანა ცვლადი სიხისტის ნახევრად უსასრულო დრეკადი ჩართვის მქონე უსასრულო ორთოტროპული ფირფიტისათვის. ამოცანა მიყვანილია სინგულარული ოპერატორთან ცვლადი კოეფიციენტის მქონე ინტეგრო–დიფერენციალურ განტოლებათა სისტემაზე. როდესაც ეს კოეფიციენტი ხარისხოვანი კანონით იცვლება, გამოკვლეულია მიღებული განტოლებათა სისტემა, აგებულია ზუსტი ამონახსნები და დადგენილია უცნობი საკონტაქტო ძაბვების ყოფაქცევა ჩართვის ბოლოების მახლობლობაში.

Introduction

The contact problems on interaction of thin-shelled elements (stringers or inclusions) of various geometric form with massive deformable bodies belong to the extensive field of the theory of contact and mixed problems of mechanics of deformable rigid bodies. A vast number of work are devoted to problems of tension and bending of finite or infinite plates with thin absolutely rigid elements or elements with constant rigidity (see, e.g., [1-5]). We investigated the contact problems of tension and bending of plates with inclusion of variable rigidity, as for unknown contact stresses we obtained the Prandtl's integral differential equation with variable coefficient. In the case when the coefficient of a singular operator tends at the ends of line of integration to zero of any order, this equation is equivalent to singular

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integral equation of the third kind. We have investigated this equation and got or approximate solutions [6-8].

1. The basic relations

An orthotropic plate is strengthened with an inclusion of variable rigidity along semi-axis $(0,\infty)$ loaded with tangential efforts of intensity $\tau_0(x)$. Let us define the distribution law of contact strains along the line of contact. Suppose that after deformation the inclusion remains rectilinear and horizontal. According to the equilibrium equations of inclusion element and Hooke's law we have

$$\begin{gathered}\frac{du^{(1)}(x)}{dx}=\frac{1}{h_1(x)E_1(x)}\int\limits_0^x[\tau^{(1)}(t)-\tau_0(t)]dt\\ \frac{dv^{(1)}(x)}{dx}=0,\quad x>0\end{gathered}\tag{1.1}$$

and the equilibrium equations of the inclusion has the form

$$\int\limits_0^\infty[\tau^{(1)}(t)-\tau_0(t)]dt=0,\quad \int\limits_0^\infty q^{(1)}(t)dt=0\tag{1.2}$$

where $u^{(1)}(x)$ and $v^{(1)}(x)$ are the horizontal and vertical displacements of inclusion points; $\tau^{(1)}(x)$ and $q^{(1)}(x)$-the skippings of tangential and normal contact strains, subjects to determinations; $E_1(x)$ and $h_1(x)$-the modulus of elasticity of the inclusion and its thickness, respectively.

Besides, it is known that the derivations of displacements on the border of anisotropic, unorthotropic semiplane, dependent on outer load, acting on semi axis, has the form

$$\begin{aligned}\frac{du^{(2)}(x)}{dx}&=\frac{A_1^{(2)}}{\pi}\int\limits_0^\infty\frac{q^{(2)}(t)dt}{t-x}+B_1^{(2)}q^{(2)}(x)+\frac{A_2^{(2)}}{\pi}\int\limits_0^\infty\frac{\tau^{(2)}(t)dt}{t-x}\\ \frac{dv^{(2)}(x)}{dx}&=\frac{A_3^{(2)}}{\pi}\int\limits_0^\infty\frac{q^{(2)}(t)dt}{t-x}+B_4^{(2)}\tau^{(2)}(x)+\frac{A_4^{(2)}}{\pi}\int\limits_0^\infty\frac{\tau^{2}(t)dt}{t-x}\end{aligned}\tag{1.3}$$

where $u^{(2)}(x)$ and $v^{(2)}(x)$ are the boundary values of horizontal and vertical displacements on the semi axes; $\tau^{(2)}(x)$, $q^{(2)}(x)$-the boundary values of tangential and normal strains, respectively.

When a body is orthotropic and the axes of anisotropy are parallel to coordinate axes, the characteristic equation is biquadrate and its roots are purely imaginary [9] ($\mu_1^{(2)} = i\nu_1^{(2)}, \mu_2^{(2)} = i\nu_2^{(2)}$), then

$$A_1^{(2)} = 0, \quad B_1^{(2)} = -\beta_{11}^{(2)}\nu_1^{(2)}\nu_2^{(2)} - \beta_{12}^{(2)}, \quad A_2^{(2)} = -\beta_{11}^{(2)}(\nu_1^{(2)} + \nu_2^{(2)})$$

$$A_3^{(2)} = -\beta_{22}^{(2)}\frac{\nu_1^{(2)} + \nu_2^{(2)}}{\nu_1^{(2)}\nu_2^{(2)}}, \quad A_4^{(2)} = 0, \quad B_4^{(2)} = \beta_{22}^{(2)}\frac{1}{\nu_1^{(2)}\nu_2^{(2)}} + \beta_{12}^{(2)}$$

$\beta_{ij}^{(2)}$-the elastic coefficients of plates material.

Due to the contact condition of the inclusion with orthotropic plate

$$\frac{du^{(1)}(x)}{dx} = \frac{du^{(2)}(x)}{dx}, \quad \frac{dv^{(1)}(x)}{dx} = \frac{dv^{(2)}(x)}{dx} \tag{1.4}$$

2. Solution of problem

By formulas (1.1-1.4) for determination of unknown contact strains one may receive the following system of singular integral differential equations:

$$\begin{gathered}\int_0^\infty \frac{\psi'(t)dt}{t-x} + \frac{\pi B_1^{(2)}}{A_2^{(2)}}\varphi(x) = \frac{\pi}{h_1(x)E_1(x)A_2^{(2)}}\psi(x) + f_1(x) \\ \int_0^\infty \frac{\varphi(t)dt}{t-x} + \frac{\pi B_4^{(2)}}{A_3^{(2)}}\psi'(x) = f_2(x), \quad x > 0\end{gathered} \tag{2.1}$$

$$f_1(x) = -\int_0^\infty \frac{\tau_0(t)dt}{t-x}, \quad f_2(x) = -\frac{\pi B_4^{(2)}}{A_3^{(2)}}\tau_0(x), \quad \varphi(x) = q^{(1)}(x) = q^{(2)}(x)$$

$$\psi(x) = \int_0^x [\tau(t) - \tau_0(t)]dt, \quad \tau(t) = \tau^{(1)}(t) = \tau^{(2)}(t).$$

Assume that $E_1(x)h_1(x) = h_0 x^\omega$, $h_0 = \text{const}$ and ω is an arbitrary real number (in point $x = 0$ the inclusion has a qualitative cusp, while in infinity is becomes rigid).

System of equations (2.1) is considered with the following boundary conditions

$$\psi(0) = 0, \quad \psi(\infty) = 0, \quad \int_0^\infty \varphi(t)dt = 0.$$

Substituting $x = e^\xi$, $t = e^\xi$, Fourier's transformation of eqs. (2.1) gives:

$$\operatorname{scth}\pi s\Psi(s) + \frac{B_1^{(2)}}{A_2^{(2)}}is\Phi(s) = k\Psi(s + i(\omega - 1)) - F_1(s)$$

$$\operatorname{scth}\pi s\Phi(s) + \frac{B_4^{(2)}}{A_3^{(2)}}is\Psi(s) = -F_2(s), \quad |s| < \infty$$

where $\Phi(s)$ and $\Psi(s)$ are Fourier's transformations for functions $\varphi_0(\xi) = \int\limits_{-\infty}^{\xi} e^{\zeta}\varphi(e^{\zeta})d\zeta$ and $\psi_0(\xi) = \psi(e^{\xi})$, respectively, while $F_1(s)$ and $F_2(s)$-for functions $f_1(e^{\xi})$ and $f_2(e^{\xi})$, $k = -\dfrac{1}{h_0 A_2^{(2)}} > 0$.

From the last system, relative to function $\Psi(s)$, there is received the boundary condition of Karleman type problem for strip

$$\frac{1}{k}G(s)\Psi(s) - \Psi(s+i(\omega-1)) = F(s), \quad |s| < \infty \tag{2.2}$$

where $G(s) = s\operatorname{cth}\pi s\Big(1+\dfrac{B_1^{(2)}B_4^{(2)}}{A_2^{(2)}A_3^{(2)}}\operatorname{th}^2\pi s\Big)$, $F(s) = \dfrac{1}{k}\Big(\dfrac{iB_1^{(2)}F_2(s)}{A_2^{(2)}\operatorname{cth}\pi s} - F_1(s)\Big)$.

After solution of the last problem we can find function $\Phi(z)$ by formula

$$\Phi(z) = -\frac{F_2(z) + iB_4^{(2)}z\Psi(z)/A_3^{(2)}}{z\operatorname{cth}\pi z}. \tag{2.3}$$

Therefore, we consider the problem: find function $\Psi(z)$, which is holomorphic in strip $0 < \operatorname{Im} z < \omega - 1$, vanishing in infinity, continuously extendable on the border of the strip and satisfying condition (2.2).

Let us represent the coefficient of problem (2.2) in the form

$$G(s) = -isG_0(s)\operatorname{cth}\pi\operatorname{sth}\frac{\pi s}{2(\omega-1)}\frac{\operatorname{sh}(\pi(s+i(\omega-1))/2(\omega-1))}{\operatorname{sh}(\pi s/2(\omega-1))}\times \\ \times\Big(1+\frac{B_1^{(2)}B_4^{(2)}}{A_2^{(2)}A_3^{(2)}}\Big) \tag{2.4}$$

where $G_0(s) = \Big(1+\dfrac{B_1^{(2)}B_4^{(2)}}{A_2^{(2)}A_3^{(2)}}\operatorname{th}^2\pi s\Big)\Big/\Big(1+\dfrac{B_1^{(2)}B_4^{(2)}}{A_2^{(2)}A_3^{(2)}}\Big)$. Using the properties of biquadrate equation, one can show that $-1 < \dfrac{B_1^{(2)}B_4^{(2)}}{A_2^{(2)}A_3^{(2)}} < 0$, therefore, function $G_0(s)$ is positive and $G_0(\pm\infty) = 1$.

Since the index of function $G_0(s)\operatorname{cth}\pi\operatorname{sth}\dfrac{\pi s}{2(\omega-1)}$ equals zero and $\ln\Big[G_0(s)\operatorname{cth}\pi\operatorname{sth}\dfrac{\pi s}{2(\omega-1)}\Big]$ is integrable on entire axis, then it is represented in the form

$$G_0(s)\operatorname{cth}\pi\operatorname{sth}\frac{\pi s}{2(\omega-1)} = \frac{X_0(s)}{X_0(s+i(\omega-1))} \tag{2.5}$$

where

$$X_0(z) = \exp\bigg\{\frac{1}{2(\omega-1)}\int\limits_{-\infty}^{\infty}\ln\Big[G_0(s)\operatorname{cth}\pi\operatorname{sth}\frac{\pi s}{2(\omega-1)}\Big]\operatorname{cth}\pi(s-z)ds\bigg\}.$$

Function $X_0(z)$ is holomorphic in the strip, continuous on the border of the strip and bounded in the closed strip $0 \le \operatorname{Im} z \le \omega - 1$ and in infinity.Putting (2.4), (2.5) into condition (2.2) and introducing notations

$$\Psi_1(z) = \frac{zX_0(z)\Psi(z)}{\operatorname{sh}(\pi z/2(\omega-1))}, \quad H_0 = 1 + \frac{B_1^{(2)}B_4^{(2)}}{A_2^{(2)}A_3^{(2)}}$$

we receive

$$\frac{H_0}{k}(\omega - 1 - is)\Psi_1(s) + \Psi_1(s + i(\omega-1)) = \\ = \frac{F(s)(s+i(\omega-1))X_0(s+i(\omega-1))}{\operatorname{sh}(\pi(s+i(\omega-1))/2(\omega-1))}. \tag{2.6}$$

The coefficient of condition (2.6) is represented in the form

$$\frac{H_0}{k}(\omega-1-is) = \frac{X_1(s+i(\omega-1))}{X_1(s)}, \quad |s| < \infty$$

where $X_1(z) = \exp\Big(-\dfrac{iz}{\omega-1}\ln\dfrac{(\omega-1)H_0}{k}\Big)\Gamma(1-\dfrac{iz}{\omega-1})$, $0 < \operatorname{Im} z < \omega - 1$. Substituting the value into (2.6), the solution of this problem takes the form [10]

$$\Psi(z) = -\frac{X(z)}{2iz(\omega-1)}\int_{-\infty}^{\infty}\frac{F(t)(t+i(\omega-1))dt}{X(t+i(\omega-1))\operatorname{sh}(\pi(t-z)/(\omega-1))} \tag{2.7}$$

where $X(z) = \dfrac{X_1(z)}{X_0(z)}\operatorname{sh}\dfrac{\pi z}{2(\omega-1)}$, $0 < \operatorname{Im} z < \omega - 1$

Function $X(z)$ satisfies condition

$$C_1|t|^{1/2} \le |X(t+i\tau)| \le C_2|t|^{3/2}, \quad 0 \le \tau \le \omega - 1.$$

When functions $F_1(t)$ and $F_2(t)$ exponentially vanish in infinity, then the integral in solution (2.7) exponentially decreases, i.e., it is continuous in closed strip $0 \le \operatorname{Im} z \le \omega - 1$ and exponentially vanishes in infinity.

Therefore, the solution of given problem is given by formula (2.7) and function $\Phi(z)$ is determined by formula (2.3).

Suppose that $\omega \ge 2$. The contact strain can be calculated by formula

$$\tau^{(1)}(x) - \tau_0(x) = \psi'(x) = \frac{x^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} it\Psi(t)e^{-it\ln x}dt =$$

$$= \frac{ix^{-1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(t+i(\omega-1))\Psi(t+i(\omega-1))e^{-i(t+i(\omega-1))\ln x}dt =$$

$$= x^{\omega-2}\tau_1(x),$$

where $\tau_1(x)$ is bounded function in the neighborhood of point $x = 0$. Function $\Phi(z)$ exponentially vanishes in infinity, it is analytically extendable in strip $0 < \operatorname{Im} z < \omega - 1$, except, perhaps, the point: $z_0 = \dfrac{i}{2}$, $z_1 = \dfrac{3i}{2}, \dots$ being poles in strip. Then according to Cauchy's formula we have

$$q^{(1)}(x) = x^{-1}\varphi_0'(\ln x) = x^{-1} \operatorname*{res}_{t=\frac{i}{2}} [it\Phi(t)e^{-it\ln x}] + x^{-1} \operatorname*{res}_{t=\frac{3i}{2}} [it\Phi(t)e^{-it\ln x}] +$$

$$+ \frac{ix^{-1}}{\sqrt{2\pi}} \int\limits_{-\infty}^{\infty} (t + i(\omega-1))\Phi(t + i(\omega-1))e^{-i(t+i(\omega-1))\ln x} dt =$$

$$= c_1 x^{-1/2} + c_2 x^{1/2} + x^{\omega-2} q_1(x),$$

where c_1 and c_2 are known constant, $q_1(x)$ is bounded function for $x \geq 0$.

For $1 < \omega < 2$, according to condition (2.2) we see that function $\Psi(z)$, defined by formula (2.7), is analytically extendable in strip $0 < \operatorname{Im} z < 1$, except point $z = \dfrac{i}{2}$. Therefore, tangential contact strain in the neighborhood of point $x = 0$ has the following estimation: $\tau^{(1)}(x) - \tau_0(x) = O(x^{-1/2})$, $x \to 0$.

Now consider the case when $\omega \leq 1$. Introducing notation: $m = -(\omega - 1)$, condition (2.2) turns in

$$\frac{1}{k} G(s)\Psi(s) - \Psi(s - im) = F(s), \quad -\infty < s < \infty. \tag{2.8}$$

The problem takes the form: find a function, which is holomorpic in strip $-m < \operatorname{Im} z < m$, vanishing in infinity, bounded in entire strip, except points $z_k^+ = t_k^+ + i\tau_k^+$ $(k = 0, 1, 2, \dots, l)$, being zeros of function $G(z)$ in upper strip.

If we solve the problem: find a function, which is holomorphic in strip $-m < \operatorname{Im} z < 0$, vanishing in infinity, continuously extendable on the border of the strip and according to condition (2.8), then the solution of preceding problem is given by function

$$\Psi_0(z) = \begin{cases} \Psi(z), & -m < \operatorname{Im} z < 0 \\ \dfrac{F(z) - \Psi(z - im)}{G(z)} k, & 0 < \operatorname{Im} z < m. \end{cases}$$

Analogously to previous reasoning for function $\Psi(z)$ we have the following representation

$$\Psi(z) = \frac{\widetilde{X}(z)}{2im} \int\limits_{-\infty}^{\infty} \frac{F(t)dt}{\widetilde{X}(t)\operatorname{sh}(\pi(t-z)/m)}, \quad -m < \operatorname{Im} z < 0$$

$$\widetilde{X}(z) = \frac{\widetilde{X}_0(z)\widetilde{X}_1(z)}{z} \operatorname{sh}\frac{\pi z}{2m}, \quad \widetilde{X}_1(z) = \Gamma\Big(1 + \frac{iz}{m}\Big) \exp\Big(\frac{iz}{m} \ln \frac{mH_0}{k}\Big)$$

$$\widetilde{X}_0(z) = \exp\left\{\frac{1}{2im}\int\limits_{-\infty}^{\infty} \ln\Big[G_0(t)\operatorname{cth}\pi \operatorname{tth}\frac{\pi t}{2m}\Big]\operatorname{cth}\pi(t-z)dt\right\}.$$

Unknown contact strain: $\tau^{(1)}(x)-\tau_0(x) = c_0 x^{\tau_0^+ -1}+\varphi_2(x)$, $c_0 = \text{const}$, $\varphi_2(x)$-is bounded function for $x \geq 0$.

For $\omega = 1$ condition (2.2) gives

$$\Psi(z) = k\frac{F(z)}{G(z)-k}$$

and for tangential strain we have: $\tau^{(1)}(x)-\tau_0(x) = O(x^{\mu-1})$, for $x \to 0$, where μ is the zero of function $G(z)-k$, closest to real axis in the upper semi-plane.

In the case of isotropic body (ν is Poisson's ratio, E-modulus of elasticity) the system of eqs. (2.1) gives

$$\begin{aligned}
&s\operatorname{th}\pi s\Psi(s)+\frac{1-2\nu}{2(1-\nu)}is\Phi(s) = k_0\Psi(s+i(\omega-1))-F_1(s),\\
&s\operatorname{th}\pi s\Phi(s)+\frac{1-2\nu}{2(1-\nu)}is\Psi(s) = -F_2(s), \quad k_0 = \frac{E}{h_0(1-\nu^2)},
\end{aligned}$$

which gives the following condition

$$\frac{1}{k_0}G(s)\Psi(s)-\Psi(s+i(\omega-1)) = F(s), \quad -\infty < s < \infty$$

where

$$\begin{aligned}
G(s) &= s\operatorname{th}\pi s\Big[1-\frac{(1-2\nu)^2}{4(1-\nu)^2}\operatorname{th}^2\pi s\Big],\\
F(s) &= \frac{1}{k_0}\Big[F_1(s)-\frac{1-2v}{2(1-v)}\operatorname{th}\pi s F_2(s)\Big].
\end{aligned}$$

Therefore, the contact problem for orthotropic plane with elastic inclusion, when the axes of anisotropy are parallel to co-ordinate axes, is being solved exactly in the same way as the analogous problem for isotropic plane.

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